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ON POINTWISE LIMITS OF SEQUENCES OF I-CONTINUOUS FUNCTIONS

Marek BALCERZAK, Ewa ŁAZAROW

Abstract: In the paper, the family $B_1(\mathcal{C}_I)$ of pointwise limits of sequences of I-continuous functions is considered. We formulate a condition necessary for a function to be in $B_1(\mathcal{C}_I)$, analogous to that given by Grande. Moreover, we show that $B_1(\mathcal{C}_I)$ essentially contains the Baire class 1 and is essentially contained in the Baire class 2.

Key words: I-continuous functions, Baire classes.

Classification : 26A21

For any family $\mathcal{F}$ of functions which map R (the real line) into R, we denote by $B_1(\mathcal{F})$ the family of pointwise limits of sequences of functions taken from $\mathcal{F}$. Then we define $B_2(\mathcal{F})=B_1(B_1(\mathcal{F}))$, $B_3(\mathcal{F})=B_1(B_2(\mathcal{F}))$ and so on.

Denote by $\mathcal{C}$ the family of all continuous functions from R into R (with the natural topologies).

In the sequel, $\mathcal{B}$ will denote the family of all subsets of R having the Baire property, I will denote the σ-ideal of sets of the first category. In [3] there were introduced notions of I-density point and I-dispersion point of a set $E \in \mathcal{B}$ (one can also consider left- or right-hand I-density points or I-dispersion points).

Let $\Phi(A)$ denote the set of I-density points of A . It turns out (see [3]) that the family $\mathcal{T}=\{A \in \mathcal{B} : A \subset \Phi(A)\}$ is a topology. It is called the I-density topology. Continuous functions mapping R with the topology $\mathcal{T}$ into R with the natural topology are called I-continuous. The family of these functions will be denoted by $\mathcal{C}_I$.

In [1] Grande investigated the family $B_1(\mathcal{A} \cap \mathfrak{D})$ where $\mathcal{A}$, $\mathfrak{D}$

denote respectively the families of approximately continuous functions, and functions whose sets of points of discontinuity have the Lebesgue measure zero. Let us consider $\mathcal{C}_I$ instead of $\mathcal{A}$, and $\mathcal{D}_I$, instead of $\mathcal{D}$, where $\mathcal{D}_I$ is the family of functions whose sets of points of discontinuity belong to I. Then we have $\mathcal{C}_I \cap \mathcal{D}_I = \mathcal{C}_I$ since $\mathcal{C}_I \subset B_1(\mathcal{C})$ (see [3]) and $B_1(\mathcal{C}) \subset \mathcal{D}_I$. Our paper shows that $B_1(\mathcal{C}_I)$ behaves similarly as $B_1(\mathcal{A} \cap \mathcal{D})$.

Grande formulated a condition necessary for a function to be in $B_1(\mathcal{A} \cap \mathcal{D})$. We shall prove the analogous result in the case of $B_1(\mathcal{C}_I)$.

For $E \subset R$, let int E, $\bar{E}$ denote, respectively, the interior and closure of E in the natural topology.

For any $x \in R$, we denote by $\mathcal{P}(x)$ the collection of all intervals [a,b] such that $x \in (a,b)$ and of all sets of the form $E = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{x\}$ where, for every n, $a_n < b_n < a_{n+1} < x < d_{n+1} < c_n < d_n$ and $x \in \Phi(E)$.

In [2], there was introduced a topology τ which consists of all sets $U \in \mathcal{T}$ such that if $x \in U$, then there exists a set $P \in \mathcal{P}(x)$ included in $\{x\} \cup \operatorname{int} U$. It was proved that τ is the coarsest topology for which all I-continuous functions are continuous.

For any subset M of R, define $\Delta(M)$ as the set of all x such that, for each $P \in \mathcal{P}(x)$, we have $\emptyset \neq P \cap M \neq \{x\}$.

<u>Lemma 1</u> ([2]). Let $M \subset R$. If $U \in \tau$ and $U \cap \Delta(M) \neq \emptyset$, then $(\operatorname{int} U) \cap M \neq \emptyset$.

If $a \in R$ and $f: R \to R$, then we write shortly $\{f < a\}$ instead of $\{x : f(x) < a\}$ and analogously, for the inequalities $>$, $\leq$, $\geq$.

