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ON SOME CONVEXITIES OF ORLICZ AND ORLICZ-BOCHNER SPACES

S. CHEN and H. HUDZIK

Abstract: In the whole paper it is assumed that μ is a non-atomic infinite measure. An elementary proof of the Akimovič theorem proved in [1], concerning a renorming of reflexive Orlicz spaces, is given. Orlicz spaces with $\delta_{L^\Phi}(1)>0$ are characterized. This yields some sufficient conditions for uniformly normal structure of Orlicz spaces. Some examples of Orlicz spaces L^Φ with $\delta_{L^\Phi}(1)>0$ and not being uniformly convex as well as of uniformly non-square Orlicz spaces L^Φ with $\delta_{L^\Phi}(1)=0$ are given. Locally uniformly non-$l_n^{(1)}$ Orlicz-Bochner spaces are also characterized. Finally, some connections between uniform non-squareness and nearly flatness of Orlicz spaces are given.

Key words and phrases: Orlicz function, Orlicz space, Orlicz-Bochner space, condition Δ_2, modulus of convexity, uniform convexity, uniform non-squareness, reflexivity, local uniform non-$l_n^{(1)}$ property, uniform smoothness, nearly flatness, flatness.

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0. Introduction. In the following, the notion of the modulus of convexity of a Banach space $(X,\|\ \|)$ will be needed. It is a function $\delta_X(\cdot):(0,2]\to[0,1]$ defined by

$$\delta_X(\varepsilon)=\inf\{1-\|\tfrac{x+y}{2}\|:\|x\|\le 1,\ \|y\|\le 1,\ \|x-y\|\ge\varepsilon\}.$$

Recall that a Banach space $(X,\|\ \|)$ is said to be uniformly convex if $\delta_X(\varepsilon)>0$ for any $\varepsilon\in(0,2]$, and it is said to be uniformly non-square if $\delta_X(\varepsilon)>0$ for some $\varepsilon\in(0,2)$, which means that there is $\delta\in(0,1)$ such that $\min(\|x+y\|,\|x-y\|)\le 2(1-\delta)$ for any $x,y\in B_X(1)$ (= the unit ball of X). Uniform non-squareness has been considered by R.C. James in [11] in connection with reflexivity of Banach spaces. Namely, it is proved there that any uniformly non-square Banach space is reflexive. For some Orlicz spaces equipped with Luxemburg norm, uniform non-squareness coincides with reflexivity (see [6]). The same holds for Orlicz spaces equipped with Orlicz norm (see [20] and [22]).

A Banach space $(X, \|\ \|)$ is called locally uniformly non-$l_n^{(1)}$ ($n \geq 2$, $n \in \mathbf{N}$) if for any $x_1 \in B_X(1)$ there is $\delta(x_1) \in (0,1)$ such that for all $x_2, \ldots, x_n \in B_X(1)$ we have $\|\frac{1}{n}(x_1 \pm x_2 \pm \ldots \pm x_n)\| \leq 1 - \delta(x_1)$ for some choice of signs (see [19]).

A Banach space $(X, \|\ \|)$ is said to be flat if there is a curve $g:[0,4] \to X$ such that $\|g(s)\| = 1$, $g(s+2) = -g(s)$ for any s in $[0,2]$ and $\|g(s)-g(s')\| = |s-s'|$ for all s and s' in $[0,2]$ (see [9] and [19]). Locally uniformly non-$l_n^{(1)}$ Banach spaces are good spaces in this sense that they are not flat (see [19]). As will be noted below, any Orlicz space over a non-atomic infinite measure is either locally uniformly non-$l_n^{(1)}$ or is flat.

We say that a Banach space $(X, \|\ \|)$ has uniformly normal structure if there exists $\delta \in (0,1)$ such that for every convex, bounded and closed set $A \subset X$ with positive diameter there is $x \in A$ such that

$$\sup_{y \in A} \|x-y\| \leq (1-\delta) \operatorname{diam} A$$

(see [2],[3],[4] and [15]). In these papers the notion of normal structure is studied. Both notions - normal structure and uniformly normal structure - are useful in the fixed point theory. It is known that X has a uniformly normal structure whenever $\delta_X(1) > 0$ (see e.g. [4]). So, it is natural and important to find a criterion for $\delta_{L^\Phi}(1) > 0$. It will be done in the case of an infinite non-atomic measure.

Now, we shall give some definitions and notations concerning Orlicz spaces. A function $\Phi: R \to [0,\infty]$ is said to be an Orlicz function if it is convex, even, vanishing and continuous at 0 and not identically equal to 0. In the following, (T,Σ,μ) denotes a space of infinite non-atomic measure. For a given Orlicz function Φ, the Orlicz space $L^\Phi(\mu)$ is defined as the set of all Σ-measurable, real-valued functions f defined on T such that $I_\Phi(\lambda f) = \int_T \Phi(\lambda f(t))d\mu < \infty$ for some $\lambda > 0$. The space $L^\Phi(\mu)$ equipped with the Luxemburg norm

$$\|f\|_\Phi = \inf\{\varepsilon > 0 : I_\Phi(f/\varepsilon) \leq 1\}$$

is a Banach space (see [14],[16] and [18]). We say that an Orlicz function Φ satisfies condition Δ_2 if there is $K > 0$ such that $\Phi(2u) \leq K\Phi(u)$ for all $u \in \mathbf{R}$. For a given Orlicz function Φ, Φ^* denotes its complementary function in the sense of Young, i.e.

$$\Phi^*(v) = \sup_{u \geq 0}\{|v|u - \Phi(u)\}$$

for any $v \in \mathbf{R}$. If φ is the right-hand derivative of Φ and $\varphi^*(v) = \sup[u \geq 0 : \varphi(u) \leq v]$ for $v \geq 0$, then $\Phi^*(v) = \int_0^{|v|} \varphi^*(s)ds$ for any $v \in \mathbf{R}$.

1. **Results.** First, we shall prove the following auxiliary

1.1. **Lemma.** For a given Orlicz function Φ, the following assertions are equivalent:

(1) Φ^* satisfies condition Δ_2.

(2) There are $a>1$ and $k\in(0,1)$ such that $\Phi(\frac{u}{a})\leq\frac{k}{a}\Phi(u)$ for all $u\in\mathbf{R}$.

(3) For any positive integer $n\geq 2$ there is $\eta\in(0,1)$ such that

$$\sum_{\pm 1}\Phi\left(\frac{u_1\pm\ldots\pm u_n}{n}\right)\leq\frac{2^{n-1}\eta}{n}\sum_{i=1}^{n}\Phi(u_i)$$

for all $u_1,\ldots,u_n\in\mathbf{R}$, where the symbol "$\sum_{\pm 1}$" stands for summation over all 2^{n-1} choices of signs.

(4) For any $a>1$ there is $k\in(0,1)$ such that $\Phi(\frac{u}{a})\leq\frac{k}{a}\Phi(u)$ for all $u\in\mathbf{R}$.

(5) For any $a>1$ there is $\xi>1$ such that $\Phi(\frac{\xi}{a}u)\leq\frac{\xi^{-1}}{a}\Phi(u)$ for all $u\in\mathbf{R}$.

