Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 3, 581--592

Persistent URL: http://dml.cz/dmlcz/106570

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28,3 (1987)

CLASSES OF GRAPHS DEFINABLE BY GRAPH ALGEBRA IDENTITIES OR QUASI-IDENTITIES

Reinhard PÖSCHEL and Walter WESSEL

Abstract Graph algebras establish a useful connection between graphs and universal algebras. A graph theoretic characterization of graph quasi-varieties and graph varieties, resp., i.e. classes of graphs definable by graph algebra quasi-identities and identities, resp., is given. The results are structure theorems of "Birkhoff type": A class of finite undirected graphs is a graph quasi-variety (graph variety, resp.) iff it is closed w.r.t. isomorphisms, induced subgraphs, finite disjoint unions and homogeneous subproducts (direct products, resp.). Some examples and applications are also considered.

Key words Graph algebra, graph variety, graph quasi-variety, term, identity, quasi-identity, homogeneous subproduct.

AMS Subject Classification 05C99, 08B99, 08A05, 05C75.

INTRODUCTION

There are many fruitful algebraic concepts in graph theory which use mostly linear algebra or group theory (automorphism groups). Less attention has been paid to connections between graph theory and universal algebra. The introduction of graph algebras [17], [7](Shallon algebras [10]) establishes one possible connection between graphs and universal algebras. This approach has been extensively used for the benefit of universal algebra (see e.g. [7], [1], where many algebras with nonfinitely based identities have been found among graph algebras; subvarieties of varieties generated by graph algebras are characterized in [5]).
In the present paper the opposite point of view is considered: We are interested in which graph theoretic results (structure theorems) can be obtained from universal algebra via graph algebras. In particular, we ask for a graph theoretic characterization of classes of (finite, undirected) graphs which can be defined by quasi-identities or identities in their corresponding graph algebras. As an answer to this question we obtain theorems which use

only graph theoretic closure operations: A class of finite undirected graphs is a graph quasi-variety (graph variety, resp.) iff it is closed w.r.t. isomorphisms, induced subgraphs, disjoint unions and homogeneous subproducts (direct products, resp.).

The present paper is a revised version of the manuscript [13]; however, after the manuscript had been finished we learned that the characterization of graph varieties had been found independently by E.W. Kiss [4](with a different proof). Therefore we shall deal with this result only briefly (in §3). Moreover, in the meanwhile this result has been extended also to directed graphs [11]. Thus, in the following we are interested mainly in the characterization of quasi-varieties (§2).

We want to call attention to the interplay between universal algebra and graph theory. Therefore, at a first step, we shall restrict ourselves to the case of finite undirected graphs, which is easier to handle. There is no doubt that all results can be generalized to directed graphs (in case of varieties see [11], for varieties of arbitrary relational systems see [12]).

In the last section of the paper we give several examples and show how the structure theorem (for graph varieties) could be applied, e.g., every finite undirected graph without loops is an induced subgraph of a suitable power of the graph G_o with two adjacent vertices and one loop.

Throughout the paper by a graph we mean a directed graph without multiple edges (i.e. a binary relation on the set of vertices).

ACKNOWLEDGEMENTS. Thanks are due to I.G. Rosenberg and A. Pultr for critical comments and many useful hints.

1. PRELIMINARIES

We recall the following terminology and notations.

1.1 Let $G=(V(G),E(G))$ be a directed graph ($V(G)$ - set of vertices, $E(G)\subseteq V(G)\times V(G)$ - set of edges). G is called undirected, if $E(G)$ is a symmetric relation. A graph G' is called an induced subgraph of G if $V(G')\subseteq V(G)$ and $E(G')=E(G)\cap V(G')\times V(G')$. Given graphs G_i ($i\in I$), the direct product $G=\prod_{i\in I} G_i$ is the graph G with $V(G)=\prod_{i\in I} V(G_i)$ (cartesian product) and $E(G)=\{(a,b)\in V(G)\times V(G) \mid \forall i\in I: (a(i),b(i))\in E(G_i)\}$; here, for $a\in V(G)$, let $a(i)$ denote the

