Luc Vrancken-Mawet

The 0-distributivity in the class of subalgebra lattices of Heyting algebras and closure algebras

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 2, 387--396

Persistent URL: http://dml.cz/dmlcz/106549

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

28,2(1987)

THE 0-DISTRIBUTIVITY IN THE CLASS OF SUBALGEBRA LATTICES OF HEYTING ALGEBRAS AND CLOSURE ALGEBRAS

L. VRANCKEN-MAWET

Abstract: Using Priestley duality, we characterize those Heyting and closure algebras whose subalgebra lattice is 0-distributive (i.e. satisfies $x \wedge y=0$ and $x \wedge z=0 \Longrightarrow x \wedge (y \vee z)=0$).

Key words: Heyting and closure algebras, subalgebra lattice, 0-distributivity, congruences on quasi-ordered topological spaces.

Classification: 06D05

Introduction. In [2],[3] and [5], we study the subalgebra lattice of Heyting algebras and closure algebras and characterize those Heyting algebras and closure algebras whose subalgebra lattice is distributive. Besides, our results characterize in the class **D** of distributive lattices those which are subalgebra lattices of Heyting algebras or closure algebras.

In this paper, we extend the class **D** to the wider class of 0-distributive (i.e. lattices which satisfy the following weakening of the distributivity law: $x \wedge y=0$ and $x \wedge z=0$ imply $x \wedge (y \vee z)=0$). To obtain these results we use a duality between closure algebras and closure spaces and the notion of congruence on quasi-ordered topological spaces. We recall these notions in the first paragraph.

§ 1 Recalls

1.1. Definitions. (a) A closure algebra $B=(B;\wedge,\vee,^{c},^{-},0,1)$ is a Boolean algebra $(B;\wedge,\vee,^{c},0,1)$ with a unary operator (closure operator) satisfying

(i) $0^{-}=0$;

(ii) $\forall\ x \in B, x \leq x^{-}=x^{-\,-}$;

(iii) $\forall x,\ y \in B, (x \vee y)^{-}=x^{-} \vee y^{-}$.

A closed element a of B is such that $a=a^{-}$. The set of all closed elements of B is a dual Heyting algebra under $x+y=(y-x)^-$. We denote it by $Cl(B)$.

(b) A <u>closure space</u> $X=(X,\tau,\leq)$ is a Boolean space (X,τ) with a quasi-order satisfying

(i) $\forall x \in X, (x]=\{y \in X \mid y \leq x\}$ (resp. $[x)=\{y \in X \mid x \leq y\}$) is closed and

(ii) for any clopen subset U of X, $(U]=\bigcup\{(x] \mid x \in U\}$ is clopen.

The set of all minimal (resp. maximal) elements of X is denoted by $MinX$ (resp. $MaxX$).

Let B be a closure algebra. The set $M(B)$ of all maximal ideals of B, endowed with the topology generated by the set $\{I \in M(B) \mid a \notin I\}$, $a \in B$, and quasi-ordered by the relation $\leq$ defined by $I \leq J \Longleftrightarrow I \cap Cl(B) \subseteq J \cap Cl(B)$, is a closure space, called dual space of B.

Conversely, if X is a closure space, then the Boolean algebra of all clopen subsets of X, denoted by $\mathcal{O}(X)$, becomes a closure algebra if one defines U^- by $(U]$.

The Stone duality extends to this more general situation as follows [3].

1.2. Proposition. There exists a dual equivalence between the category **CA** of closure algebras and the category **CS** of closure spaces whose morphisms are the continuous maps $f:X \longrightarrow X'$ such that $f([x))=[f(x))$, for all $x \in X$.

1.3. Definition. A <u>congruence</u> on the closure space $X=(X,\tau,\leq)$ is an equivalence such that

(i) if $(x,y) \notin \Theta$, then there exists a Θ-saturated (i.e. union of Θ-classes) clopen subset U of X with $x \in U$ and $y \in -U$;

(ii) if $x \,\Theta\, y \leq z$, then there exists $t \in X$ such that $x \leq t \,\Theta\, z$.

The set of all congruences of X, ordered by inclusion is a lattice denoted by $Con(X)$.

1.4. Examples. Let X be a closure space.

(a) The identity ω and the universal equivalence are congruences.

(b) The equivalence $\varsigma=\{\Theta(p,q) \mid p \leq q \leq p\}$ is a congruence.