Proof. (1) $\Longrightarrow$ (5). First note that if Φ^* satisfies condition Δ_2, then for any $b>1$ there is $c>1$ such that $\Phi^*(cu)\leq b\,\Phi^*(u)$ for all $u\in\mathbf{R}$. Indeed, the function f defined by

$$f(c)=\sup_{u\neq 0}\{\Phi^*(cu)/\Phi^*(u)\}$$

for $c\geq 1$, is convex and has finite values. Moreover, $f(1)=1$ and $f(c)\geq c$ for $c\geq 1$. Since f is continuous, it has Darboux property. Thus, for any $b>1$ there exists $c>1$ such that $f(c)=b$. It means that

(+) $\quad\Phi^*(cu)\leq b\,\Phi^*(u)$

for all $u\in\mathbf{R}$. Observe that this method may also be applied in the case of an arbitrary Banach space X and an arbitrary Orlicz function Φ^* defined on X. By (+), there is $\xi>1$ such that

$$\Phi^*(\xi^2 v)\leq a\xi\Phi^*(v)$$

for all $v\geq 0$. So,

$$\Phi\left(\frac{\xi}{a}u\right)=\sup_{v\geq 0}\left\{\frac{\xi}{a}|u|v-\Phi^*(v)\right\}\leq\sup_{v\geq 0}\left\{\frac{\xi}{a}|u|v-\frac{1}{\xi a}\Phi^*(\xi^2 v)\right\}=$$

$$=\sup_{v\geq 0}\left\{\frac{\xi^2}{\xi a}|u|v-\frac{1}{\xi a}\Phi^*(\xi v)\right\}=\frac{1}{\xi a}\Phi(u).$$

The implications (5) $\Longrightarrow$ (4) and (3) $\Longrightarrow$ (2) are obvious.

(4) $\Longrightarrow$ (3). By (4), there is $k\in(0,1)$ such that $\Phi(\frac{u}{n})\leq\frac{k}{n}\Phi(u)$ for all $u\in\mathbf{R}$. If $u_1,\ldots,u_n\in\mathbf{R}$, then $|u_1\pm\ldots\pm u_n|\leq\max|u_i|$ for some choice of signs, and

$$\Phi\left(\frac{u_1\pm\ldots\pm u_n}{n}\right)\leq\Phi\left(\frac{\max|u_i|}{n}\right)\leq\frac{k}{n}\Phi(\max|u_i|)\leq\frac{k}{n}\sum_{i=1}^{n}\Phi(u_i)$$

for this choice of signs. By convexity of Φ, we have

$$\Phi\left(\frac{u_1\pm\ldots\pm u_n}{n}\right)\leq\frac{1}{n}\sum_{i=1}^{n}\Phi(u_i)$$

for all remaining choices of signs. Hence it follows that

$$\sum_{\pm 1}\Phi\left(\frac{u_1\pm\ldots\pm u_n}{n}\right)\leq\frac{k+2^{n-1}-1}{n}\sum_{i=1}^{n}\Phi(u_i),$$

i.e. (3) holds with $\eta=1-(1-k)/2^{n-1}$.

(2) $\Rightarrow$ (1). We have

$$\Phi^*\left(\frac{u}{k}\right)\leq\sup_{v\geq 0}\left\{\frac{|u|}{k}v-\frac{a}{k}\Phi\left(\frac{v}{a}\right)\right\}=\frac{a}{k}\sup_{v\geq 0}\left\{|u|\frac{v}{a}-\Phi\left(\frac{v}{a}\right)\right\}=\frac{a}{k}\Phi^*(u)$$

for all $u\in\mathbf{R}$. Since k^{-1} is greater than 1, it means that Φ^* satisfies condition Δ_2.

Note. Conditions (1),(2),(4) and (5) are also equivalent for any Orlicz function Φ defined on an arbitrary Banach space. The function Φ^* is then defined on the dual X^* of X, by the formula

$$\Phi^*(x^*)=\sup_{x\in X}\{|x^*(x)|-\Phi(x)\}.$$

An analogous lemma to that given above has been recently obtained independently by A. Kamińska and B. Turett in [13].

1.2. Theorem. Let $L^{\Phi}(\mu)$ be a reflexive Orlicz space. Then there exists an Orlicz function Φ_1 equivalent to Φ such that $(L^{\Phi_1}(\mu), \|\ \|_{\Phi_1})$ and $(L^{\Phi_1^*}(\mu), \|\ \|_{\Phi_1^*})$ are uniformly convex and uniformly smooth spaces. The same holds for the Orlicz norms $\|\ \|^0_{\Phi_1}$ and $\|\ \|^0_{\Phi_1^*}$ instead of the Luxemburg norms $\|\ \|_{\Phi_1}$ and $\|\ \|_{\Phi_1^*}$, respectively.

Proof. By assumptions, Φ satisfies condition (4) from Lemma 1.1. Denote $\psi(u)=\Phi(u)/u$ for $u>0$. Then for any $a>1$ there is $k<1$ such that

$(*)$ $\psi\left(\frac{u}{a}\right)\leq k\psi(u)$ for all $u\geq 0$. Let ψ^* denote the generalized inverse function of ψ, i.e. $\psi^*(s)=\sup\{t\geq 0:\psi(t)\leq s\}$ for every $s\geq 0$. Since Φ satisfies condition Δ_2, for any $b>1$ there is $a>1$ such that $\Phi(au)\leq b\Phi(u)$ for all $u\in\mathbf{R}$. Hence

$$\psi(au)\leq\frac{\Phi(au)}{u}\leq b\frac{\Phi(u)}{u}=b\psi(u)$$

for all $u>0$. Therefore

(**) $\psi^*(\frac{v}{b})=\sup\{u\geq 0: \psi(u)\leq \frac{v}{b}\}\leq \frac{1}{a}\sup\{au: \psi(au)\leq v\}=\frac{1}{a}\psi^*(v)$.

Define $\Phi_1(u)=\int_0^{|u|}\psi(t)dt$ for all $u\in \mathbf{R}$. Then $\Phi_1^*(v)=\int_0^{|u|}\psi^*(s)\,ds$ for all $v\in\mathbf{R}$. The inequalities $\Phi(u/2)\leq\Phi_1(u)\leq\Phi(u)$ hold for all $u\in\mathbf{R}$. Therefore, $L^{\Phi_1}(\mu)=L^{\Phi}(\mu)$ and $L^{\Phi_1^*}(\mu)=L^{\Phi^*}(\mu)$. Moreover, $\frac{1}{2}\|f\|_\Phi\leq\|f\|_{\Phi_1}\leq\|f\|_\Phi$ for any $f\in L^{\Phi}(\mu)$. By conditions (*) and (**), the functions Φ_1 and Φ_1^* are uniformly convex (see [1]). Therefore, both spaces $(L^{\Phi_1}, \|\ \|_{\Phi_1})$ and $(L^{\Phi_1^*}(\mu), \|\ \|_{\Phi_1^*})$ are uniformly convex (see [12] and [16]). The spaces $L^{\Phi_1}(\mu)$ and $L^{\Phi_1^*}(\mu)$ are also uniformly convex under Orlicz norms $\|\ \|^0_{\Phi_1}$ and $\|\ \|^0_{\Phi_1^*}$, respectively (for definition of Orlicz norm see [14]). This follows by the results of Milnes [17]. Thus, both dual spaces $(L^{\Phi_1}, \|\ \|_{\Phi_1})$ and $(L^{\Phi_1}, \|\ \|^0_{\Phi_1^*})$ are uniformly convex (and so also uniformly smooth). The same holds for the pair $(L^{\Phi_1}, \|\ \|^0_{\Phi_1}),(L^{\Phi_1^*}, \|\ \|^0_{\Phi_1^*})$. The last statement follows by the fact that the criteria for uniform convexity of Orlicz norm and of Luxemburg norm are the same. This follows also by the above considerations and by the fact that $\|\ \|_{\Phi_1}$, $\|\ \|^0_{\Phi_1^*}$ and $\|\ \|^0_{\Phi_1^*}$, $\|\ \|_{\Phi_1^*}$ are pairs of mutually associated norms.

