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EVERYWHERE REGULARITY THEOREMS FOR MAPPINGS WHICH MINIMIZE p-ENERGY

Martin FUCHS

Abstract: We consider functions $u:D^n \to \overline{M}$ defined on some n-dimensional region taking values in the closure of a smooth domain M located in Euclidean space or a Riemannian manifold which locally minimize a degenerate functional of the form $\int_D |Du|^p$ ($p \geq 2$) under this nonlinear side condition. While the partial regularity theory was developed in [F1,2], we study here geometric conditions on M which exclude singular points.

Key words: p-harmonic problems for vector functions, degenerate functionals, regularity of minimizers, removable singularities, blow-up technique, obstacle problems.

Classification: Primary 49
Secondary 35D10

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0. Introduction and results. In this section we fix our assumptions and state the main results: Let D denote a bounded open subset of an n-dimensional manifold X, $n \geq 2$; moreover we are given an N-dimensional Riemannian manifold Y embedded in a Euclidean space R^L. Suppose further that M is a domain with smooth boundary and compact closure in Y. For a real number $p \geq 2$ and functions u in the Sobolev space $H^{1,p}(D,R^L)$ we introduce the p-energy

$$E_p(u,D) := \int_D |Du|^p$$

and look at local minimizers of this functional under the nonlinear side condition $Im(u) \subset \overline{M}$ a.e., the set Y-M playing the role of an obstacle for the admissible comparison functions. To be precise, we define the restricted Sobolev space

$$H^{1,p}(D,\overline{M}) := \{u \in H^{1,p}(D,R^L) : u(x) \in \overline{M} \text{ a.e.}\}$$

and the class

$$K := \{u \in H^{1,p}(D,\overline{M}) : E_p(u,D) \leq E_p(v,D) \text{ for all } v \in H^{1,p}(D,\overline{M}),\ spt(u-v) \subset\subset D\}$$

of local minimizers.

In [F1,2] we showed that $u \in \mathbf{K}$ is of class C^1 up to a closed ("singular") set $S \subset D$ with

$$\begin{cases} \text{H-dim}(S) \leq n-[p]-1 & \text{for } n>p+1, \\ S \text{ is discrete} & \text{for } n-1 \leq p < n, \\ S=\emptyset & \text{for } p \geq n. \end{cases}$$

Obviously one is interested in geometric configurations which are general enough to study contact problems and on the other hand being suitably restricted to exclude singular points. For constrained minimizers of the Dirichlet´s integral (i.e. $p=2$) it turns out that the star-shapedness of the domain M is a necessary and sufficient condition for proving everywhere regularity. This result was obtained in [F3,4] where we also constructed a set M being in a certain sense a limit of star-shaped domains and for which singular minima occur.

The purpose of this note is to extend the regularity result to the p-case: the methods developed in [F,3,4] do not apply since we made use of the Green´s function for the linear Euler operator occurring on the left-hand-side of the system satisfied by a local minimizer, and for $p>2$ this Euler operator is highly nonlinear in Du. To overcome this difficulty we proceed in the following way: fixing a point $x_0 \in D$ and blowing up the minimizer $u \in \mathbf{K}$ at x_0 we get a radially independent limit u_0 satisfying a certain differential inequality which implies that u_0 has to be constant. From this it is easy to deduce that u is regular in a neighborhood of x_0.

In order to formulate our theorem we assume that $\mathbf{B}:=\mathbf{B}_R(P)$ is a regular ball in the target manifold Y (see [H] for a definition) containing M. We say that M is <u>geodesically star-shaped</u> with respect to the center P of $\mathbf{B}$ if for any point Q in the boundary of M the unique geodesic $w: [0,1] \to \mathbf{B}$ joining P and Q stays in the interior of M for all $t<1$.

Theorem: Let M be geodesically star-shaped with respect to the center P of the regular ball $\mathbf{B}$. Then any local minimizer $u \in \mathbf{K}$ is of class C^1 on the whole domain D.

Corollary (<u>unconstrained Riemannian case</u>): Assume that $u \in H^{1,p}(D,Y) := \{w \in H^{1,p}(D,\mathbb{R}^L) : w(x) \in Y \text{ a.e.}\}$ has the property $\mathrm{Im}(u) \subset \mathbf{B}$ for a regular ball $\mathbf{B}$ in Y and $E_p(u,D) \leq E_p(v,D)$ for all $v \in H^{1,p}(D,Y)$ such that $\mathrm{spt}(u-v) \subset\subset D$. Then u is of class $C^1(D)$.

Remarks: 1) If the sectional curvature of Y is ≤ 0, then each ball is regular (see [H]). 2) In the flat case $Y=\mathbb{R}^N\subset\mathbb{R}^L$ the condition of the theorem reduces to the fact that M is a bounded star-shaped subset of $\mathbb{R}^N$ with smooth boundary.

1. Proof of the theorem. We may assume that D is the unit ball $B=B_1^n(0)$ in $\mathbb{R}^n$ equipped with the flat metric, the general case requires some minor modifications which we indicate at the end of this section. We show that $u\in K$ is regular in a neighborhood of $0\in B$. According to the basic regularity theorem 3.1 in [F2] this follows from

$$(1)\qquad \liminf_{r\to 0}\ r^{p-n}\int_{B_r(0)}|Du|^p\,dx=0.$$

In order to prove (1) we fix a sequence $r_i\longrightarrow 0$ of positive numbers r_i and consider the local minimizers $u_i(z):=u(r_iz)$, $z\in B$. Quoting [F2], Lemma 4.3, we see (after passing to a subsequence) $u_i\longrightarrow u_0$ in $H^{1,p}_{loc}(B,\mathbb{R}^L)$ for some function u_0 which is radially independent. Clearly (1) follows if we can show that Du_0 vanishes.

