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Mario Zuluaga Uribe

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ON A FIXED POINT THEOREM AND APPLICATIONS TO A TWO POINT BOUNDARY VALUE PROBLEM

Mario ZULUAGA

Abstract: In this paper we present a fixed point theorem which is an extension of a well known theorem due to Krasnosel'skii. As a consequence, we give an application to a two point boundary value problem.

Key words: Fixed point, boundary value problem.

Classification: 34B15

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1. Introduction and Notations. We are going to study the two point B.V.P.

$$\text{(I)} \qquad \begin{aligned} &u''(t)+g(u(t))=0 \\ &u(a)=u(b)=0 \end{aligned}$$

We will prove the following

Theorem 1. Let $g: \mathbb{R} \to \mathbb{R}$ be such that:

a) g is increasing,

b) $|g(u)-g(v)| \leq \lambda_1 |u-v|$ where λ_1 is the first eigenvalue for the problem

$$\text{(II)} \qquad \begin{aligned} &u''(t)+\lambda u(t)=0 \\ &u(a)=u(b)=0. \end{aligned}$$

c) There exists $c', c'' \geq c_0$ (where c_0 is going to be defined below) such that

$$g(c')=\lim_{t\to\infty} \frac{g(-c''t)}{t}$$

d) At least one of the following equalities holds:

$$g(-c'')=\lim_{t\to\infty} \frac{g(-c't)}{t}$$

or

$$g(-c'')=\lim_{t\to\infty} \frac{g(c't)}{t}.$$

Hence, problem (I) has at least a solution.

The theorem 1 will be a consequence of the following fixed point theorem.

Theorem 2. Let $f:H \longrightarrow H$ be a compact mapping defined on a Hilbert space H. Let $\alpha(\rho)$ be the real valued function $\alpha: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\alpha(\rho) \geq 1$ and $\lim_{\rho \to \infty} \alpha(\rho) = 1$ and suppose that

a) for each $u \in H$, $\|u\| = \rho > 0$, $\langle f(u), u \rangle \leq \alpha(\rho) \|u\|^2$

b) $f(u) - \|u\| f(\frac{u}{\|u\|}) = o(\|u\|)$ if $\|u\| \to \infty$.

Then f has a fixed point in H.

Remarks. Theorem 2 is an extension of the well known theorem due to Krasnosel'skii, for example see [2] p. 271. If $\alpha(\rho) = 1$ for some $\rho > 0$, condition b) in Theorem 2 is superfluous. In Theorem 1 we consider the case $g'(0) \leq \lambda_1$. This case has attracted much attention recently; for example see [3] for references and for the interesting case $\lambda_k \leq g'(0) < \lambda_{k+1}$. Our results are based on simpler arguments and thus our publication seems to be worthwhile.

Preliminary results. Let $H = H_0^1[a,b]$ be the Sobolev space of square integrable functions on $[a,b]$ vanishing on $\{a,b\}$ with generaralized first derivative in $L^2[a,b]$. The inner product and norm in H are given by

$$\langle u,v \rangle_1 = \int_a^b u'(t) \cdot v'(t)\, dt,$$

$$\|u\|_1^2 = \langle u,u \rangle_1 .$$

We indicate with $\langle \, , \rangle_0$ and $\| \ \|_0$ the inner product and norm in $L^2[a,b]$. According to the Sobolev's lemma (see [1] p.95) H can be imbedded in the space of continuous functions defined on $[a,b]$. Thus there exists a real number $c_0 > 0$ such that

$$(1,1) \qquad \max_{t \in [a,b]} |u(t)| \leq c_0 \|u\|_1 ,$$

for all $u \in H$.

By Poincaré's inequality we have

$$(1,2) \qquad \lambda_1 \|u\|_0^2 \leq \|u\|_1^2$$

for all $u \in H$.

2. Proof of the theorems

Proof of Theorem 2. Suppose that for all $\rho > 0$, f has no fixed points in $\bar{B}(0,\rho)=\{x\in H/\|x\|\leq\rho\}$. Then for all $\rho>0$ the Leray-Schauder degree

$$d[I-f,B(0,\rho),0]=0.$$

We consider the homotopy, $H(x,t)=x-tf(x)$, $\|x\|\leq\rho$ and $0\leq t\leq 1$. There exists $t\in(0,1]$ and $x\in H$, $\|x\|=\rho$ such that $x=tf(x)$. Let $\{t_n\}\subset(0,1]$ and $\{x_n\}\subset H$, $\|x_n\|=\rho_n$ such that $\rho_n\to\infty$ and

(2,1) $$x_n=t_nf(x_n).$$

By (2.1) and condition a) in Theorem 2 we have

(2,2) $$1\geq t_n\geq\frac{1}{\alpha(\rho_n)}.$$

From (2,1) and condition b) in Theorem 2 we have

(2,3) $$\lim_{n\to\infty}\left[\frac{1}{t_n}\frac{x_n}{\|x_n\|}-f\left(\frac{x_n}{\|x_n\|}\right)\right]=0.$$