Note. This theorem was obtained in a slightly weaker version by V. Akimovič [1]. However, the proof given here is different and simpler than that given in [1]. In an analogous manner, the proof of Theorem 1.1.7 concerning Musielak-Orlicz spaces given in [10] may be simplified.

Before we shall give the next theorem, we introduce the following parameter for an Orlicz function Φ :

$$p(\Phi)=\sup\{a\in(0,1): \sup_{u>0}[2\Phi(u+au)/2)/(\Phi(u)+\Phi(au))]<1\},$$

where the convention $\sup\emptyset=0$ is used. It is evident that if $p(\Phi)>0$, then for any $a<p(\Phi)$ there is $\delta=\delta(a)\in(0,1)$ such that

$$\Phi(\tfrac{1}{2}(u+au))\leq\tfrac{1}{2}(1-\delta)\{\Phi(u)+\Phi(au)\}$$

for all $u\in\mathbf{R}$. In particular, if $p(\Phi)=1$, then Φ is uniformly convex (see [12] and [16]).

1.3. Theorem. For a given Orlicz function Φ the following assertions are equivalent:

(1) $\delta_L\Phi(1)>0$.

(ii) Φ satisfies condition Δ_2 and $p(\Phi)>\frac{1}{3}$.

(iii) Φ satisfies condition Δ_2 and the following condition:

(a) there is $\sigma\in(0,1)$ such that for any $u,v\in\mathbf{R}$ satisfying $|u-v|\geq\frac{1-\sigma}{2}|u+v|$, we have $\Phi(\frac{1}{2}(u+v))\leq\frac{1-\sigma}{2}\{\Phi(u)+\Phi(v)\}$.

Proof. First, note that for any Orlicz function Φ satisfying condition Δ_2, we have

(1.1) $b=\sup_{u>0}\{2\Phi(\frac{1}{2}(u+p(\Phi)u))/(\Phi(u)+\Phi(p(\Phi)u))\}=1.$

Indeed, by condition Δ_2 it follows that for any $\eta>1$ there exists $\xi>1$ such that

(1.2) $\Phi(\xi u)\leq\eta\Phi(u)$

for all $u\in\mathbf{R}$ (see the proof of Lemma 1.1). Assume that (1.1) does not hold, i.e. $b<1$. Let $\xi>1$ be the number satisfying (1.2) with $\eta=1/\sqrt{b}$, and let $r=(\xi(1+p(\Phi))-1)/p(\Phi)$. Then $r>1$ and $\xi(1+p(\Phi))=1+rp(\Phi)$. Thus,

$$\Phi(\tfrac{1}{2}(1+rp(\Phi))u)=\Phi(\tfrac{1}{2}\xi(1+p(\Phi))u)\leq\frac{1}{\sqrt{b}}\Phi(\tfrac{1}{2}(1+p(\Phi))u)\leq$$

$$\leq\frac{b}{\sqrt{b}}\,\frac{\Phi(u)+\Phi(p(\Phi)u)}{2}\leq\frac{\sqrt{b}}{2}\{\Phi(u)+\Phi(rp(\Phi)u)\}$$

for all $u\in\mathbf{R}$. This contradicts the definition of $p(\Phi)$. Now, we shall prove the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii). Assume that (ii) does not hold. If condition Δ_2 is not satisfied, then (iii) does not hold . So, assume that $p(\Phi)\leq\frac{1}{3}$ and denote

$$f_\Phi(\alpha)=\sup_{u>0}\{2\Phi(\tfrac{1}{2}(u+\alpha u))/(\Phi(u)+\Phi(\alpha u))\}$$

for all $\alpha\in[0,1]$. We have by (1.1) that $f_\Phi(\frac{1}{3})=1$. Since the function f_Φ is non-decreasing in $[0,1]$ (see [1]), we have

(1.3) $\sup\{2\Phi(\frac{1}{2}(u+v))/(\Phi(u)+\Phi(v)):u,v>0,v\leq\frac{u}{3}\}=$

$$=\sup\{2\Phi(\tfrac{1}{2}(u+v))/(\Phi(u)+\Phi(v)):u,v>0,u-v\geq\tfrac{u+v}{2}\}=1.$$

It means that condition (a) from (iii) is not satisfied. Thus, the implication (iii) $\Rightarrow$ (ii) is proved.

(ii) $\Rightarrow$ (i). Only the fact that $p(\Phi)>\frac{1}{3}$ implies (a) needs to be proved. If $p(\Phi)>\frac{1}{3}$, then there is $\alpha>\frac{1}{3}$ such that $f_\Phi(\alpha)<1$. Hence it follows that

(1.4) $\sup\{2\Phi(\frac{1}{2}(u+v))/(\Phi(u)+\Phi(v)):u,v\geq 0,u+v>0,v\leq\alpha u\}=$

$$=\sup\{2\Phi(\tfrac{1}{2}(u+v))/(\Phi(u)+\Phi(v)):u,v\geq 0,u+v>0,u-v\geq\tfrac{1-\alpha}{1+\alpha}(u+v)\}<1.$$

Now, assume that $u\cdot v\leq 0$. The number $\delta=f_\Phi(0)$ is smaller than 1, and we have

(1.5) $\Phi(\frac{1}{2}(u+v))\leq\Phi(\frac{1}{2}\max(|u|,|v|))\leq\frac{\delta}{2}\Phi(\max(|u|,|V|))\leq\frac{\delta}{2}\{\Phi(u)+\Phi(v)\}.$

Since $(1-\alpha)/(1+\alpha)<\frac{1}{2}$, combining (1.4) and (1.5), we obtain condition (a) with some $\sigma\in(0,1)$. In virtue of condition Δ_2 there exists a function $p(0,1)\to(0,1)$ such that $\|f\|_\Phi\le 1-p(\varepsilon)$, whenever $I_\Phi(f)\le 1-\varepsilon$, $\varepsilon\in(0,1)$. Namely, it suffices to put

$$p(\varepsilon)=\sup\{\sigma\in(0,1):\ \sup_{u>0}[\Phi(u/(1-\sigma))/\Phi(u)]\le\frac{1}{1-\varepsilon}\}$$