i-th component: $a=(a(i))_{i\in I}$. Assume the sets $V(G_i)(i\in I)$ to be pairwise disjoint (otherwise make the sets $V(G_i)$ disjoint, e.g. use $V(G_i)\times\{i\}$), then the disjoint union $G=\dot{\bigcup}_{i\in I} G_i$ is simply the union of the graphs G_i, i.e., $V(G)=\bigcup_{i\in I} V(G_i)$, $E(G)=\bigcup_{i\in I} E(G_i)$. Products or unions are called finite, if I is finite. A mapping $h: V(G) \longrightarrow V(G')$ is a homomorphism from a graph G into a graph G' if, for all $a,b\in V(G)$, $(a,b)\in E(G)$ implies $(h(a),h(b))\in E(G')$. The homomorphism h is called strong if also $(a,b)\notin E(G)$ implies $(h(a),h(b))\notin E(G')$. An isomorphism (graph isomorphism) is a bijective strong homomorphism. Hom(G,G') denotes the set of all homomorphisms from G into G'.
For a class $\mathcal{K}$ of graphs, let $I\mathcal{K}$, $S\mathcal{K}$, $P\mathcal{K}$, $P_f\mathcal{K}$, $U\mathcal{K}$, $U_f\mathcal{K}$ denote the class of all isomorphic copies, induced subgraphs, direct products, finite direct products, disjoint unions, finite disjoint unions of members of $\mathcal{K}$, respectively. Let $\mathcal{G}_{df}$ and $\mathcal{G}_{uf}$ be the class of finite directed graphs and undirected graphs, resp., without multiple edges. We omit the index f (=finite), if also infinite graphs are to be considered.

1.2 For all graphs G under consideration, let ∞ be a fixed element such that $\infty\notin V(G)$. Given a graph G, we define a binary operation $\cdot$(expressed by juxtaposition) on $V(G)\cup\{\infty\}$ by setting $ab=a$ if $(a,b)\in E(G)$ and $ab=\infty$ otherwise (in particular $a\infty=\infty a=\infty$ for $a\in V(G)$). In general, this operation is neither commutative nor associative. The algebra

$$G^{\#}=\langle V(G)\cup\{\infty\};\cdot,\infty\rangle$$

is called the graph algebra of G ([7], Shallon algebra [10]); here ∞ denotes a nullary operation. Remark: $G^{\#}$ can be considered as the one-point completion of the partial first projection of E(G) together with the constant operation ∞. For graphs G_1 and G_2, $h:G_1\longrightarrow G_2$ is a strong homomorphism iff $h:G_1^{\#}\longrightarrow G_2^{\#}$ (extended by $h(\infty)=\infty$) is a homomorphism of the corresponding graph algebras.

1.3 Let T(X) be the set of all terms over the the alphabet $X=\{x_0,x_1,x_2,\dots\}$ using juxtaposition and the symbol ∞. T(X) is defined inductively as follows:
(i) every x_i $(i=0,1,2,\dots)$(also called variable) and ∞ is a term;

(ii) if t and t' are terms, then (tt') is a term;

(iii) T(X) is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

The leftmost variable of a term t is denoted by Left(t). A term in which the symbol ∞ occurs is called trivial. Let T'(X) be the set of all non-trivial terms. To every non-trivial term t we assign a directed graph G(t)=(V(t),R(t)) where V(t) is the set of all variables in t and R(t) is defined inductively by $R(t)=\emptyset$ if $t\in X$ and $R((tt'))=R(t)\cup R(t')\cup\{(Left(t),Left(t'))\}$. Note that G(t) always is a connected graph.

Example: For $t=((x_0x_1)(x_2x_3))$ we have $Left(t)=x_0$, $V(t)=\{x_0,x_1,x_2,x_3\}$ and $R(t)=\{(x_0,x_1), (x_2,x_3), (x_0,x_2)\}$.

1.4 Given $t\in T(X)$, $G\in \mathcal{G}_d$ and a mapping (assignment) $h:V(t)\longrightarrow V(G)\cup\{\infty\}$, let h(t) denote the value of t in $G^{\#}$ if every variable $x\in V(t)$ is substituted by h(x). Let $t,t'\in T(X)$. A graph G satisfies the identity t=t' (or the identity holds in G), notation $G\models t=t'$, if h(t)=h(t') holds in $G^{\#}$ (notation $G\models h(t)=h(t')$) for every assignment $h:V(t)\longrightarrow V(G)\cup\{\infty\}$. For a set $\sum$ of identities and an assignment $h:V(\sum)\longrightarrow V(G)\cup\{\infty\}$ ($V(\sum)$ denotes the set of variables occuring in $\sum$) we write $G\models h(\sum)$ if, for all $(t=t')\in\sum$, $G\models h(t)=h(t')$; and we write $G\models\sum$ if $G\models t=t'$ for all $(t=t')\in\sum$.

