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THE COARSEST TOPOLOGY FOR I-APPROXIMATELY CONTINUOUS FUNCTIONS

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Abstract: In this paper we examine functions $f:R \to R$ which are I-approximately continuous on R. The topology, labelled the I-density topology $\mathcal{T}_I$ has been presented in [2]. There has been shown that with respect to $\mathcal{T}_I$ the I-approximately continuous functions are continuous. We shall define a completely regular topology $\tau \subseteq \mathcal{T}_I$ making all I-approximately continuous functions continuous.

Key words: I-density topology, I-approximately continuous functions.

Classification: 26A21

Throughout this paper, $\mathcal{B}$ will denote the family of all subsets of R having the Baire property, I will denote the sigma ideal of sets of the first category. For $a \in R$ and $A \subset R$ we denote $a \cdot A = \{ax : x \in A\}$ and $A - a = \{x - a : x \in A\}$. Recall [2] that 0 is an I-density point of a set $A \in \mathcal{B}$ if and only if $\chi_{n \cdot A \cap [-1,1]} \xrightarrow[n \to \infty]{I} 1$, i.e. if and only if for every increasing sequence $\{n_m\}_{m \in N}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in N}$ such that $\chi_{n_{m_p} \cdot A \cap [-1,1]} \xrightarrow[p \to \infty]{} 1$ except for a set belonging to I. A point x_0 is an I-density point of $A \in \mathcal{B}$ if and only if 0 is an I-density point of $A - x_0$. The set of all I-density points of A will be denoted by $\Phi(A)$. The notions of right-hand, left-hand I-density points and of I-dispersion points are defined in an obvious manner. The topology $\mathcal{T}_I$ is the family of all sets $A \in \mathcal{B}$ such that $A \subset \Phi(A)$.

Definition 1. Let f be any function defined in some neighbourhood of x_0 and having there the Baire property.

$\text{I-}\liminf_{x \to x_0} f(x) = \sup\{\alpha : \{x : f(x) < \alpha\}$ has x_0 as an I-dispersion point$\}$,

$I\text{-}\lim\sup_{x\to x_0} f(x)=\inf\{\alpha:\{x:f(x)>\alpha\}$ has x_0 as an I-dispersion point$\}$.

We say that f is I-approximately continuous at x_0 if and only if $I\text{-}\lim\inf_{x\to x_0} f(x)=I\text{-}\lim\sup_{x\to x_0} f(x)=f(x_0)$.

Throughout this paper $\mathcal{T}$ will denote the natural topology, cl(A) (int(A)) will denote closure (interior) of the set A with respect to $\mathcal{T}$.

Definition 2. For $x\in R$, by $\mathcal{P}(x)$ we will define the family of all closed intervals $[a,b]$ such that $x\in(a,b)$ and of all interval sets $\bigcup_{n=1}^{\infty}[a_n,b_n]\cup\bigcup_{n=1}^{\infty}[c_n,d_n]\cup\{x\}$ where for all n, $b_{n-1}< a_n<b_n<x$ and $x<c_n<d_n<c_{n-1}$, and $x\in\mathcal{C}(\bigcup_{n=1}^{\infty}[a_n,b_n]\cup\bigcup_{n=1}^{\infty}[c_n,d_n])$.

It is obvious that if $P\in\mathcal{P}(x)$ then P is perfect with respect to the natural topology.

Lemma 1 [1]. Let $G\subset R$ be an open set with respect to $\mathcal{T}$. Then 0 is an I-dispersion point of G if and only if for every natural number n, there exist a natural number k and a real number $\delta>0$ such that, for each $h\in(0,\delta)$ and for each $i\in\{1,\dots,n\}$ there exist two natural numbers $j_r, j_l\in\{1,\dots,k\}$ such that

$$G\cap\left(\left(\frac{i-1}{n}+\frac{j_r-1}{n\cdot k}\right)h,\ \left(\frac{i-1}{n}+\frac{j_r}{n\cdot k}\right)h\right)=\emptyset$$

and

$$G\cap\left(-\left(\frac{i-1}{n}+\frac{j_l}{n\cdot k}\right)h,\ -\left(\frac{i-1}{n}+\frac{j_l-1}{n\cdot k}\right)h\right)=\emptyset .$$

We shall use the above lemma for $x\in R$ by translating the set, if necessary.