<u>Theorem 1</u>. Let $f \in B_1(\mathcal{C}_I)$. Then the following condition $(I\mathcal{C}_1)$ holds:

For any $a, b \in R$, $a < b$, and nonempty sets U, V, if

(1) $U \subset \{f < a\}$,

(2) $V \subset \{f > b\}$,

(3) $U \subset \Delta(\bar{U})$ and $V \subset \Delta(\bar{V})$,

then $U \setminus \bar{V} \neq \emptyset$ or $V \setminus \bar{U} \neq \emptyset$.

Remark. The condition $U \subset \Delta(\bar{U})$ means that the closure of U in the topology τ is a perfect set in this topology (see [2]).

Proof. Suppose on the contrary that there are $a, b \in R$, $a < b$, and nonempty sets U and V fulfilling conditions (1),(2),(3), such that $U \setminus \bar{V} = \emptyset$ and $V \setminus \bar{U} = \emptyset$. These equations easily give $\bar{U} = \bar{V}$. Let $f(x) = \lim_{n \to \infty} f_n(x)$, $x \in R$, where $f_n \in \mathcal{C}_I$ for every n. We have

$$\{f \le a\} = \bigcap_{m=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcup_{n=r}^{\infty} \{f_r < a + \frac{1}{m}\}$$

and $\{f_r < a + \frac{1}{m}\} \in \tau$ for all m, r. So, $\{f \le a\}$ can be expressed in the form $\bigcap_{k=1}^{\infty} U_k$ where $U_k \in \tau$ for all k. Analogously, $\{f \ge b\}$ can be expressed in the form $\bigcap_{k=1}^{\infty} V_k$ where $V_k \in \tau$ for all k. Let $F = \Delta(\bar{U})$. Observe that each of the sets $\bar{F} \cap \operatorname{int} U_k$, $\bar{F} \cap \operatorname{int} V_k$, $k = 1, 2, \ldots$, is dense in $\bar{F}$ with the natural topology.
Indeed, let G be an open set in the natural topology, such that $G \cap \bar{F} \neq \emptyset$. Then $G \cap F \neq \emptyset$ and from Lemma 1 it follows that $G \cap U \neq \emptyset$, $G \cap V \neq \emptyset$. Conditions (1),(2),(3) imply that

$$U \subset \{f < a\} \cap F, \quad V \subset \{f > b\} \cap F.$$

Consequently, for all k, we have

$$\emptyset \neq G \cap U \subset G \cap \{f < a\} \cap F \subset G \cap U_k \cap F,$$

$$\emptyset \neq G \cap V \subset G \cap \{f > b\} \cap F \subset G \cap V_k \cap F.$$

Thus, in virtue of Lemma 1, we obtain

$$\operatorname{int}(G \cap U_k) \cap U \neq \emptyset, \quad \operatorname{int}(G \cap V_k) \cap V \neq \emptyset,$$

for all k, and, using (3), we easily deduce that

$$G \cap (\operatorname{int} U_k) \cap \bar{F} \neq \emptyset, \quad G \cap (\operatorname{int} V_k) \cap \bar{F} \neq \emptyset, \text{ for all } k.$$

So, we have proved that $\bar{F} \cap \operatorname{int} U_k$, $\bar{F} \cap \operatorname{int} V_k$ are dense in $\bar{F}$ for all k. Now, the Baire Category Theorem implies

$$\bigcap_{k,l} (\operatorname{int} U_k \cap \operatorname{int} V_l) \cap \bar{F} \neq \emptyset,$$

which gives a contradiction since $\{f \le a\}$, $\{f \ge b\}$ are disjoint.

<u>Corollary</u>. $B_1(\mathcal{C}_I) \subsetneq B_2(\mathcal{C})$.

Proof. Since $\mathcal{C}_I \subset B_1(\mathcal{C})$, the inclusion $B_1(\mathcal{C}_I) \subset B_2(\mathcal{C})$ is obvious. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then $f \in B_2(\mathcal{C})$. On the other hand, f does not satisfy condition $(I\mathcal{C}_1)$ of Theorem 1 (it suffices to consider $a=\frac{1}{4}$, $b=\frac{3}{4}$, U equal to the set of all irrational numbers, $V=R\setminus U$). Thus $f\notin B_1(\mathcal{C}_I)$.

For any interval I with endpoints a, b ($a<b$), let us denote: $l(I)=a$, $r(I)=b$, $|I|=b-a$.