1.5 Proposition (cf. also [7],[5],[11]). *Let G be a graph.*

(1) *For $t\in T(X)$ and an assignment $h:V(t)\longrightarrow V(G)$ the following are equivalent:*

(a) $h(t)\neq\infty$,

(b) $h(t)=h(Left(t))$,

(c) *h is a homomorphism from G(t) into G.*

Note in particular, if the image of h is not connected, then h is not a homomorphism and we have $h(t)=\infty$.

(2) *For $t,t'\in T'(X)$ we have $G\models t=t'$ iff $Hom(G(t),G)=Hom(G(t'),G)=:H$ and $h(Left(t))=h(Left(t'))$ for all $h\in H$.*

(note that $V(t)\neq V(t')$ implies $H=\emptyset$, i.e. $G\models t=\infty$ & $G\models t'=\infty$).

(3) *We have $G\models t=t'$ $(t,t'\in T'(X))$ for every graph G iff $G(t)=G(t')$ and $Left(t)=Left(t')$.*

(4) An undirected graph G can be represented as $G=G(t)$ for some term $t\in T'(X)$ iff G is finite and connected.

Proof. (1) and (4) follow from the definitions, (2) follows from (1), and (3) from (2).■

Remark: For every trivial term $t\in T(X)$ and any graph G we have $G\models t=\infty$. Thus one can use $T'(X)\cup\{\infty\}$ instead of $T(X)$ in all further considerations. Moreover, for $h:V(t)\longrightarrow V(G)\cup\{\infty\}$ with $t\in T'(X)$ and $h(x)=\infty$ for some $x\in V(t)$, we have $G\models h(t)=\infty$.

1.6 A quasi-identity q is a finite set $\Sigma=\{(t_1=t_1'),\dots,(t_n=t_n')\}$ of identities together with an identity $t=t'$; we use the notation $\Sigma\longrightarrow t=t'$ or $t_1=t_1'\wedge\dots\wedge t_n=t_n'\longrightarrow t=t'$. A graph G satisfies the quasi-identiy q, notation $G\models q$, if for every assignment $h: V(q)\longrightarrow V(G)\cup\{\infty\}$ ($V(q)$ denotes the set of variables occuring in q) the following implication holds:

$$h(t_1)=h(t_1')\wedge\dots\wedge h(t_n)=h(t_n') \Rightarrow h(t)=h(t').$$

This will be denoted by $G\models h(\Sigma)\longrightarrow h(t)=h(t')$ (or $G\models h(q)$). Note that every identity $t=t'$ can be considered as a quasi-identity because $G\models t=t' \iff G\models \infty=\infty\longrightarrow t=t'$. For $\mathcal{K}\subseteq\mathcal{G}_d$ and a set Φ of quasi-identities (or identities) we write $\mathcal{K}\models\Phi$ if $G\models q$ for all $q\in\Phi$ and $G\in\mathcal{K}$.

2. CHARACTERIZATION OF GRAPH QUASI-VARIETIES

2.1 Definitions. For a set Φ of quasi-identities and a class of graphs let $\Phi^*=\{G\in\mathcal{G}_d \mid G\models\Phi\}$ and $Qid(\mathcal{K})=\{q\mid q$ is a quasi-identity and $\mathcal{K}\models q\}$. We set $Qvar(\mathcal{K})=(Qid(\mathcal{K}))^*$ and, for a given class $\mathcal{G}$ of graphs, $Qvar_{\mathcal{G}}(\mathcal{K})=\mathcal{G}\cap Qvar(\mathcal{K})$. A class of this form is called quasi-equational or a graph quasi-variety in $\mathcal{G}$, in particular, $Qvar_{\mathcal{G}}(\mathcal{K})$ is the graph quasi-variety generated by $\mathcal{K}$ in $\mathcal{G}$.

2.2 Remark. In general, graph quasi-varieties are not quasi-varieties in the usual universal-algebraic sense; e.g. the direct product of graph algebras is not a graph algebra again. Clearly, $Qvar_{\mathcal{G}}(\mathcal{K})$ consists of graphs from $\mathcal{G}$ whose graph algebras belong to the quasi-variety $ISPP_u\mathcal{K}^\#$ (cf.[2; p. 219, Thm. 2.25], $\mathcal{K}^\#=\{G^\#\mid G\in\mathcal{K}\}$). However, the most algebras in $ISPP_u\mathcal{K}^\#$ are not graph algebras.