(see [7]). Let $f,g\in L^\Phi(\mu)$; $\|f\|_\Phi\le 1$, $\|g\|_\Phi\le 1$, $\|f-g\|_\Phi\ge 1$. Then we have $I_\Phi(f)\le 1$, $I_\Phi(g)\le 1$, $I_\Phi(f-g)\ge 1$. Define

$$A=\{t\in T:|f(t)-g(t)|\ge\frac{1-\sigma}{2}|f(t)+g(t)|\}.$$

We have $I_\Phi((f-g)\chi_{T\setminus A})\le 1-\sigma$. Thus, $I_\Phi((f-g)\chi_A)\ge\sigma$. Applying condition Δ_2, we get

$$I_\Phi((f-g)\chi_A)\le\frac{1}{2}\{I_\Phi(2f\chi_A)+I_\Phi(2g\chi_A)\}\le\frac{K}{2}\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}.$$

Using this estimation and applying condition (a) from (iii) for $t\in A$, we obtain

$$1-I_\Phi(\tfrac{1}{2}(f+g))\ge\frac{1}{2}\{I_\Phi(f)+I_\Phi(g)\}-I_\Phi(\tfrac{1}{2}(f+g))\ge$$

$$\ge\frac{1}{2}\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}-I_\Phi(\tfrac{1}{2}(f+g)\chi_A)\ge$$

$$\ge\frac{1}{2}\ I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}-\frac{1-\sigma}{2}\{I_\Phi(f\chi_A)+I_\Phi(g\chi_A)\}=$$

$$=\frac{\sigma}{2}\{I_\Phi(f)\chi_A)+I_\Phi(g\chi_A)\}\ge\frac{\sigma^2}{K}.$$

Hence it follows that $I_\Phi(\frac{1}{2}(f+g))\le 1-\sigma^2/K$, i.e. $\|\frac{1}{2}(f+g)\|_\Phi\le 1-p(\sigma^2/K)$. This means that $\delta_{L^\Phi}(1)\ge p(\sigma^2/K)>0$.

For the proof of the implication (i) $\Rightarrow$ (iii) assume first that condition Δ_2 is not fulfilled. Then $L^\Phi(\mu)$ contains an isometric copy of l^∞ and so $\delta_{L^\Phi}(1)=0$ (even $\delta_{L^\Phi}(2)=0$), i.e. (i) does not hold. Assume now that property (a) does not hold, i.e. for any $\sigma\in(0,1)$ there exist $u,v\in\mathbf{R}$ such that

$$|u-v|\ge\frac{1-\sigma}{2}|u+v|\ \text{ and }\ \Phi(\tfrac{1}{2}(u+v))>\frac{1-\sigma}{2}\{\Phi(u)+\Phi(v)\}.$$

Choose a set $B\in\Sigma$ such that $(\Phi(u)+\Phi(v))\mu(B)=2$ and let C be a subset of B such that $\mu(C)=\mu(B\setminus C)$. Define

$$f(t)=u\chi_C(t)+v\chi_{B\setminus C}(t),\quad g(t)=\chi_C(t)+u\chi_{B\setminus C}(t).$$

We have $I_\Phi(f)=I_\Phi(g)=1$. So, $\|f\|_\Phi=\|g\|_\Phi=1$. Moreover,

$$I_\Phi((f-g)/(1-\sigma)^2)\ge(1-\sigma)^{-1}I_\Phi((f+g)/2)=(1-\sigma)^{-1}\Phi(\tfrac{1}{2}(u+v))\mu(B)>$$

$$>\frac{1}{2}\{\Phi(u)+\Phi(v)\}\mu(B)=1.$$

Hence, $\|f-g\|_\Phi\ge(1-\sigma)^2$. We have also

$$I_\Phi((f+g)/2(1-\sigma))\ge(1-\sigma)^{-1}I_\Phi(\tfrac{1}{2}(f+g))=(1-\sigma)^{-1}\Phi(\tfrac{1}{2}(u+v))\mu(B)>$$

$$> \frac{1}{2}\{\Phi(u)+\Phi(v)\}\mu(B)=1.$$

Thus, $\|\frac{1}{2}(f+g)\|_{\Phi} \geq 1-\sigma$. This means that $\delta_{L^{\Phi}}((1-\sigma)^2)=0$ for any $\sigma \in (0,1)$. Since $\delta_{L^{\Phi}}$ is a continuous function in the interval $[0,2)$, this yields $\delta_{L^{\Phi}}(1)=0$. This finishes the proof of the implication (i) $\Longrightarrow$ (iii) and of our theorem.

To give the examples mentioned in Abstract, two lemmas are needed.

1.4. Lemma. Let Φ be an Orlicz function with the right-hand derivative φ on $R_+=[0,\infty)$ and assume that there are constants a and δ in $(0,1)$ such that

(1.6) $\Phi(\frac{1}{2}(u+au)) \leq \frac{1-\delta}{2}\{\Phi(u)+\Phi(au)\}$

for all $u \in \mathbf{R}$. Then $\varphi(au) \leq (1+\delta)^{-1}\varphi(u)$ for all $u \geq 0$.

Proof. The proof is analogous to that given in [1], but we shall give it for the sake of completeness. Assume that there exist a, δ in $(0,1)$ such that (1.6) holds. Let $\varepsilon > 0$ be such that $a=1/(1+\varepsilon)$. We obtain

$$\Phi(\tfrac{1}{2}(u+u/(1+\varepsilon))) \leq \tfrac{1}{2}(1-\delta)\{\Phi(u)+\Phi(u/(1+\varepsilon))\}$$

for all $u \in \mathbf{R}$. Putting $u=(1+\varepsilon)v$, we get

$$\Phi(\frac{(1+\varepsilon)v+v}{2}) \leq \frac{1}{2}(1-\delta)\{\Phi((1+\varepsilon)v)+\Phi(v)\},$$

i.e.

(1.7) $2\Phi(u+\frac{\varepsilon}{2}u) \leq (1-\delta)\{\Phi((1+\varepsilon)u)+\Phi(u)\}$

for all $u,v \in \mathbf{R}$. We have also

(1.8) $\frac{\varepsilon u}{2}\varphi(u) \leq \Phi(u+\frac{\varepsilon}{2}u)-\Phi(u) \leq \Phi(u+\varepsilon u)-\Phi(u+\frac{\varepsilon}{2}u) \leq \frac{\varepsilon u}{2}\varphi(u+\varepsilon u)$

for all $u \geq 0$. Hence, we obtain

(1.9) $$\frac{\frac{\varepsilon u}{2}\varphi(u+\varepsilon u))}{\frac{\varepsilon u}{2}\varphi(u)} \geq \frac{\Phi(u+\varepsilon u)-\Phi(u+\varepsilon u/2)}{\Phi(u+\varepsilon u/2)-\Phi(u)}.$$

Combining (1.7) and (1.9), we get

$$\frac{\varphi((1+\varepsilon(u)}{\varphi(u)} \geq \frac{\Phi(u+\frac{\varepsilon}{2}u)-\Phi(u)+\delta\{\Phi(u+\frac{\varepsilon}{2}u)+\Phi(u)}{\Phi(u+\frac{\varepsilon}{2}u)-\Phi(u)} \geq 1+\delta$$

for any $u>0$. Putting $u=v/(1+\varepsilon)=av$, we obtain

$$\varphi(av) \leq (1+\delta)^{-1}\varphi(v)$$

for all $v>0$. This is the desired result.