In the sequel, we shall say that $A\subset R$ is a right-hand (left-hand) interval set at a point x_0 if and only if $A=\bigcup_{n=1}^{\infty}(a_n,b_n)$ where $b_{n+1}<a_n<b_n$ for all n and $a_n\searrow x_0$, $b_n\searrow x_0$ ($a_n<b_n<a_{n+1}$ for all n and $a_n\nearrow x_0$, $b_n\nearrow x_0$). A right-hand (left-hand) interval set will be called normal if and only if, for every n, the intervals (a_{n+1},b_{n+1}), (x_0,a_n) (resp. (a_{n+1},b_{n+1}), (b_n,x_0)) have the same centres.

In [4] there was given an example of a right-hand interval set $\bigcup_{n=1}^{\infty}(a_n,b_n)\subset(0,1)$ at the point 0, such that 0 is its right-hand I-dispersion point. Then, obviously, the set $\bigcup_{n=1}^{\infty}(b_{n+1},a_n)$ is a right- hand interval set at the point 0, and 0 is its right- hand I-density point. In a similar way, for any point x, we can construct a right-hand (left-hand) interval set at x for which x is a right-hand (left-hand) I-density point.

<u>Lemma 2</u>. There exist right-hand interval sets $A=\bigcup_{n=1}^{\infty}(c_n,d_n)$, $A^*=\bigcup_{n=1}^{\infty}(c_n^*,d_n^*)$ at the point 0, such that A^* is normal, $[c_n,d_n]\subset(c_n^*,d_n^*)\subset(0,1)$ for all n, and 0 is a right-hand I-density point of A.

Proof. We shall base ourselves on the construction described in [4]. Let $(a_1,b_1)\subset(0,1)$, $a_1>0$ be an arbitrary interval and let $q_1=E(a_1^{-1})+1$ ($E(x)$ stands for the entier of x). Choose $b_2\in(0,1)$ such that $q_1\cdot b_2=2^{-2}$ and put $a_2=\frac{2}{3}b_2$, $q_2=E(a_2^{-1})+1$. Suppose that we have already defined numbers a_i, b_i, q_i for $i=1,2,\dots,k$. Choose $b_{k+1}\in(0,1)$ such that $q_k\cdot b_{k+1}=2^{-k-1}$ and put $a_{k+1}=\frac{k+1}{k+2}b_{k+1}$, $q_{k+1}=E(a_{k+1}^{-1})+1$. Thus, by induction, we have defined the numbers a_n, b_n, q_n for each integer $n\geq 1$. Consider the set $D=\bigcup_{n=1}^{\infty}(a_n,b_n)$. As in [4] we can show that 0 is a right-hand I-dispersion point of 0. Thus 0 is a right-hand I-density point of the set $\bigcup_{n=1}^{\infty}(b_{n+1},a_n)$. Put $c_n=b_{n+1}$, $d_n=a_n$, $n=1,2,\ldots$. Observe that the construction implies

$$b_{n+1}<2^{-n}(E(a_n^{-1})+1)^{-1}<2^{-n}a_n<n^{-1}a_n=b_n-a_n.$$

for all n. Let $\varepsilon_n=\frac{1}{2}(b_n-a_n-b_{n+1})$, $n=1,2,\ldots$. Define a sequence $\{r_n\}$ by induction as follows: Let $0<r_1<\varepsilon_1$ and having defined $r_k>0$, $k=1,\ldots,n$, such that $\sum_{i=k}^{n} r_i<\varepsilon_k$ for $k=1,\ldots,n$, choose $r_{n+1}>0$ such that $\sum_{i=k}^{n+1} r_i<\varepsilon_k$ for $k=1,\ldots,n+1$. Next, put $b_n^*=b_n-\sum_{k=n}^{\infty} r_k$, $n=1,2,\ldots$. For all n, we have $b_n^*-b_{n+1}^*=b_n-b_{n+1}-r_n>b_n-b_{n+1}-2\varepsilon_n=a_n$. Let $c_n^*=b_{n+1}^*$, $d_n^*=b_n^*-b_{n+1}^*$, $n=1,2,\ldots$ It is easy to check that the sets $A=\bigcup_{n=1}^{\infty}(c_n,d_n)$, $A^*=\bigcup_{n=1}^{\infty}(c_n^*,d_n^*)$ fulfil the assertion.