Therefore it is reasonable to ask for an *internal* characterization of $\mathrm{Qvar}_{\mathcal{G}}(\mathcal{K})$ using operations on binary relations (=graphs) only (which avoid ultraproducts and infinite graphs).

2.3 Definition. A graph G is called a *homogeneous subproduct* of a family $(G_i)_{i\in I}$ of graphs, if
(1) G is an induced subgraph of $\prod_{i\in I} G_i$,
(2) For all $a,b\in V(G)$ holds either $\forall i\in I: (a(i),b(i))\in E(G_i)$ or $\forall i\in I: (a(i),b(i))\notin E(G_i)$.

For a set $\mathcal{K}$ of graphs, let $P_h\mathcal{K}$ ($P_{hf}\mathcal{K}$, resp.) denote the set of all homogeneous subproducts of families (finite families, resp.) of members of $\mathcal{K}$.

2.4 Proposition. *Let $\mathcal{K}$ be a class of graphs and G be a graph with $|V(G)|\geq 2$. Then $G\in IP_h\mathcal{K}$ iff for all $a,b\in V(G)$ with $a\neq b$ there exists a $K\in\mathcal{K}$ and a strong homomorphism $\varphi:G\longrightarrow K$ such that $\varphi(a)\neq\varphi(b)$. Moreover, if G is finite, then $G\in IP_h\mathcal{K}$ implies $G\in IP_{hf}\mathcal{K}$. In case $|V(G)|=1$, $G\in IP_h\mathcal{K}$ iff there is a strong homomorphism $\varphi:G\longrightarrow K$.*

Proof. "$\Longrightarrow$": If $G\leq\prod_{i\in I} G_i$ is a homogeneous subproduct, then every projection $p_i:G\longrightarrow G_i : a\longmapsto a(i)$ is a homomorphism which is strong by 2.3(2). Distinct $a,b\in V(G)$ must differ in at least one component $i\in I$, i.e. $p_i(a)\neq p_i(b)$.
"$\Longleftarrow$": Let $I=\{(a,b)\mid a\neq b,\ a,b\in V(G)\}$ and assume that there is a strong homomorphism $\varphi_{(a,b)}:G\longrightarrow K_{(a,b)}\in\mathcal{K}$ with $\varphi_{(a,b)}(a)\neq\varphi_{(a,b)}(b)$ for all $(a,b)\in I$. Let B be the induced subgraph of $\prod_{i\in I}K_i$ with $V(B)=\{(\varphi_i(c))_{i\in I}\mid c\in V(G)\}$. Then B is a homogeneous subproduct since $\exists i\in I: (\varphi_i(c),\varphi_i(c'))\in E(K_i) \Rightarrow (c,c')\in E(G) \Rightarrow \forall i\in I: (\varphi_i(c),\varphi_i(c'))\in E(K_i)$. Moreover, G is isomorphic to B because $c\longmapsto(\varphi_i(c))_{i\in I}$ is an isomorphism by construction, i.e. $G\in IP_h\mathcal{K}$. If G is finite, this construction gives also a finite set I. ∎

Remark. Sometimes it is useful to know special operations which can be represented as homogeneous subproducts. E.g. every induced subgraph of G belongs to $P_h\{G\}$. Further, let O_n be an induced subgraph of $G\in\mathcal{G}_d$ which is an *anticlique* (i.e. $E(O_n)=\emptyset$) such that all vertices in $V(O_n)$ have the same neighbours in G (i.e. O_n is an *autonomous* subgraph of G in the sense of [8]). Substitute O_n

in G by another anticlique O_m (preserving the same neighbourhood for all vertices) and let $G[O_n/O_m]$ be the resulting graph. Then we have $n \geq 2 \Rightarrow G[O_n/O_m] \in IP_h\{G\}$, $(n,m \in \{1,2,3,\dots\})$. Analogously one can use autonomous cliques K_n (with loops) instead of anticliques (then, for $n \geq 2$, $G[K_n/K_m] \in IP_h\{G\}$). Moreover, every homogeneous subproduct can be obtained from one of its factors by using the just described operations (without restriction to n). In fact, if $G \in IP_h\mathcal{K}$, by 2.4 there exists at least one strong homomorphism $\varphi: G \longrightarrow K$ $(K \in \mathcal{K})$; but the image of φ is a graph which arises from G by identifying vertices with equal neighbourhoods.