1.5. Lemma. Let Φ be an Orlicz function with right-hand derivative φ on $\mathbf{R}_+$ and let the inequality $\varphi(bu) \leq k\varphi(u)$ be satisfied for some b, k in (0,1) and for all $u \geq 0$. If $a \in (0,1)$ is such that $1+a=2b$, then $p(\Phi) > a$.

Proof. We have

$$\frac{\Phi(u)+\Phi(au)}{2\Phi(\frac{1}{2}(u+au)} = \frac{2\Phi(\frac{1}{2}(u+au))+\int_{\frac{u+au}{2}}^{u}\varphi(t)dt-\int_{au}^{\frac{u+au}{2}}\varphi(t)dt}{2\Phi(\frac{1}{2}(u+au))} =$$

$$=1+\frac{\int_{au}^{\frac{u+au}{2}}[\varphi(\xi+\frac{1-a}{2}u)-\varphi(\xi)]d\xi}{2\Phi(\frac{1}{2}(u+au))}.$$

We have $\xi \leq \frac{1+a}{2}u$, i.e. $\frac{1-a}{1+a}\xi \leq \frac{1-a}{2}u$. Thus $\xi+\frac{1-a}{2}u \geq \xi+\frac{1-a}{1+a}\xi = \frac{2}{1+a}\xi = b^{-1}\xi$. Hence

$$\frac{\Phi(u)+\Phi(au)}{2\Phi(\frac{1}{2}(u+au))} \geq 1+\frac{(k^{-1}-1)\int_{au}^{\frac{u+au}{2}}\varphi(t)dt}{2\Phi(\frac{1}{2}(u+au))} =$$

$$= 1+\frac{(k^{-1}-1)\{\Phi(\frac{1}{2}(u+au))-\Phi(au)\}}{2\Phi(\frac{1}{2}(u+au))} =$$

$$= 1+(k^{-1}-1)\{\frac{1}{2}-\frac{\Phi(au)}{2\Phi(\frac{1}{2}(u+au))}\} \geq 1+\frac{1}{2}(k^{-1}-1)(1-\frac{2a}{1+a}) = \eta > 1.$$

To obtain the last inequality, the estimation

$$\Phi(\frac{1+a}{2}u) = \Phi(\frac{1+a}{2a}au) \geq \frac{1+a}{2a}\Phi(au)$$

for all $u \in \mathbf{R}$ is applied. The proof is finished.

Note. From Lemmas 1.4 and 1.5 the result of V. Akimovič [1] concerning the characterization of uniform convexity of Orlicz functions in the terms of their right-hand derivatives follows.

1.6. Example. There is an Orlicz function Φ such that $\delta_{L^{\Phi}}(1) > 0$ and $\delta_{L^{\Phi^*}}(1) > 0$, and the spaces $L^{\Phi}(\mu)$ and $L^{\Phi^*}(\mu)$ are not strictly convex.

To see this, define

$$\varphi(t)=\begin{cases}(\frac{4}{3})^{n-1} & \text{for } t\in[(\frac{4}{3})^{n-1},(\frac{4}{3})^{n}),\ n=1,2,\dots\\ (\frac{4}{3})^{-n} & \text{for } t\in[(\frac{4}{3})^{-n},(\frac{4}{3})^{-n+1}),\ n=1,2,\dots\\ 0 & \text{for } t=0.\end{cases}$$

We have $\varphi(\frac{4}{3}u)=\frac{4}{3}\varphi(u)$, i.e. $\varphi(\frac{3}{4}u)=\frac{3}{4}\varphi(u)$ for all $u\geq 0$. By simple calculations, we obtain for $s\geq 0$:

$$\varphi^*(s)=\sup\{t\geq 0:\varphi(t)\leq s\}=\begin{cases}(\frac{4}{3})^{n} & \text{for } s\in[(\frac{4}{3})^{n-1},(\frac{4}{3})^{n}),\ n=1,2,\dots\\ (\frac{4}{3})^{-n+1} & \text{for } s\in[(\frac{4}{3})^{-n},(\frac{4}{3})^{-n+1}),\ n=1,2,\dots\\ 0 & \text{for } t=0.\end{cases}$$

We have also $\varphi^*(\frac{4}{3}u)=\frac{4}{3}\varphi^*(u)$, i.e. $\varphi^*(\frac{3}{4}u)=\frac{3}{4}\varphi^*(u)$ for all $u\geq 0$. Define $\Phi(u)=\int_0^{|u|}\varphi(t)dt$ for all $u\in\mathbf{R}$. Then $\Phi^*(v)=\int_0^{|v|}\varphi^*(s)ds$ for all $v\in\mathbf{R}$. Both functions Φ and Φ^* satisfy condition Δ_2 for all $u\in\mathbf{R}$. We shall prove it only for Φ, because the proof for Φ^* is the same. We have

$$\frac{u}{2}\varphi(\frac{u}{2})\leq\Phi(u)\leq u\varphi(u) \text{ and } \varphi(2u)\leq\varphi((\frac{4}{3})^3u)=(\frac{4}{3})^3\varphi(u)$$

for all $u\geq 0$. Hence,

$$\Phi(2u)\leq 2u\varphi(2u)\leq 2u(\frac{4}{3})^3\varphi(u)\leq 2u(\frac{4}{3})^6\varphi(\frac{u}{2})\leq 4(\frac{4}{3})^6\Phi(u),$$

i.e. Φ satisfies condition Δ_2. The assumptions of the last lemma are satisfied with $b=k=\frac{3}{4}$. By this lemma it follows also that $p(\Phi)\geq a$, where $a=2b-1=\frac{3}{2}-1=\frac{1}{2}$. In the same way we obtain $p(\Phi^*)\geq\frac{1}{2}$. By Theorem 1.3 it follows that $\delta_{L^\Phi}(1)>0$ and $\delta_{L^{\Phi^*}}(1)>0$. Since the functions Φ and Φ^* are not strictly convex, the spaces $L^\Phi(\mu)$ and $L^{\Phi^*}(\mu)$ are not strictly convex.

1.7. Example. There is an Orlicz function Φ such that both spaces $L^\Phi(\mu)$ and $L^{\Phi^*}(\mu)$ are uniformly non-square and $\delta_{L^\Phi}(1)=\delta_{L^{\Phi^*}}(1)=0$.