In the proof of the following theorem we try to apply the scheme presented by Grande (see [1], the proof of Th. 3). However, while he uses an arbitrary perfect nowhere dense set of measure zero, we use some special perfect nowhere dense set.

Let $2^{<\omega}$ be the set of all finite sequences with terms from $\{0,1\}$ (including the empty sequence $\emptyset$). For $\sigma\in 2^{<\omega}$, let $|\sigma|$ denote the number of terms in σ . If n is 0 or 1, then $\sigma^\frown n$ stands for the member of $2^{<\omega}$, with length $|\sigma|+1$, whose first $|\sigma|$ terms form the sequence σ and the last term is n.

<u>Theorem 2</u>. $B_1(\mathcal{C})\subsetneq B_1(\mathcal{C}_I)$.

<u>Proof</u>. Since the inclusion $B_1(\mathcal{C})\subset B_1(\mathcal{C}_I)$ is obvious, we ought to show that the equality does not hold here.

We shall start from the construction of some perfect nowhere dense set. Let $A=\bigcup_{n=1}^{\infty}(c_n,d_n)$, $A^*=\bigcup_{n=1}^{\infty}(c_n^*,d_n^*)$ be the sets obtained in Lemma 2. Put $P_\emptyset=[0,1]$ and let $P_{\langle 0\rangle}$, $P_{\langle 1\rangle}$ be closed intervals such that $l(P_{\langle 0\rangle})=0$, $r(P_{\langle 1\rangle})=1$, $|P_{\langle 0\rangle}|=|P_{\langle 1\rangle}|=c_1^*$. The set $P_\emptyset\setminus(P_{\langle 0\rangle}\cup P_{\langle 1\rangle})$ is an open interval denoted by $V_\emptyset$. Let $n\geq 1$ and assume that the intervals P_σ have already been defined for all $\sigma\in 2^{<\omega}$, $|\sigma|=n$. Fix an arbitrary $\sigma\in 2^{<\omega}$, $|\sigma|=n$.

Let $P_{\sigma^\wedge k}$, k=0,1, be closed intervals such that $l(P_{\sigma^\wedge 0})=l(P_\sigma)$; $r(P_{\sigma^\wedge 1})=r(P_\sigma)$ and $|P_{\sigma^\wedge k}|=c^*_{n+1}$ for k=0,1. The set $P_\sigma \setminus (P_{\sigma^\wedge 0} \cup P_{\sigma^\wedge 1})$ is an open interval denoted by V_σ. In this way, we define by induction intervals P_σ, V_σ for all $\sigma \in 2^{<\omega}$. Let $P=\bigcap_{m=1}^{\infty} \bigcup_{|\sigma|=m} P_\sigma$. It is easy to verify that P is a perfect nowhere dense set.

Let H denote the set of endpoints of all intervals P_σ, $\sigma \in 2^{<\omega}$, excluding the points 0 and 1. For each $x \in H$, we shall define two sequences $\{I_n(x)\}$, $\{J_n(x)\}$ of intervals such that $I_n(x)$ are closed, $J_n(x)$ are open, $I_n(x) \subset J_n(x) \subset (0,1)\setminus P$ for all n, and x is an I-density point of the set $\bigcup_{n=1}^{\infty} I_n(x)$. Thus, let $x \in H$ and assume, for instance, that $x=l(P_\sigma)$, $|\sigma|=m$. Choose two left-hand interval sets $\bigcup_{i=1}^{\infty}(a_i,b_i)$, $\bigcup_{i=1}^{\infty}(a^*_i,b^*_i)$ at the point x, such that $[a_i,b_i] \subset (a^*_i,b^*_i) \subset (0,x)\setminus P$ for all n and x is a left-hand I-density point of $\bigcup_{i=1}^{\infty}(a_i,b_i)$. Denote $\sigma_1=\sigma$ and, for each $i \geq 1$, let $\sigma_{i+1}=\sigma_i{}^\wedge 0$. Observe that the construction implies that

$V_{\sigma_i}=(x+c^*_{m+i},\ x+d^*_{m+i})$ for i=1,2,... . Let

$U_{\sigma_i}=(x+c_{m+i},\ x+d_{m+i})$, i=1,2,... .