2.5 Lemma. Let $t,t' \in T'(X)$ and let $B \leq \prod_{i\in I} G_i$ be a homogeneous subproduct of $(G_i)_{i\in I}$. Then for an assignment $h: V(t) \cup V(t') \longrightarrow V(B) \cup \{\infty\}$ we have $B \models h(t)=h(t') \Leftrightarrow \forall i \in I: G_i \models h_i(t)=h_i(t')$, where h_i is the composition of h and the i-th projection p_i $(h_i(x)=p_i(h(x))=(h(x))(i)$, and $h_i(x)=\infty$ if $h(x)=\infty)$.

Proof. Since the p_i are strong homomorphisms, h_i is a homomorphism iff h is a homomorphism. We are done by 1.5(1). ∎

2.6 Lemma. Let $s,s' \in T'(X)$, $G \models G_1 \dot\cup G_2$, $h: V(s) \cup V(s') \longrightarrow V(G) \cup \{\infty\}$ be an assignment and let $h_i: V(s) \cup V(s') \longrightarrow V(G_i) \cup \{\infty\}$ be defined by $h_i(x)=h(x)$ if $h(x) \in V(G_i)$ and $h_i(x)=\infty$ otherwise $(i \in \{1,2\})$. Then $(G \models h(s)=h(s')) \Leftrightarrow (G_1 \models h_1(s)=h_1(s')$ and $G_2 \models h_2(s)=h_2(s'))$.

Proof. Obviously $h(s)=\infty$ if $h(V(s)) \nsubseteq V(G_1)$ and $h(V(s)) \nsubseteq V(G_2)$ (by 1.5(1)). Moreover, $h_i(s)=h(s)$ if $h(V(s)) \subseteq V(G_i) \cup \{\infty\}$ and $h_i(s)=\infty$ if $h(V(s)) \nsubseteq V(G_i)$. Using these properties it is easy to show $h(s)=h(s') \Leftrightarrow (h_1(s)=h_1(s') \,\&\, h_2(s)=h_2(s'))$. ∎

2.7 Lemma. For a class $\mathcal{K}$ of graphs we have $IUP_h\mathcal{K} \subseteq Qvar\,\mathcal{K}$.

Proof. 1) Let $B \leq \prod_{i\in I} G_i$ be a homogeneous subproduct, $G_i \in \mathcal{K}$ $(i \in I)$, and let $q=(\sum \longrightarrow t=t') \in Qid\,\mathcal{K}$. We are going to show $B \models q$. Let $h: V(q) \longrightarrow V(B) \cup \{\infty\}$ be some assignment such that $B \models h(\sum)$. We have to show $B \models h(t)=h(t')$. By 2.5, $\forall i \in I: G_i \models h_i(\sum)$ (h_i defined as in 2.5). Thus $\forall i \in I: G_i \models h_i(t)=h_i(t')$ because $G_i \models q$ (note $G_i \in \mathcal{K}$). Again by 2.5, $B \models h(t)=h(t')$. Consequently $B \models q$.
2) Let $G = G_1 \dot\cup G_2$ be the disjoint union of $G_1, G_2 \in \mathcal{K}$ and let

$q=(\sum \longrightarrow t=t')\in Qid(\mathcal{K})$. We are going to show $G\models q$. Let $h:V(q)\longrightarrow V(G)\cup\{\infty\}$ be an assignment and assume $G\models h(\sum)$. We have to show $G\models h(t)=h(t')$. We define assignments $h_1:V(q)\longrightarrow V(G_1)\cup\{\infty\}$, $h_2: V(q)\longrightarrow V(G_2)\cup\{\infty\}$ setting $h_i(x)=h(x)$ if $h(x)\in V(G_i)$ and $h_i(x)=\infty$ otherwise ($i\in\{1,2\}$). By 2.6, $G\models h(\sum)$ implies $G_i\models h_i(\sum)$, consequently $G_i\models h_i(t)=h_i(t')$ (since $G_i\models q$), and, again by 2.6, $G\models h(t)=h(t')$.