(c) The dual atoms of Con(X) are equivalences $\Phi(U)$ with two classes U and -U where U is a clopen subset of X satisfying one of the following conditions:

(i) $(U \cap \mathrm{Max}X)^{\xi} = (-U \cap \mathrm{Max}X)^{\xi}$;

(ii) U and -U are both increasing and decreasing;

(iii) U is increasing and contains MaxX.

(d) Let E be a closed subset of X and let us denote by $\Theta(E)$ the equivalence generated by $E \times E$. If

(i) either E is such that $x \in [E) \Rightarrow y \leq x$, for all $y \in E$, or

(ii) E is increasing,

then $\Theta(E)$ is a congruence of X. In particular, $\Theta(\mathrm{Max}X) \in \mathrm{Con}(X)$. If $E=\{p,q\}$, we write $\Theta(p,q)$ instead of $\Theta(\{p,q\})$.

1.5. Propositions. (a) Let $X \in$ **CS** and $B \in$ **CA**, the dual closure algebra. Then the subalgebra lattice of B is dually isomorphic to Con(X). [3].

(b) Let $X \in$ **CS**, B its dual closure algebra and $\Theta \in \mathrm{Con}(X)$. Then $X/\Theta \in$ **CS**. In particular, X/ξ is a pospace (i.e. partially ordered topological space) whose Priestley dual ([2]) is Cl(B).

(c) Partially ordered closure spaces and dual Heyting spaces ([2]) coincide. In particular, if B is generated by Cl(B), the subalgebra lattice of B is isomorphic to that of Cl(B).

Consequently, our study of the congruence lattice of closure spaces leads to the corresponding properties for the subalgebra lattice of closure algebras and also of Heyting algebras.

1.6. Definitions. (a) A <u>clique</u> is a set Y with a quasi-order $\leq$ defined by $x,y \in Y \Rightarrow x \leq y$.

An n-<u>clique</u> is a clique of cardinal n and is denoted by $n\uparrow$.

(b) Let X, Y be quasi-ordered sets. Then X+Y (resp. $X \oplus Y$) denotes the cardinal (resp. ordinal) sum of X and Y.

(c) An order-connected component of a quasi-ordered space X (abbreviated o.c.c.) is a subset Y of X such that $(Y] = Y$ and $[Y) = Y$ and which is minimal for this property.

We now investigate the Heyting and closure algebras whose subalgebra lattice is 0-distributive, that is, satisfies the following property:

$$x \wedge y = 0 \text{ and } x \wedge z = 0 \text{ imply } x \wedge (y \vee z) = 0.$$

Clearly, this is equivalent to study the closure space whose congruence lattice is 1-distributive, i.e. such that

$$x \vee y=1 \text{ and } x \vee z=1 \text{ imply } x \vee (y \wedge z)=1.$$

We separate here the case when X is partially ordered from the case when X is not partially ordered.

In what follows, we denote by S(B) the subalgebra lattice of a Boolean algebra B. These lattices and their order duals have been characterized by Sachs in [4].

§ 2. Heyting algebras

2.1. Theorem. Let $X \in CS$ be partially ordered. Then the following assertions are equivalent.

(i) Con(X) is 1-distributive;

(ii) there exist bounded chains C and C' and a (possibly empty) antichain Y such that X is order-isomorphic either to $C \oplus (C'+Y)$ or to $C+1$;

(iii) there exist Boolean algebras B and B' such that B is complete and atomic and Con(X) is isomorphic either to $B \times (S(B')+1)$ or to B.

Proof. (i) $\Longrightarrow$ (ii). Let $X \in$ **CS** be such that X is partially ordered and Con(X) is 1-distributive. The ordered type of X is deduced from the following observations.

α) Necessarily, $X-(\mathrm{Min}X \cup \mathrm{Max}X)$ is a chain and $|\mathrm{Min}X-\mathrm{Max}X| \leq 2$. If not, let $x,y \in X-(\mathrm{Min}X \cup \mathrm{Max}X)$ (resp. $x,y \in \mathrm{Min}X-\mathrm{Max}X$) and $t \in \mathrm{Min}X-\mathrm{Max}X-\{x,y\}$. Denote by V and U increasing clopen subsets containing MaxX such that $y \in V$, $x \in U$, $\{x,t\} \cap V=\emptyset$, $\{y,t\} \cap U=\emptyset$. We have $\Phi(V) \vee \Theta(V \cup U)=1$, $\Phi(U) \vee \Theta(V \cup U)=1$ and $(\Phi(V) \wedge \Phi(U)) \vee \Phi(V \cup U) \neq 1$, which contradicts the 1-distributivity of Con(X).