Definition 3. Let τ be the collection of all subsets U of R such that

1. $U\in\mathcal{T}_I$,
2. if $U\neq\emptyset$ and $x\in U$ then there exists the set $P\in\mathcal{P}(x)$ such that $P\subset \operatorname{int} U\cup\{x\}$.

Theorem 1. τ is a topology on R and $\mathcal{T}\subsetneq\tau\subsetneq\mathcal{T}_I$.

Proof. Let $U_1,U_2\in\tau$. Then $U_1,U_2\in\mathcal{T}_I$ and $U_1\cap U_2\in\mathcal{T}_I$. Let $U_1\cap U_2\neq\emptyset$ and $x\in U_1\cap U_2$. Then there exist the sets $P_1,P_2\in\mathcal{P}(x)$

such that $P_1 \subset \text{int } U_1 \cup \{x\}$ and $P_2 \subset \text{int } U_2 \cup \{x\}$. Since there exists $P \subset P_1 \cap P_2$ such that $P \in \mathcal{P}(x)$ and $P_1 \cap P_2 \subset \text{int}(U_1 \cap U_2) \cup \{x\}$, therefore $U_1 \cap U_2 \in \tau$.

Next, suppose that $U_t \in \tau$ for each $t \in T$ and $U = \bigcup_{t \in T} U_t$. Then $\bigcup_{t \in T} U_t \in \mathcal{T}_I$ and for each $x \in U$ there exists $P \in \mathcal{P}(x)$, $P \subset \text{int } U_{t_0} \cup \{x\}$ such that $x \in U_{t_0}$. Therefore $P \subset \text{int } U \cup \{x\}$ and $U \in \tau$.

Since $\emptyset$ and R belong to τ, therefore τ is a topology on R and $\mathcal{T} \subset \tau \subset \mathcal{T}_I$.

Let A be the set of all irrational numbers of R. Then $A \in \mathcal{T}_I$ and $A \notin \tau$. Now, let $G_1 = \bigcup_n (a_n, b_n)$, $G_2 = \bigcup_n (c_n, d_n)$ such that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$, $0 < a_n < b_n < a_{n-1}$ and $c_n < d_n < c_{n+1} < 0$ for all $n \in N$ and 0 is a right-hand and left-hand I-dispersion point of G_1 and G_2, respectively (see [3]). Let

$$P = (R \setminus (G_1 \cup G_2)) \cap [c_1, b_1].$$

Then $P \in \mathcal{P}(0)$. Let $G \in \mathcal{T}$ be such that

$$G = \bigcup_{n=1}^{\infty} \left(\frac{2b_{n+1}+a_{n+1}}{3}, \frac{2a_n+b_n}{3}\right) \cup (-\infty, 0),$$ then $P \subset G \cup \{0\}$ and $0 \notin G$.

Then $G \cup \{0\} \in \tau$ and $G \cup \{0\} \notin \mathcal{T}$. So the proof of Theorem is completed.

<u>Theorem 2</u>. If f is any I-approximately continuous function then f is a continuous function with respect to τ.

<u>Proof</u>. In the first part of the proof we shall show that if $0 \in \{x : f(x) > 0\}$ then there exists a set $P \in \mathcal{P}(0)$ such that $P \subset \text{int } \{x : f(x) > 0\} \cup \{0\}$.