3) By 2) we have $U_f\mathcal{K}\subseteq Qvar\,\mathcal{K}$. We get (cf. 3.3): $G\in U\mathcal{K}\Rightarrow S_f\{G\}\subseteq UP_h\mathcal{K}\Rightarrow S_f\{G\}\subseteq U_fP_h\mathcal{K}\Rightarrow S_f\{G\}\subseteq Qvar\,\mathcal{K}\Rightarrow G\in Qvar\,\mathcal{K}$. ∎

Now we are ready to formulate the main theorem for quasi-varieties in $\mathcal{G}_{uf}$.

<u>2.8</u> Characterization Theorem. <u>Let</u> $\mathcal{K}$ <u>be a non-empty subclass of</u> $\mathcal{G}_{uf}$. <u>Then</u> $Qvar_{\mathcal{G}_{uf}}(\mathcal{K}) = IU_fP_{hf}\mathcal{K}$. (For infinite graphs see 3.3.)

Proof. Because of 2.7 and 2.4 it suffices to show $Qvar_{\mathcal{G}_{uf}}(\mathcal{K})\subseteq IU_fP_h\mathcal{K}$. Since every graph is the disjoint union of its connected components it remains to show $G\in IP_h\mathcal{K}$ for every given connected undirected graph $G\in Qvar_{\mathcal{G}_{uf}}(\mathcal{K})$. Let $V(G)=\{a_0,\dots,a_n\}$. Consider the following set $\sum$ of identities

$$\sum=\{x_ix_j=x_i \mid (a_i,a_j)\in E(G)\}\cup\{x_ix_j=\infty \mid (a_i,a_j)\notin E(G)\}.$$

$\sum$ is finite since G is finite. Obviously, under the canonical assignment $\iota:x_i\longmapsto a_i$ ($i=0,\dots,n$) we have $G\models\iota(\sum)$. Thus for given $a_i,a_j\in V(G)$, $a_i\neq a_j$, the quasi-identity $\sum\longrightarrow x_i=x_j$ does not hold in G. Since $G\in Qvar(\mathcal{K})$ there is some $K\in\mathcal{K}$ with $K\not\models\sum\longrightarrow x_i=x_j$, i.e. there must be some assignment $h:V(\sum)\longrightarrow V(K)\cup\{\infty\}$ such that $K\models h(\sum)$ but $K\not\models h(x_i)=h(x_j)$. By construction of $\sum$ and connectedness of G, for every two variables $x,y\in V(\sum)$ there is some sequence $z_1x=z_1,\dots,z_mz_{m-1}=z_m$, $yz_m=y$ of identities from $\sum$, ($z_1,\dots,z_m\in V(\sum)$). Consequently $h(x)=\infty$ would imply $h(y)=\infty$ (note $z\infty=\infty$). Since $h(x_i)\neq h(x_j)$ we have $h(x)\neq\infty$ for all $x\in V(\sum)$, i.e. h maps $V(\sum)$ into $V(K)$. By construction of $\sum$, $h'=\iota^{-1}h$: $a_i\longmapsto h(x_i)$ is a strong homomorphism from G into K; in fact, $(a_k,a_l)\in E(G)\Rightarrow x_kx_l=x_k\in\sum\Rightarrow K\models h(x_k)h(x_l)=h(x_k)$, i.e. $(h'(a_k),h'(a_l))\in E(K)$; analogously one shows $(a_k,a_l)\notin E(G)\Rightarrow(h'(a_k),h'(a_l))\notin E(K)$; $k,l\in\{0,\dots,n\}$.
By 2.4 we can conclude $G\in IP_h\mathcal{K}$. ∎

3. CHARACTERIZATION OF GRAPH VARIETIES

In this section we present a characterization theorem for classes of undirected graphs which can be defined by identities. Definitions and results can be developed analogously to the case of quasi-varieties (cf. introduction).

3.1 Definitions. For a set $\sum$ of identities and classes $\mathcal{K}$ and $\mathcal{G}$ of graphs let $\sum^* = \{G \in \mathcal{G}_d \mid G \models \sum\}$, $Id(\mathcal{K}) = \{t=t' \mid \mathcal{K} \models t=t',\ t,t' \in T'(X) \cup \{\infty\}\}$, $Var\,\mathcal{K} = (Id(\mathcal{K}))^*$ (the graph variety generated by $\mathcal{K}$), $Var_{\mathcal{G}}(\mathcal{K}) = \mathcal{G} \cap Var\,\mathcal{K}$ (the graph variety generated by $\mathcal{K}$ in $\mathcal{G}$).