In particular, this means that there exist at most two o.c.c. not reduced to a singleton and at most one o.c.c. which meets $X-(\mathrm{Min}X \cup \mathrm{Max}X)$. Precisely, X must satisfy the following condition.

β) There exists at most one o.c.c. which is not reduced to a singleton. Let C_1, C_2 be o.c.c. such that $|C_1| \geq 2$, $|C_2| \geq 2$ and x_i ($i=1,2$) the element of $\mathrm{Min}C_i-\mathrm{Max}C_i$. Let U (resp. V) be an increasing clopen set which is decreasing (resp. contains MaxX) and such that $C_1 \subseteq U$, $C_2 \cap U = \emptyset$ (resp. $x_1 \notin V$, $x_2 \notin V$). The congruences $\Phi(U)$, $\Phi(V)$, $\Theta(V \cup U)$ contradict as in α) the 1-distributivity of Con(X).

In fact, there exist at most two o.c.c. since the following condition γ) is necessary for Con(X) to be 1-distributive.

γ) Two elements of MaxX which are not in the same o.c.c. constitute MaxX. Let x, y be maximal elements of different o.c.c. We may suppose that the o.c.c. of y is reduced to $\{y\}$. If $z \in \text{MaxX} - \{x,y\}$, let U be a clopen subset of X which is increasing, decreasing and such that $\{x,z\} \subseteq U \subseteq -\{y\}$. We have $\Phi(U) \vee \Theta(x,y) = 1$, $\Phi(U) \vee \Theta(z,y) = 1$, $\Phi(U) \vee (\Theta(x,y) \wedge \Theta(z,y)) \neq 1$, a contradiction to the 1-distributivity of Con(X).

δ) There exists at most one minimal element which is not maximal. If not, let $x \neq y \in \text{MinX} - \text{MaxX}$. First, we have $[x) \cap (X - \text{MaxX}) - \{x\} = [y) \cap (X - \text{MaxX}) - \{y\}$. Indeed, let $z \in [y) \cap (X - \text{MaxX}) - ([x) \cap (X - \text{MaxX}))$ and U, V be increasing clopen sets such that $\text{MaxX} \cup \{x\} \subseteq V$, $\text{MaxX} \cup \{z\} \subseteq U$, $\{y,z\} \subseteq -V$ and $\{x,y\} \subseteq -U$. We have $\Phi(V) \vee \Theta(V \cup U) = 1$, $\Phi(U) \vee \Theta(V \cup U) = 1$ and $\Theta(U \cup V) \vee (\Phi(V) \wedge \Phi(U)) \neq 1$, which is impossible.

It follows from this that $\alpha = \Theta(x,y) \cup \Theta(\text{MaxX})$ is a congruence. If U' and V' are increasing clopen subsets containing MaxX and such that $x \in U - V$ and $y \in V - U$, the congruences $\Phi(U)$, $\Phi(V)$ and α induce a contradiction to the 1-distributivity of Con(X).

If X is not order-connected, then X is the cardinal sum of a chain and a singleton. We shall now investigate the case when X is order-connected.

If $X - \text{MaxX} \neq \emptyset$, $\cap \{[x) \mid x \in Y\} \neq \emptyset$, for each finite subset Y of X-MaxX. By a compactness argument, we deduce $\cap \{[x) \mid x \in X - \text{MaxX}\} \neq \emptyset$. Hence there exists $x_0 \in \text{MaxX}$ such that $(x_0] - \{x_0\} = X - \text{MaxX}$. The conclusion follows from the necessary condition ε).

ε) If $x_1, x_2 \in \text{MaxX} - \{x_0\}$, then $(x_1] \neq \{x_1\}$ and $(x_2] \neq \{x_2\}$ imply $(x_1] - \{x_1\} = (x_2] - \{x_2\}$. If not, suppose z maximal in $(X - \text{MaxX}) \cap (x_1] - (x_2]$ (if such z does not exist, we interchange x_1 and x_2). Let U be a clopen subset of X which contains x_0, x_2 and not x_1 and let V be a clopen subset of X containing x_1 and x_2 and disjoint from $U \cap [z)$. Consider the congruences $\alpha = \Theta([z))$, $\beta = \Theta(U \cap \text{MaxX}) \cup \Theta((z]) \cup \Theta(-U \cap \text{MaxX})$, $\gamma = \Theta(V \cap \text{MaxX}) \cup \Theta((z]) \cup$