To see this, define $\Phi(u)=\int_0^{|u|}\varphi(t)dt$, where

$$\varphi(t)=\begin{cases}4^{n-1} & \text{for } t\in[4^{n-1},4^{n}),\ n=1,2,\dots\\ 4^{-n} & \text{for } t\in[4^{-n},4^{-n+1}),\ n=1,2,\dots\\ 0 & \text{for } t=0.\end{cases}$$

We have

$$\varphi^*(s)=\begin{cases} 4^n & \text{for } s\in[4^{n-1},4^n),\ n=1,2,\dots \\ 4^{-n+1} & \text{for } s\in[4^{-n},4^{-n+1}),\ n=1,2,\dots \\ 0 & \text{for } s=0. \end{cases}$$

We have also $\varphi(4t)=4\varphi(t)$, i.e. $\varphi(\frac{t}{4})=\frac{1}{4}\varphi(t)$ and $\varphi^*(4s)=4\varphi^*(s)$, i.e. $\varphi^*(\frac{s}{4})=\frac{1}{4}\varphi^*(s)$ for all $s,t\geq 0$. Moreover, $\Phi^*(v)=\int_0^{|v|}\varphi^*(s)ds$ for all $v\in\mathbf{R}$. Therefore, both functions Φ and Φ^* satisfy condition Δ_2 (see the discussion concerning Example 1.6). Hence the conditions $p(\Phi)>0$ and $p(\Phi^*)>0$ follow. We shall prove for example that $p(\Phi)>0$. By Lemma 1.1, there exists $\xi=1+\alpha$ with $\alpha\in(0,1)$ such that $\Phi(\frac{\xi}{2}u)\leq\frac{\xi^{-1}}{2}\Phi(u)$ for all $u\in\mathbf{R}$. Hence it follows that

$$\Phi\left(\frac{1+\alpha}{2}u\right)\leq\frac{\xi^{-1}}{2}\{\Phi(u)+\Phi(\alpha u)\}$$

for all $u\in\mathbf{R}$. This means that $p(\Phi)\geq\alpha>0$. In virtue of Theorem 1.2 in [6], we conclude that both spaces $L^{\Phi}(\mu)$ and $L^{\Phi^*}(\mu)$ are uniformly non-square. We have $\varphi(bu)\geq\varphi(u)$ and $\varphi^*(bu)\geq\varphi^*(u)$ for any $b>\frac{1}{4}$ and some $u>0$. Applying Lemma 1.4, we get $p(\Phi)\leq\frac{1}{4}$ and $p(\Phi^*)\leq\frac{1}{4}$. In virtue of Theorem 1.3 it means that $\delta_{L^{\Phi}}(1)=\delta_{L^{\Phi^*}}(1)=0$.

1.8. Remark. Also in $\mathbf{R}^2$ there exist norms $\|\ \|$ and $|||\ |||$ such that $\|\ \|$ is not strictly convex and $\delta_{\|\ \|}(1)>0$, $|||\ |||$ is uniformly non-square, but $\delta_{|||\ |||}(1)=0$.

It suffices to put

$$\|x\|=\max(|x_1|+(\sqrt{2}-1)|x_2|,(\sqrt{2}-1)|x_1|+|x_2|),$$
$$|||x|||=\max(|x_1|,\ \tfrac{1}{3}|x_1|+\tfrac{2}{3}|x_2|).$$

Indeed, the unit sphere corresponding to the norm $\|\ \|$ is a sum of eight intervals of the same length smaller than 1. Hence it follows that $\delta_{\|\ \|}(1)>0$. It is evident that $\|\ \|$ is not strictly convex.

The unit sphere corresponding to the norm $|||\ |||$ has six extremal points: $(1,1)$, $(0,\frac{3}{2})$, $(-1,1)$, $(0,-\frac{3}{2})$, $(-1,-1)$, $(1,-1)$. So, the number of extremal points is equal to $2n+2$, where $n=2$ is the dimension of $\mathbf{R}^2$. Applying Theorem 1.5 from [5], we conclude that $|||\ |||$ is uniformly non-square. Taking $x=(1,1)$, $y=(1,-1)$, we have $|||x-y|||=\frac{4}{3}>1$ and $|||\frac{x+y}{2}|||=1$. This yields $\delta_{|||\ |||}(1)=0$.

We shall investigate below the local uniform non-$l_n^{(1)}$ property for

Orlicz-Bochner spaces. Let $(X, \|\ \|)$ be a Banach space and let $F_\mu(T,X)$ denote the space of all (equivalence classes) of strongly Σ-measurable functions from T into X. For a given Orlicz function Φ the convex modular $I_\Phi : F_\mu(T,X) \to [0,\infty]$ is defined by $I_\Phi(f)= \int_T \Phi(\|f(t)\|)d\mu$. The Orlicz-Bochner space is the set $L^\Phi(\mu,X)$ of all functions from $F_\mu(T,X)$ for which $I_\Phi(\lambda f)<\infty$ for some $\lambda > 0$. This space will be considered with Luxemburg norm $\|\ \|_\Phi$ defined by $\|f\|_\Phi = \inf\{\varepsilon > 0: I_\Phi(f/\varepsilon)\le 1\}$.

To characterize locally uniformly non-$l_n^{(1)}$ Orlicz-Bochner spaces, the following characterization of the local uniform non-$l_n^{(1)}$ property for normed linear spaces will be useful.

1.9. Lemma. A normed linear space $(X, \|\ \|)$ is locally uniformly non-$l_n^{(1)}$ if and only if for any $x_1 \in X$ there exists $\varepsilon(x_1)$ in $(0,1)$ such that for all $x_2,\dots,x_n$ in X, we have

$$\left\|\frac{1}{n}(x_1 \pm \dots \pm x_n)\right\| \le \frac{1}{n}(\|x_1\| + \dots + \|x_n\|) - \varepsilon(x_1)\min\{\|x_i\| : 1 \le i \le n\}$$

for some choice of signs.

Proof. Suppose X is locally uniformly non-$l_n^{(1)}$ and $x_1 \ne 0$ is in X. Then there is $\delta\left(\frac{x_1}{\|x_1\|}\right)$ in $(0,1)$ such that for all $x_2,\dots,x_n \in X\setminus\{0\}$, we have

$$(1.10)\qquad \left\|\frac{1}{n}\left(\frac{x_1}{\|x_1\|} \pm \dots \pm \frac{x_n}{\|x_n\|}\right)\right\| \le 1-\delta\left(\frac{x_1}{\|x_1\|}\right)$$

for some choice of signs. Denote $\delta\left(\frac{x_1}{\|x_1\|}\right)= \varepsilon(x_1)$ and assume that $\|x_k\| = \min\{\|x_i\| : i=1,\dots,n\}$, $1\le k \le n$. Without loss of generality we may assume that all signs in the left-side of (1.10) are equal 1. Thus, we have

$$1-\varepsilon(x_1) \ge \left\|\frac{1}{n}\left(\frac{x_1}{\|x_1\|} + \dots + \frac{x_1}{\|x_n\|}\right)\right\| = \left\|\frac{1}{\|x_k\|}\frac{x_1+\dots+x_n}{n} - \sum_{i\ne k}\left(\frac{1}{\|x_k\|} - \frac{1}{\|x_i\|}\right)\frac{x_i}{n}\right\| \ge$$

$$\ge \frac{1}{\|x_k\|}\left\|\frac{x_1+\dots+x_n}{n}\right\| - \frac{1}{n}\sum_{i\ne k}\left(\frac{1}{\|x_k\|} - \frac{1}{\|x_i\|}\right)\|x_i\| .$$