3.2 Characterization Theorem. Let $\mathcal{K} \subseteq \mathcal{G}_{uf}$ be a non-empty class of finite undirected graphs. Then $Var_{\mathcal{G}_{uf}}\mathcal{K} = IU_fSP_f\mathcal{K}$.

We omit our original proof given in [13] and refer to [4] or [11] (in [11] the general case of directed graphs is treated and 3.2 is an easy consequence). Note that if $\mathcal{K}$ is finite then $Var\,\mathcal{K}$ is also generated by a single graph.

3.3 Let us consider what happens if we want to treat also infinite graphs. Since every finitely generated subalgebra of a graph algebra is finite, we have $Var\,\mathcal{K} = Var\,S_f\mathcal{K}$ and $Qvar\,\mathcal{K} = Qvar\,S_f\mathcal{K}$ for a given class $\mathcal{K}$ of graphs, where $S_f\mathcal{K}$ denotes the class of all finite induced subgraphs of members of $\mathcal{K}$. Thus, for arbitrary $\mathcal{K} \subseteq \mathcal{G}_u$, we have $Var_{\mathcal{G}_{uf}}\mathcal{K} = IU_fS_fP_f\mathcal{K}$ and $Qvar_{\mathcal{G}_{uf}}\mathcal{K} = IS_fU_fP_{hf}\mathcal{K} = IU_fP_{hf}S_f\mathcal{K}$ by 3.2 and 2.8. Moreover, we have $G \in Qvar\,\mathcal{K} \Leftrightarrow S_f\{G\} \subseteq Qvar_{\mathcal{G}_{df}}\mathcal{K}$ and $G \in Var\,\mathcal{K} \Leftrightarrow S_f\{G\} \subseteq Var_{\mathcal{G}_{df}}\mathcal{K}$. This characterizes general graph quasi-varieties and graph varieties; our restriction to finite graphs was not very essential.

3.4 Remark. By a result of H. Werner (cf.[1]) almost all varieties generated by graph algebras of undirected graphs are nonfinitely based. This situation changes if we consider graph varieties. Many classes defined by forbidden subgraphs can be characterized by finitely many identities. E.g., the class $\mathcal{K}_m = \{G \in \mathcal{G}_{uf} \mid G$ has no loops and $\omega(G) \leq m\}$ ($\omega(G)$ = clique number = cardinality of a maximum sized clique) is finitely based with respect to graph varieties (cf.[10]), e.g. $\mathcal{K}_2 = \{G \in \mathcal{G}_{uf} \mid G \models (x_0(x_1(x_2x_0))) = \infty\}$. However, for $m \geq 2$, $\mathcal{K}_m$ contains a chain with 4 vertices, the graph algebra of which generates a nonfinitely based variety (cf. [1]).

4. EXAMPLES AND APPLICATIONS

Many graph theoretic properties can be expressed as identities or quasi-identities. We mention here some examples which may be of graph theoretic interest, too.

4.1 Examples of graph varieties. Σ shall denote a set of identities such that Σ^* is the indicated class of graphs:

a) Graphs without loops: $\Sigma = \{x_0x_0 = \infty\}$;

b) Undirected graphs: $\Sigma = \{x_0(x_1x_0) = x_0x_1\}$;

c) Posets (reflexive, antisymmetric and transitive relations):
$\Sigma = \{x_0x_0 = x_0,\ x_0(x_1x_0) = x_1(x_0x_1),\ x_0(x_1x_2) = (x_0x_2)(x_1x_2)\}$;

d) Disjoint unions of complete graphs wtih loops:
$\Sigma = \{x_0x_0 = x_0,\ x_0(x_1x_0) = x_0x_1,\ (x_0x_1)x_2 = x_0(x_1x_2)\}$;

e) Undirected graphs with bounded clique number (cf. 3.4, there is one identity $t = \infty$ such that $\mathcal{K}_m = \{G \in \mathcal{G}_{uf} \mid G \models t = \infty\}$).

f) Undirected graphs with bounded chromatic number, in particular bipartite graphs (it is known that a Σ exists which consists of identities of the form $t = \infty$ only, but no Σ is explicitely known, except for bipartite graphs, cf. [11]).

Every graph variety is also a graph quasi-variety. The converse is not true, and we mention here some graph quasi-varieties (of undirected graphs without loops (4.1a,b) which are not graph varieties.