$\cup\ \Theta(-V\cap \mathrm{Max}X)$. It is clear that $\alpha \vee \beta = \alpha \vee \gamma = 1$. Since $\beta \cap \gamma$ is not a congruence (for each $t \in (x_2] \cap (x_0]$, $z(\beta \cap \gamma)t \leq x_2$ would imply the existence of $u \in [z) \cap U \cap V$ such that $z \leq u(\beta \cap \gamma)x_2$), and that $\beta \wedge \gamma|_{\mathrm{Max}X} = \beta \cap \gamma|_{\mathrm{Max}X}$, we have necessarily $(z,t') \notin \beta \cap \gamma$ for some $t' \in (z]$. It is clear that $(z,t') \notin \alpha$ from what we deduce the contradiction $\alpha \vee (\beta \wedge \gamma) \neq 1$.

This completes the proof of (i) $\Longrightarrow$ (ii) (take $C' = (x_0] - (x_1]$ and $Y = \mathrm{Max}X - \{x_0\}$).

(ii) $\Longrightarrow$ (iii). If X is either a chain or the cardinal sum of a chain and a singleton, then Con(X) is a complete and atomic Boolean algebra ([2]).

Suppose that X is order-isomorphic to $C \oplus (C'+Y)$ where C and C' are bounded chains and Y is a non empty antichain.Let B (resp. B') be isomorphic to Con(C) (resp. Con(C')) ([2]). By an argument similar to that of Theorem 2.1 in [5], it is clear that Con(X) is isomorphic to $B \times B' \times \mathrm{Con}(1 \oplus (1+Y))$. It is also easy to check that $\mathrm{Con}(1 \oplus (1+Y))$ is isomorphic to $(\mathrm{Con}(1+Y)) \oplus 1$; now 1+Y is a Boolean space whose congruence lattice is of the form $S(B')$, whence the proof is complete.

The implication (iii) $\Longrightarrow$ (i) is clear.

Denote by $\mathcal{H}$ the class of all Heyting algebras which are Boolean products of chains, all 2-elements chains except perhaps one. From the duality and the proposition 1.5, we deduce the following corollary of Theorem 2.1.

2.2. Corollary. Let A be a Heyting algebra. Then the following assertions are equivalent.

(i) The subalgebra lattice Sub(A) of A is 0-distributive.

(ii) There exist $H \in \mathcal{H}$ and a chain C such that A is isomorphic either to $H \oplus C$ or to $C \times 2$ or to C.

(iii) There exist Boolean algebras B and B' such that B is complete and atomic and Sub(A) is isomorphic either to $B \times (0 \oplus S(B'))$ or to B.

2.3. Remark. From 1.5 it follows that the subalgebra lattice of a closure algebra generated by its closed elements is 0-distributive if and only if the order-dual of Cl(B) satisfies (ii) of 2.2.

2.4. Corollary. Let L be a 0-distributive lattice. Then the following assertions are equivalent.

(i) There exist a Heyting algebra A such that L is isomorphic to the subalgebra lattice of A.

(ii) There exists a closure algebra A generated by its closed elements such that L is isomorphic to the subalgebra lattice of A.

(iii) There exist Boolean algebras B and B' such that B is complete and atomic and L is isomorphic either to B or to $B \times \times (0 \oplus S(B'))$.

Proof. We have (i) $\Rightarrow$ (ii) by 1.5 and (i) $\Rightarrow$ (iii) by 2.2. Conversely, if B is a complete and atomic Boolean algebra, there exists a chain C which is a Heyting algebra such that $B \simeq \mathrm{Sub}(C)$. If B' is a Boolean algebra, we have $\mathrm{Sub}(B' \oplus C) \simeq B \times (0 \oplus S(B'))$. This completes the proof of (iii) $\Rightarrow$ (i).

§ 3. Closure algebras

3.1. Theorem. Let $X \in$ **CS**. Then Con(X) is 1-distributive if and only if X satisfies one of the following conditions.

(i) There exist an upper bounded chain C (possibly empty), a bounded chain C', a clique Y and an equivalence Y' (in other words, Y' is the cardinal sum of cliques) such that X is order-isomorphic to $Y \oplus C \oplus (C'+Y')$.

(ii) There exist an upper bounded chain C, a clique Y and an equivalence Y' such that X is order-isomorphic to $Y \oplus C \oplus Y'$ and $(V \cap \mathrm{Max}X)^{\xi} \neq (-V \cap \mathrm{Max}X)^{\xi}$, for all clopen subsets V of X.

(iii) There exist an upper bounded chain C and cliques Y and Y' such that X is order-isomorphic to $(Y \oplus C)+Y'$.