Therefore

$$\frac{1}{\|x_k\|}\left\|\frac{x_1+\dots+x_n}{n}\right\| \le 1-\varepsilon(x_1) - \frac{n-1}{n} + \frac{1}{n\|x_k\|}\sum_{i\ne k}\|x_i\| =$$

$$= -\varepsilon(x_1) + \frac{1}{n} + \frac{1}{n\|x_k\|}\sum_{i\ne k}\|x_i\| = \frac{1}{n}\sum_{i=1}^{n}\frac{\|x_i\|}{\|x_k\|} - \varepsilon(x_1),$$

i.e.

$$\left\|\frac{1}{n}(x_1+\ldots+x_n)\right\| \le \frac{1}{n}\sum_{i=1}^{n}\|x_i\| - \varepsilon(x_1)\|x_k\| = \frac{1}{n}\sum_{i=1}^{n}\|x_i\| - \varepsilon(x_1)\min\|x_i\| .$$

Conversely, let the inequality from the lemma hold for some choice of signs and let $x_1,\ldots,x_n \in B_X(1)$. If $\min\|x_i\| \le \frac{1}{2}$, then $\left\|\frac{1}{n}(x_1+\ldots+x_n)\right\| \le 1-\frac{1}{2n}$. If $\min\|x_i\| > \frac{1}{2}$, then by the assumed inequality, we have

$$\left\|\frac{1}{n}(x_1 \pm \ldots \pm x_n)\right\| \le \frac{1}{n}(\|x_1\| +\ldots+\|x_n\|) - \frac{1}{2}\varepsilon(x_1) \le 1-\frac{1}{2}\varepsilon(x_1)$$

for some choice of signs. So, in general

$$\left\|\frac{1}{n}(x_1 \pm \ldots \pm x_n)\right\| \le 1-\frac{1}{2}\min\left(\varepsilon(x_1),\frac{1}{n}\right)$$

for some choice of signs, which means that X is locally uniformly non-$l^{(1)}_n$.

Note. The proof of this lemma is analogous to that of the lemma concerning a characterization of uniform non-$l^{(1)}_n$ property given in [13].

1.10. Theorem. The following assertions are equivalent:

(a) An Orlicz-Bochner space $L^{\Phi}(\mu,X)$ is locally uniformly non-$l^{(1)}_n$.

(b) Both spaces $L^{\Phi}(\mu,\mathbf{R})$ and $(X, \|\ \|)$ are locally uniformly non-$l^{(1)}_n$.

(c) Φ is linear in no neighbourhood of zero, Φ satisfies condition Δ_2 (for all $u \in \mathbf{R}$), and $(X, \|\ \|)$ is locally uniformly non-$l^{(1)}_n$.

Proof. The implication (a) $\Longrightarrow$ (b) follows by the fact that both spaces X and $L^{\Phi}(\mu,\mathbf{R})$ can be isometrically embedded into $L^{\Phi}(\mu,X)$. The implication (b) $\Longrightarrow$ (c) is proved in [8]. Now, we shall prove the implication (c) $\Longrightarrow$ (a). Assume that (c) holds and $\|f_i\|_{\Phi}=1$ for $i=1,2,\ldots,n$ (we may restrict ourselves only to elements f_i from the unit sphere of $L^{\Phi}(\mu,X)$). Then $I_{\Phi}(f_i)=1$ for $i=1,2,\ldots,n$. Let $c>0$ be such that assuming

$$A_1=\{t\in T: c^{-1} \le \|f_1(t)\| \le c\},$$

we have $I_{\Phi}(f_1\chi_{A_1}) \ge \frac{7}{8}$. Let $d>0$ be such that $\Phi(c)/\Phi(d) < \frac{1}{8(n-1)}$, and let

$$A_i=\{t\in T: \|f_i(t)\| \le d\},\ i=2,\ldots,n.$$

We have $\Phi(d)\mu(T\setminus A_i) < I_{\Phi}(f_i\chi_{T\setminus A_i}) \le 1$, whence we get $\mu(T\setminus A_i) \le \frac{1}{\Phi(d)}$ for $i=1,2,\ldots,n$. So,

$$I_{\Phi}(f_1\chi_{A_1\setminus A_i}) \le \Phi(c)\mu(A_1\setminus A_i) \le \Phi(c)/\Phi(d) \le \frac{1}{8(n-1)} .$$

Defining $D=\bigcap_{i=1}^{n} A_i$, we have

$$\frac{7}{8} \le I_{\Phi}(f_1\chi_{A_1}) \le I_{\Phi}(f_1\chi_{\bigcup_{i=2}^{n}(A_1\setminus A_i)}) + I_{\Phi}(f_1\chi_D) \le$$

$$\leq \sum_{i=2}^{m} I_\Phi(f_1\chi_{A_1\setminus A_i})+I_\Phi(f_1\chi_D)\leq \frac{1}{8}+I_\Phi(d_1\chi_D).$$

Hence,

(1.11) $\quad I_\Phi(f_1\chi_D)\geq \frac{3}{4}.$

By assumption (c), we have $\Phi(\frac{u}{n})<\frac{\Phi(u)}{n}$ for all $u>0$. Hence it follows that $\Phi(\frac{n-1}{n}u)<\frac{n-1}{n}\Phi(u)$ for all $u>0$. Since Φ is a continuous function on the interval $[c^{-1},nd]$, there exists $\eta\in(0,1)$ such that

(1.12) $\quad \Phi(\frac{n-1}{n}u)\leq \eta\,\frac{n-1}{n}\Phi(u)$

for all $u\in[c^{-1},nd]$. In virtue of condition Δ_2, there is $a\in(0,1)$ such that $\Phi((1+a)u)\leq(\sqrt{\eta})^{-1}\Phi(u)$ for all $u\in R$ (see the proof of Lemma 1.1). Combining this with (1.12), we obtain

(1.13) $\quad \Phi((1+a)\frac{n-1}{n}u)\leq(\sqrt{\eta})^{-1}\Phi(\frac{n-1}{n}u)\leq\sqrt{\eta}\,\frac{n-1}{n}\Phi(u)$

for all $u\in[c^{-1},nd]$. Denote $\ell=\sqrt{\eta}$. Note that a and ℓ depend only on f_1 and Φ . Define for $x\in B_X(1)$:

$$\delta(x)=\inf\{1-\min_{\pm 1}\|\frac{1}{n}(x\pm x_2\pm\ldots\pm x_n)\|\},$$

where the infimum is taken over all $x_2,\ldots,x_n$ in $B_X(1)$. The function

$\delta(f_1(t)/\|f_1(t)\|)=\alpha(t)$ is Σ-measurable. By Lemma 1.9 we have

(1.14) $\quad \min_{\pm 1}\|\frac{1}{n}(f_1(t)\pm\ldots\pm f_n(t))\|\leq\frac{1}{n}(\|f_1(t)\|+\ldots+\|f_n(t)\|)-$

$$-\alpha(t)\min_i\|f_i(t)\|=\frac{1}{n}(\|f_1(t)\|+\ldots+\|f_n(t)\|)\times$$

$$\times(1-\alpha(t)\,n\,\min\|f_i(t)\|/(\|f_1(t)\|+\ldots+\|f_n(t)\|)).$$

We shall consider two cases.