To this purpose we fix one of the blown up functions $v:=u_i$ and introduce normal coordinates on $\mathbf{B}$ with center P. The ball $\mathbf{B}$ is mapped on the Euclidean ball $B_R^N(0)$ and M is transformed in a smooth open subset K of $B_R^N(0)$. With respect to these coordinates the representative of a minimizer $\in K$ is a local minimizer of the functional

$$\int_B (g_{ik}(w)D_\alpha w^i\,D_\alpha w^k)^{p/2}dx$$

(Greek (Latin) indices repeated twice are summed from 1 to (N)) in the class $H^{1,p}(B,\overline{K})$, (g_{ik}) denoting the metric tensor of Y. Keeping the symbol v for the coordinate representative of this function we get on account of [F2], Theorem 2.1, for all test-vectors $\Phi\in \overset{\circ}{H}{}^{1,p}\cap L^\infty(B,\mathbb{R}^N)$:

$$\int_B p\ g(v,Dv)g_{ij}(v)D_\alpha v^iD_\alpha\,\Phi^j\,dx+\int_B \tfrac{p}{2}g(v,Dv)D_jg_{il}(v)D_\alpha v^iD_\alpha v^l\,\Phi^jdx=$$

$$=\int_{B\cap[v\in\partial K]}\nu_K(v)\cdot\Phi f\,dx,$$

where $[v\in\partial K]:=\{x\in B:v(x)\in\partial K\}$, $g(v,Dv):=(g_{ij}(v)D_\alpha v^iD_\alpha v^j)^{p/2-1}$; f denotes a non-negative function growing of order $|Du|^p$ and ν_K is the interior normal vector field to the boundary of K.

Let (g^{ij}) denote the inverse of (g_{ij}) and replace Φ by the test-vector with components $g^{lj}(v)\,\Phi^j$. Following the lines of [H], proof of Theorem 1.4, we obtain:

(2) $\int_B pg(v,Dv)\,[Dv\cdot D\phi - \gamma^i_{lk}(v)D_\alpha v^l D_\alpha v^k \phi^i]\,dx = \int_B g^{kj}(v)\,\phi^j(v)\,\chi_{[v\in\partial K]}\,dx,$

(γ^i_{lk}) being the Christoffel symbols on Y. Since M is geodesically star-shaped with respect to P it is easy to show (using [H], Lemma 5.2) that

$g^{kj}(y)y^j\nu^k_K(y)=y^j\nu^j_K(y)\le 0$

holds for all points y in the boundary of K (compare [F4], Section 3). In a final step we use the test-vector $\phi=\varphi v$ with $\varphi\in C^1_0(B)$, $\varphi\ge 0$. In this case the left-hand-side of (2) is non-positive so that (2) turns into a differential inequality which is also valid for the limit function u_0:

(3) $\int_B pg(u_0,Du_0)\,[Du_0\cdot D(\varphi u_0)-\gamma^i_{lk}(u_0)D_\alpha u^l_0 D_\alpha u^k_0\varphi u^i_0]\,dx\le 0.$

We choose $\varphi(x):=\varphi(|x|)$. Since u_0 is radially independent, the inequality (3) reduces to

(4) $\int_B pg(u_0,Du_0)\,\varphi\,[|Du_0|^2-\gamma^i_{lk}(u_0)D_\alpha u^l_0 D_\alpha u^k_0 u^i_0]\,dx\le 0.$

According to [H], the inequality (6.11), the quantity [...] occurring in (4) is bounded below by a constant times $|Du_0|^2$ (recall that u_0 takes its values in the regular ball B) so that (4) immediately implies $Du_0=0$. □

If D is a domain in some n-dimensional Riemannian manifold X we introduce local coordinates $B^n_1(0)\to U(x_0)$ on a suitable neighborhood $U(x_0)\subset D$ of a point $x_0\in D$. Then a coordinate representative of $w\in K$ belongs to the class $H^{1,p}(B,\overline{K})$ and locally minimizes the functional

$$\int_B (g_{lk}(w)a^{\alpha\beta}D_\alpha w^l D_\beta w^k)^{p/2}\sqrt{a}\,dx.$$

$(a_{\alpha\beta})$ being the metric on X, $(a^{\alpha\beta}):=(a_{\alpha\beta})^{-1}$, $a:=\det(a_{\alpha\beta})$. After a change of coordinates we can arrange $a_{\alpha\beta}(0)=\delta_{\alpha\beta}$ and a slight extension of the blow-up lemma 4.3 in [F2] gives $Du_0=0$ along the same lines as before. Alternatively we can quote [F1], chapter 3, or the paper [F,F]: since $\overline{K}$ is diffeomorphic to the closed ball $\overline{B^N_1(0)}$ we proceed as in [F,F], proof of Theorem 1.3, to see that $u_0\in H^{1,p}(B,\overline{K})$ is actually a local minimizer of

$$\int_B (g_{lk}(w)D_\alpha w^l D_\alpha w^k)^{p/2}dx$$

in the restricted Sobolev space so that the Euler system (2) is valid for the limit function u_0. □

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