(iv) There exists a clique Y such that X is order-isomorphic to 1+Y.

(v) X is isomorphic to $2\!\uparrow$.

Proof. Let $X \in$ **CS** be such that Con(X) is 1-distributive. Since $\mathrm{Con}(X/\xi\ )$ is isomorphic to $\{\varphi \in \mathrm{Con}(X) \mid \xi \leq \varphi\}$ (by the third isomorphism theorem), it is also 1-distributive and by 2.1, there exist bounded chains C and C' and an antichain Y such that $X/\xi \simeq C \oplus (C'+Y)$ or $X/\xi \simeq C+1$. To determine the form of the ξ -classes, we proceed in four steps.

α) <u>The cliques which are not reduced to a singleton are either minimal or maximal.</u> Let $p^\xi \supseteq \{p,q\}$ be a clique which is neither minimal nor maximal. Its projection into the quotient space $Y=X/\Theta(\mathrm{Max}X)$ is again neither minimal nor maximal. Moreover, $\mathrm{Con}(Y)=\{\varphi \in \mathrm{Con}(X) \mid \Theta(\mathrm{Max}X) \le \varphi\}$ is also 1-distributive.

(a) In the special case when there exists $y \in Y$ such that $y^\xi \prec p^\xi$ (that means $y<z<p$ implies $z \in y^\xi \cup p^\xi$), consider an increasing clopen subset V of X containing p but not y. We have the contradiction $\Theta(\{p\} \cup y^\xi) \vee \Phi(V)=1$, $\Theta(\{q\} \cup y^\xi) \vee \Phi(V)=1$, $\Phi(V) \vee [\Theta(\{p\} \cup y^\xi) \wedge \Theta(\{q\} \cup y^\xi)] = \Phi(V)$.

(b) For the general situation, let $x,y \in Y$ be such that $y^\xi \prec x^\xi \le p^\xi$ and let V be an increasing clopen subset which separates x^ξ from y^ξ. By (a), we may suppose $x^\xi = \{x\}$ and $x^\xi \neq p^\xi$. Consider a clopen subset 0 of Y such that $y^\xi \cup \{p\} \subseteq 0 \subseteq -\{q\} \cap \cap -\{x\}$. The equivalence $\alpha = \Theta(0 \cap [y,p]) \cup \Theta(-0 \cap [y,p])$ is a congruence such that $\alpha \vee \Phi(V)=1$.

Since we have $\Theta(\{x\} \cup y^\xi) \vee \Phi(V)=1$ and $\Phi(V) \vee [\Theta(\{x\} \cup \cup y^\xi) \wedge \alpha)] = \Phi(V)$, $\mathrm{Con}(X)$ cannot be 1-distributive.

β) <u>If</u> $|\mathrm{Max}(X/\xi)| \ge 2$, <u>then</u> $(V \cap \mathrm{Max}X)^\xi \neq (-V \cap \mathrm{Max}X)^\xi$, <u>for all clopen subsets</u> V <u>of</u> X. If not, $\Phi(V)$ is a dual atom of $\mathrm{Con}(X)$. Let p^ξ and q^ξ be distinct elements of $\mathrm{Max}(X/\xi)$. We have $\Phi(V) \vee \vee \Theta(p^\xi)=1$, $\Phi(V) \vee \Theta(q^\xi)=1$, $\Phi(V) \vee (\Theta(p^\xi) \wedge \Theta(q^\xi)) = \Phi(V)$, which is impossible.

γ) <u>If</u> $X/\xi - \mathrm{Max}(X/\xi) \neq \emptyset$ <u>admits a unique upper bound</u> $x_0^\xi \in \in \mathrm{Max}(X/\xi)$ (this corresponds to the case when the chain C' of 2.1 is not reduced to a singleton), <u>then</u> $x_0^\xi = \{x_0\}$. Indeed, we argue as in α), replace p by x_0 and choose clopen increasing subsets V containing $\mathrm{Max}X$ in both cases (a) or (b).

So far, we have examined the closure spaces X such that $X \neq \neq \mathrm{Max}X$ and $\mathrm{Con}(X)$ is 1-distributive.