1^0. $\min\|f_i(t)\|>a(\|f_1(t)\|+\ldots+\|f_n(t)\|-\min\|f_i(t)\|)$. Then $\min\|f_i(t)\|\geq(a/(a+1))(\|f_1(t)\|+\ldots+\|f_n(t)\|)$. Thus, combining this with (1.14), we get

$$\Phi(\min_{\pm 1}\|\frac{1}{n}(f_1(t)\pm\ldots\pm f_n(t))\|)\leq\Phi((1-\frac{na}{1+a}\alpha(t))\frac{\|f_1(t)\|+\ldots+\|f_n(t)\|}{n})\leq$$

$$\leq(1-\frac{na}{1+a}\alpha(t))\frac{1}{n}\sum_{i=1}^{m}\Phi(\|f_i(t)\|),$$

because $\frac{na}{1+a}\alpha(t)\leq 1$.

2^0. opposite to 1^0. Assume for example (but without loss of generality) that $\min \|f_i(t)\| = \|f_1(t)\|$. We have for μ - a.a. $t\in T$ and any choice of signs,

$$\Phi(\|\frac{1}{n}(f_1(t)\pm\ldots\pm f_n(t))\|)\le\Phi(\frac{1}{n}(\|f_1(t)\|+\ldots+\|f_n(t)\|)) =$$

$$=\Phi(((1+\frac{\|f_1(t)\|}{\|f_2(t)\|+\ldots+\|f_n(t)\|})/\frac{n}{n-1})\cdot\frac{\|f_2(t)\|+\ldots+\|f_n(t)\|}{n-1})\le$$

$$\le\Phi(\frac{n-1}{n}(1+a)\frac{\|f_2(t)\|+\ldots+\|f_n(t)\|}{n-1})\le\frac{n-1}{n}\ell\Phi(\frac{\|f_2(t)\|+\ldots+\|f_n(t)\|}{n-1})\le$$

$$\le\frac{\ell}{n}\sum_{i=1}^{n}\Phi(\|f_i(t)\|).$$

Denoting $\beta(t)=1-\frac{na}{1+a}\alpha(t)$, and $\gamma(t)=\max(1-\frac{1-\beta(t)}{2^{n-1}},\ell)$, we get by the above two cases:

$$\sum_{\pm 1}\Phi(\|\frac{f_1(t)\pm\ldots\pm f_n(t)}{n}\|)\le\frac{2^{n-1}\gamma(t)}{n}\sum_{i=1}^{n}\Phi(\|f_i(t)\|),$$

where "$\sum_{\pm 1}$" stands here and in the following for the summation over all choices of signs. Define

$$B_k=\{t\in D\colon \gamma(t)\le 1-\frac{1}{k}\}.$$

By Σ-measurability of γ, $B_k\in\Sigma$ for $k=1,2,\ldots$. There is $k\in\mathbf{N}$ such that $I_\Phi(f_1\chi_{B_k})\ge\frac{1}{2}$. In the sequel we shall write B in place of B_k. Denote $\sigma=1-1/k$, where k is the number defining B. We have for $t\in B$:

$$\sum_{\pm 1}\Phi(\|\frac{1}{n}(f_1(t)\pm\ldots\pm f_n(t)\|)\le\frac{2^{n-1}\sigma}{n}\sum_{i=1}^{n}\Phi(\|f_i(t)\|).$$

Integrating this inequality both-sides over B, we get

$$\sum_{\pm 1}I_\Phi(\frac{1}{n}(f_1\pm\ldots\pm f_n)\chi_B)\le\frac{2^{n-1}\sigma}{n}\sum_{i=1}^{n}I_\Phi(f_i\chi_B).$$

Hence, we obtain

$$2^{n-1}-\sum_{\pm 1}I_\Phi(\frac{1}{n}(f_1\pm\ldots\pm f_n))\ge\frac{2^{n-1}}{n}\sum_{i=1}^{n}I_\Phi(f_i\chi_B)-\sum_{\pm 1}I_\Phi(\frac{1}{n}(f_1\pm\ldots\pm f_n)\chi_B)\ge$$

$$\ge\frac{2^{n-1}-2^{n-1}\sigma}{n}\sum_{i=1}^{n}I_\Phi(f_i\chi_B)\ge\frac{2^{n-1}(1-\sigma)}{n}I_\Phi(f_1\chi_B)\ge\frac{2^{n-1}(1-\sigma)}{2n}.$$

Hence

$$\sum_{\pm 1}I_\Phi(\frac{1}{n}(f_1\pm\ldots\pm f_n))\le 2^{n-1}-\frac{2^{n-1}(1-\sigma)}{2n}=2^{n-1}(1-\frac{1-\sigma}{2n})=2^{n-1}q,$$

where $q=1-\frac{1-\sigma}{2n}$ belongs to $(0,1)$. It must be $I_\Phi(\frac{1}{n}(f_1\pm\ldots\pm f_n))\le q$ for some choice of signs. Denoting by p the function from $(0,1)$ into $(0,1)$ such that

$\|f\|_\Phi \le 1-p(\varepsilon)$ whenever $I_\Phi(f) \le 1-\varepsilon$ (such function p exists by condition Δ_2, see the proof of Theorem 1.3), we get

$$\left\|\frac{1}{n}(f_1 \pm \ldots \pm f_n)\right\|_\Phi \le 1-p(1-q)$$

for some choice of signs, where p(1-q) depends only on f_1. The proof is finished.

The girth of S_X (= the unit sphere of X) is defined as double infimum of lengths of curves on S_X with antipodal endpoints (see [19]) and is denoted by $\mathrm{Girth}(S_X)$.

1.11. Theorem. Every Orlicz space $L^\Phi(\mu)$ of real-valued functions is either uniformly non-square or is nearly flat, i.e. $\mathrm{Girth}(S_{L^\Phi})=4$.

Proof. If $\mathrm{Girth}(S_{L^\Phi}) > 4$ then $L^\Phi(\mu)$ is reflexive (see [19]). But it is known that for any non-atomic infinite measure μ, reflexivity of $L^\Phi(\mu)$ coincides with uniform non-squareness (see [6]).

We say that a Banach space $(X, \|\ \|)$ is non-$l_n^{(1)}$ ($n \ge 2$ and integer) if for any $x_1,..,x_n$ in B_X, we have $\|x_1 \pm x_2 \pm \ldots \pm x_n\| < n$ for some choice of signs.

We say that an Orlicz function Φ satisfies condition Δ_2 at ∞ if there exist positive constants K, a such that $\Phi(2u) \le K\Phi(u)$ for all u satisfying $\Phi(u) \ge a$.

1.12. Theorem. Any Orlicz space $L^\Phi(\mu)$ of real-valued functions over a non-atomic finite measure μ is either non-$l_n^{(1)}$ for some integer $n \ge 2$ or is flat.

Proof. If $L^\Phi(\mu)$ is non-$l_n^{(1)}$ for no integer $n \ge 2$, then either Φ does not satisfy condition Δ_2 at ∞ or Φ is linear on the whole $\mathbf{R}_+$ (see [8]). Then $L^\Phi(\mu)$ is flat (see [9]).

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