Now, since $\left\{\frac{x_n}{\|x_n\|}\right\}$ is bounded, then there exists $\left\{\frac{x_{n_k}}{\|x_{n_k}\|}\right\}\subset$ $\subset\left\{\frac{x_n}{\|x_n\|}\right\}$ and $x\in H$ such that $\frac{x_{n_k}}{\|x_{n_k}\|}\rightharpoonup x$ (here, $\rightharpoonup$ denotes the weak convergence). From (2,2) and considering that $\alpha(\rho)\to 1$ if $\rho\to\infty$ we have that $\frac{1}{t_{n_k}}\cdot\frac{x_{n_k}}{\|x_{n_k}\|}\rightharpoonup x$. Since f is compact, $f\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right)\to f(x)$. By (2,3) we have $f(x)=x$. This fact, however, contradicts the assumption that f has no fixed points, and so the proof is completed.

<u>Proof of Theorem 1</u>. The function $u(t)\in H_0^1[a,b]$ is a generalized solution of (I) if for all $v(t)\in H_0^1[a,b]$

(2,4) $$\int_a^b u'(t)v'(t)dt=\int_a^b g(u(t))\cdot v(t)dt.$$

First we will find generalized solutions, u(t), of (I). By the standard regularity theory it follows that u(t) is a solution of (I).

By condition b) of Theorem 1, by the fact that $i:H_0^1[a,b]\hookrightarrow$ $\hookrightarrow L^2[a,b]$ is a compact inclusion, by (1.2) and the Riesz's theorem we can consider the function $f:H_0^1[a,b]\to H_0^1[a,b]$

defined by

(2,5) $$\langle f(u),v\rangle_1 = \langle g(u),v\rangle_0$$

for all $v(t) \in H_0^1[a,b]$.

This function f is compact and from (2,4), $u \in H_0^1[a,b]$ is a generalized solution of (I) if and only if $f(u)=u$. By condition b) of Theorem 1 if $\|u\|_0^2 \leq \left(\frac{1}{\lambda_1}\right)^2$, then $\|g(u)-g(0)\|_0^2$ 1, hence, by (1,2), we have (see [4] p. 26):

$$\langle f(u),u\rangle_1 \leq \|g(u)\|_0 \ \|u\|_0 \leq$$

(2,6)
$$\leq \left[\left(1+ \frac{1}{\lambda_1 \|u\|_1^2}\right)^{1/2} + \frac{\|g(0)\|_0}{\sqrt{\lambda_1}\|u\|_1}\right] \|u\|_1^2 .$$

We denote $\alpha(\rho) = \left(1+\frac{1}{\lambda_1 \rho^2}\right)^{1/2} + \frac{\|g(0)\|_0}{\sqrt{\lambda_1}\cdot\rho}$. It is clear that

$\alpha(\rho) \geq 1$ and $\lim_{\rho\to\infty} \alpha(\rho)=1$. Thus $\langle f(u),u\rangle_1 \leq \alpha(\rho)\cdot\|u\|_1^2$, $\|u\| = \rho$.

Finally, to prove that condition b) of Theorem 2 is fulfilled, it is sufficient to see that $\|g(u)-g\left(\frac{u}{\|u\|_1}\right)\|u\|_1\ \|_0 = o(\|u\|_1)$ if

$\|u\|_1 \longrightarrow \infty$. In fact: From (1,1) and the condition that $c',c'' \geq$ $\geq c_0$ we have

(2,7) $$- c'' \leq \frac{'u}{\|u\|_1} \leq c' .$$

Since g is increasing

(2,8) $$g(-c''\|u\|_1) \leq g(u) \leq g(c'\|u\|_1).$$

Also, from (2.7) we have

(2,9) $$g(-c'')\ \|u\|_1 \leq g\left(\frac{u}{\|u\|_1}\right) \|u\|_1 \leq g(c')\ \|u\|_1 .$$

From (2,8) and (2,9) we have

$$g(-c''\|u\|_1)-g(c')\|u\|_1 \leq g(u)-g\left(\frac{u}{\|u\|_1}\right)\cdot \|u\|_1 \leq$$

(2,10)
$$\leq g(c'\|u\|_1)-g(-c'')\cdot\|u\|_1 .$$

Also

$$g(-c'')\ \|u\|_1-g(-c'\|u\|_1) \leq g\left(\frac{u}{\|u\|_1}\right)\|u\|_1-g(u) \leq$$

(2,11)
$$\leq g(c')\ \|u\|_1-g(-c''\|u\|_1).$$

From (2,10),(2,11) and condition d) of Theorem 1 we have

$$\lim_{\|u\|_1 \to \infty} \frac{\left| g\left(\frac{u}{\|u\|_1}\right) \|u\|_1 - g(u) \right|}{\|u\|_1} = 0.$$

Thus the proof of Theorem 1 is complete.

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Departamento de Matemática, Universidad Nacional, Bogotá, D.E.
Colombia, Sud America

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