It follows from α), β), γ) that if $X \neq \mathrm{Max}X$, then X must satisfy one of the conditions (i),(ii) or (iii). Finally, we have

δ) <u>if</u> X/ξ <u>is an antichain, then</u> X <u>satisfies (iv) or (v).</u> Since $|X/\xi| \le 2$ (by 2.1), the condition β) shows that there exists at most one clique which is not reduced to a singleton. If $|X/\xi|=1$ and $|X| \ge 2$, let $\{U_1,U_2,U_3\}$ be a partition of X in clopen subsets. We have $\Theta(U_1 \cup U_2) \vee \Theta(U_2 \cup U_3)=1$, $\Theta(U_1 \cup U_2) \vee$

$\vee\ \Theta(U_1 \cup U_3)=1$ and $\Theta(U_1 \cup U_2) \vee (\Theta(U_2 \cup U_3) \wedge \Theta(U_1 \cup U_3)) \neq 1$, which contradicts the 1-distributivity of Con(X). Hence $X=2\uparrow$. The remaining possibility is (v).

This completes the characterization of closure spaces whose congruence lattice is 1-distributive.

Conversely, suppose that X satisfies one of the conditions (i),(ii),(iii),(iv) or (v). If $X=2\uparrow$, then Con(X) is isomorphic to the 2-element chain. In the other cases, there exists no dual atom $\Phi(V)$ with $(V \cap \mathrm{Max}X)^{\S} = (-V \cap \mathrm{Max}X)^{\S}$. Let $\alpha, \beta, \gamma \in$ $\in$ Con (X) be such that $\alpha \vee \beta = 1$, $\alpha \vee \gamma = 1$ and $\alpha \vee (\beta \wedge \gamma) \neq 1$. Since Con(X) is dually atomic ([3]), there exists (by 1.4) an increasing subset V of X which is both α-saturated and $(\beta \wedge \gamma)$-saturated and such that $\Phi(V)$ is a congruence. We distinguish two possibilities.

α) If V is decreasing, then X is not order-connected and V coincides with one of the two o.c.c. of X. By changing V into -V, we may suppose that V is not reduced to a clique or that $|V|=1$. Let t be the greatest element of V. Since $\alpha \vee \beta = 1$ (resp. $\alpha \vee \gamma = 1$), there exists u (resp. v) $\in \mathrm{Max}X - \{t\}$ such that $t\ \beta\ u$ (resp. $t\ \gamma\ v$) from what we deduce $\Theta(\mathrm{Max}X) \subseteq \beta \cap \gamma$ and the contradiction $\Theta(\mathrm{Max}X) \leq \Phi(V)$.

β) If V contains MaxX, let r be a minimal element of V which is not in the o.c.c. eventually reduced to a clique and s a maximal element of -V. There exists a least congruence ψ such that $\Theta(r,s) \subseteq \psi$ (if $\Theta(r,s) \notin$ Con(X), $\psi = \Theta(r,s) \cup \varphi(\mathrm{Max}X)$). From $\alpha \vee \beta = 1$ (resp. $\alpha \vee \gamma = 1$), we deduce $(r,s) \in \beta$ (resp. $(r,s) \in$ $\in \gamma$)). It follows that $(r,s) \in \beta \cap \gamma$ and $\Theta(r,s) \subseteq \psi \subseteq \beta \wedge \gamma \subseteq$ $\subseteq \Phi(V)$, which is impossible and concludes the proof.

In [3] and [5], we explain how to dualize the notions of chain, clique, cardinal sum and ordinal sum of closure spaces.

Since the condition $(V \cap \mathrm{Max}X)^{\S} \neq (-V \cap \mathrm{Max}X)^{\S}$ for all clopen subsets V of $X \in \mathbf{CS}$ becomes

$$\forall a \in B \in \mathbf{CA}, a^- = 1 \Longrightarrow (a^c)^- \neq 1$$

in **CA**, it is possible to translate Theorem 3.1 in **CA** and characterize the closure algebras whose the subalgebra lattice is 0-distributive.

References

[1] GRÄTZER G.: General Lattice Theory, Math. Reihe, Birkhäuser Verlag Basel und Stuttgart (1978).

[2] HANSOUL G. and VRANCKEN-MAWET L.: The subalgebra lattice of a Heyting algebra, Czech. Math. J. 37(112)(1987).

[3] HANSOUL G. and VRANCKEN-MAWET L.: Subalgebras of closure algebras, to appear in Period. Math. Hungaria.

[4] SACHS D.: The lattice of subalgebras of a Boolean algebra, Canad. J. Math. 14(1962), 451-460.

[5] VRANCKEN-MAWET L.: On the subalgebra lattice of a Heyting algebra, submitted.

Institut de Mathématique, Avenue des Tilleuls, 15, B-4000 Liège, Belgique

(Oblatum 3.3. 1987)