4.2 Example. Let $C \in \mathcal{G}_{uf}$ be a connected graph without loops. Up to isomorphism we can assume $V(C) \subseteq X = \{x_0, x_1, \ldots\}$ and $(x_0, x_1) \in E(C)$. Let $\mathcal{K}$ be the class of all undirected graphs without loops which contain no induced subgraph isomorphic to C or to a strong homomorphic image of C (if different vertices of C have different neighboorhoods then there are no strong homomorphic images of C except C itself). Then $\mathcal{K}$ is a graph quasi-variety characterized by $\mathcal{K} = \Phi^*$ with $\Phi = \{x_0x_0 = \infty,\ x_0(x_1x_0) = x_0x_1,\ \Sigma(C) \rightarrow x_0 = x_1\}$ where
$$\Sigma(C) = \{xy = x \mid (x,y) \in E(C)\} \cup \{xy = \infty \mid (x,y) \notin E(C),\ x,y \in V(C)\}.$$

4.3 Example. a) Perfect graphs (for definition see e.g. [3]) form a graph quasi-variety in $\mathcal{G}_{uf}$, since they are closed w.r.t. P_{hf} and U_f (this can be shown without difficulties, use e.g.

[3; p.53, 3.1] and the remark after 2.4). A concrete system of quasi-identities characterizing perfect graphs is not known.
b) If the Strong Perfect Graph Conjecture (cf. [3; p.71]) were true, then perfect graphs are those which contain no induced subgraph isomorphic to C_{2k+1} (odd cycles) or $\bar{C}_{2k+1}$ (the complementary graph) for $k \geq 2$. These graphs, however, can be characterized as a graph quasi-variety by the following set Φ of quasi-identities if we apply 4.2: $\Phi = \{x_0 x_0 = \infty,\ x_0(x_1 x_0) = x_0 x_1\} \cup \{\sum(C_{2k+1}) \longrightarrow x_0 = x_1 \mid k \geq 2\} \cup \{\sum(\bar{C}_{2k+1}) \longrightarrow x_0 = x_2 \mid k \geq 2\}$. Here we assume $V(C_{2k+1}) = V(\bar{C}_{2k+1}) = \{x_0, x_1, \dots, x_{2k}\}$, $E(C_{2k+1}) = \{(x_i, x_{i+1}) \mid i = 0, \dots, 2k\} \cup \{(x_{i+1}, x_i) \mid i = 0, \dots, 2k\}$ (indices take modulo 2k+1), $E(\bar{C}_{2k+1}) = \{(x,y) \in V(C_{2k+1})^2 \mid (x,y) \notin E(C_{2k+1})\}$.

4.4 Let $G_o \in \mathcal{G}_{uf}$ be the graph 

, i.e. $V(G_o) = \{0,1\}$, $E(G_o) = \{(0,1),(1,0),(1,1)\}$. From a result in [7; p.211] follows that $\mathrm{Var}_{\mathcal{G}_{uf}}\{G_o\}$ contains all undirected graphs without loops. Together with 3.2 this gives the following proposition:

4.5 Proposition. Every finite undirected graph without loops is isomorphic to an induced subgraph of a finite direct power of G_o. ∎

The graph G_o and the result 4.5 are considered more or less explicitely also in [13],[17],[9],[4],[14]. Moreover, the graph G_o appears in connection with investigations of subdirect irreducibles of "productive classes" of graphs (i.e. closed w.r.t. direct products) in papers of A. Pultr and J. Vinárek (e.g. [15],[16]).

4.6 Definition. Let $G \in \mathcal{G}_{uf}$ be without loops. The least number n such that G is isomorphic to an induced subgraph of G_o^n is called the G_o-dimension of G and it is denoted by $\dim_{G_o} G$.

Proposition 4.5 ensures that every $G \in \mathcal{G}_{uf}$ without loops has a finite G_o-dimension. In [6], lower and upper bounds for $\dim_{G_o} G$ are given, and the G_o-dimension is exactly determined for some classes of graphs. The G_o-dimension is a special case of the dimension proposed in [15; p.77], where the general problem to investigate the various kinds of dimension is posed. Let us note that 2.8 and 3.2 provide structure theorems for every concretely given graph quasi-variety or variety and give rise to numerical numbers characterizing the complexity of the structure, like e.g. the G_o-dimension. Here are many interesting problems of research.

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(Oblatum 30.6. 1986, revisum 21.4. 1987)