Since 0 is a right-hand I-density point of $\bigcup_{n=1}^{\infty}(c_n,d_n)$, therefore x is a right-hand I-density point of $\bigcup_{i=1}^{\infty} U_{\sigma_i}$. At last, let $\{I_n(x)\}$ consist of all intervals $[a_i,b_i]$, $\overline{U}_{\sigma_i}$ i=1,2,... , and let $\{J_n(x)\}$ consist of (a^*_i,b^*_i), V_{σ_i}, i=1,2,... . These sequences have the required properties.

Now, we construct a function $f \in B_1(\mathscr{C}_I)\setminus B_1(\mathscr{C})$. Let f be the characteristic function of the set H. Evidently, $f \notin B_1(\mathscr{C})$. We shall show that $f \in B_1(\mathscr{C}_I)$. Let $H=\{x_1,x_2,\dots\}$. Let $\{J_n^{1,1}\}$ be the sequence of all intervals taken from $\{J_n(x_1)\}$ which are included in $(x_1-1/2,x_1+1/2)$. Assume that $i \geq 1$ and that we have already defined sequences $\{J_n^{i,j}\}_{n\geq 1}$, j=1,2,...,i. Put

$$\delta_i=\frac{1}{2}\min\{1/(i+1);\ |x_k-x_l| \text{ for } k,l \in \{1,2,\dots,i+1\},\ k\neq l\}.$$

For each j=1,2,...,i+1, let $\{J_n^{i+1,j}\}_{n\geq 1}$ be the sequence of all intervals taken from $\{J_n(x_j)\}_{n\geq 1}$ which are included in

$(x_j - \sigma_i, x_j + \sigma_i)$. In such a way, we define by induction a family of closed intervals $\{J_n^{i,j}\}$, where $i,n=1,2,\ldots$, and $j=1,2,\ldots,i$, which has the following properties (comp. [1]):

(1) for fixed i and j, we have

$$\sup\{|x-x_j| : x \in J_n^{i,j}\} \to 0 \quad \text{if } n \to \infty \ ;$$

(2) for fixed i and j, the intervals $J_n^{i,j}$ are pairwise disjoint;

(3) for fixed i and j, the diameter of $\bigcup_{n=1}^{\infty} J_n^{i,j}$ does not exceed $1/i$;

(4) for fixed i and j, x_j is an I-density point of $\bigcup_{n=1}^{\infty} J_n^{i,j}$;

(5) $P \cap \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} \bigcup_{n=1}^{\infty} J_n^{i,j} = \emptyset$;

(6) for fixed $j \leq i$,

$$\bigcup_{n=1}^{\infty} J_n^{i+1,j} \subset \bigcup_{n=1}^{\infty} J_n^{i,j} \quad \text{if } i=1,2,\ldots;$$

(7) for fixed i,

$$\bigcup_{n=1}^{\infty} J_n^{i,j_1} \cap \bigcup_{n=1}^{\infty} J_n^{i,j_2} = \emptyset \quad \text{if } j_1 \neq j_2.$$

For each interval $J_n^{i,j}$, denote by $I_n^{i,j}$ that term of the sequence $\{I_k(x_j)\}_{k \geq 1}$ which is contained in $J_n^{i,j}$. By the construction, x_j is an I-density point of any set $\bigcup_{n=1}^{\infty} I_n^{i,j}$, $i \geq j$. For $i=1,2,\ldots$, define

$$f_i(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{j=1}^{i} \left(\bigcup_{n=1}^{\infty} I_n^{i,j} \cup \{x_j\} \right) \\ 0 & \text{if } x \notin \bigcup_{j=1}^{i} \left(\bigcup_{n=1}^{\infty} J_n^{i,j} \cup \{x_j\} \right) \\ \text{extended linearly on } J_n^{i,j} \setminus I_n^{i,j}, & \text{for } j=1,\ldots,i \text{ and } n=1,2,\ldots \ . \end{cases}$$

It is easy to verify that all the functions f_i belong to $\mathcal{C}_I$, and $\lim_{i \to \infty} f_i(x) = f(x)$ for each $x \in R$. This ends the proof.

Now, we may ask about a characterization of the class $B_1(\mathcal{C}_I)$; in particular, we may ask whether each function $f \in B_2(\mathcal{C})$ having the property $(I\mathcal{C}_1)$ belongs to $B_1(\mathcal{C}_I)$.

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