By assumption, there exists a natural number p such that $f(0) > \frac{1}{p}$ and $0 \in \mathcal{C}(\{x : f(x) > \frac{1}{p}\})$. The set $\{x : f(x) > \frac{1}{p}\} \in \mathcal{B}$ and therefore we have $\{x : f(x) > \frac{1}{p}\} = F \triangle I_0$ where F is a closed set in the natural topology, $I_0 \in I$ and $0 \in \mathcal{C}(F)$. Therefore it is nearly obvious (by Lemma 1) that, for a natural number n, there exist a natural number k and a real number $\delta > 0$ such that for each $h \in (0, \delta)$ and for each $i \in \{1, \ldots, n\}$, there exists $j(h,i) \in \{1, \ldots, k\}$ such that

$$\left[\frac{(i-1)k + j(h,i) - 1}{n \cdot k} \cdot h, \frac{(i-1)k + j(h,i)}{n \cdot k} \cdot h\right] \subset F.$$

Now, we shall define the family of sets $\{P_m^{ij}\}$ where $m \in N$,

$i \in \{1,\dots,n\}$ and $j \in \{1,\dots,k\}$. For each natural number $i \in \{1,\dots,n\}$ we shall say that $h \in P_m^{ij}$ if and only if j is the above described natural number $j(h,i)$ and $m \in N$ such that

$$\left(\frac{(i-1)k+j-1}{(i-1)k+j}\right)^m \cdot \delta \leq h < \left(\frac{(i-1)k+j-1}{(i-1)k+j}\right)^{m-1} \cdot \delta .$$

Observe that the sets $P_m^{ij}, m \in N, i \in \{1,\dots,n\}, j \in \{1,\dots,k\}$ have the following properties:

(i) $\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{k} P_m^{ij} = (0,\delta)$ for all $i \in \{1,\dots,n\}$,

(ii) if $h_1, h_2 \in P_m^{ij}$ then

$$\left[\frac{(i-1)k+j-1}{n \cdot k} h_1, \frac{(j-1)k+j}{n \cdot k} h_1\right] \cap \left[\frac{(i-1)k+j-1}{n \cdot k} h_2, \frac{(i-1)k+j}{n \cdot k} h_2\right] \neq \emptyset,$$

(iii) if $P_m^{ij} \neq \emptyset$ and $a_m^{ij} = \inf P_m^{ij}$, $b_m^{ij} = \sup P_m^{ij}$ then

$$\mathrm{cl}\left(\bigcup_{h \in P_m^{ij}} \left[\frac{(i-1)k+j-1}{n \cdot k} h, \frac{(i-1)k+j}{n \cdot k} h\right]\right) = \left[\frac{(i-1)k+j-1}{n \cdot k} a_m^{ij}, \frac{(j-1)k+j}{n \cdot k} b_m^{ij}\right],$$

(iv) for each $x \in \left[\frac{(i-1)k+j-1}{n \cdot k} a_m^{ij}, \frac{(i-1)k+j}{n \cdot k} b_m^{ij}\right]$, $f(x) \geq \frac{1}{p}$ where a_m^{ij}, b_m^{ij} are described above.

The statements (i) and (ii) are obvious. To prove the next statements let $r=(i-1)k+j$.

Let $x \in \left(\frac{r-1}{n \cdot k} a_m^{ij}, \frac{r}{n \cdot k} b_m^{ij}\right)$ and $r \neq 1$. Then there exist $h', h'' \in P_m^{ij}$ such that $a_m^{ij} \leq h' \leq \min\left(\frac{n \cdot k}{r-1} x, b_m^{ij}\right)$ and $\max\left(h', \frac{n \cdot k}{r} x\right) \leq h'' \leq b_m^{ij}$. Therefore $x \in \left[\frac{r-1}{n \cdot k} h', \frac{r}{n \cdot k} h''\right] = \left[\frac{r-1}{n \cdot k} h', \frac{r}{n \cdot k} h'\right] \cup \left[\frac{r-1}{n \cdot k} h'', \frac{r}{n \cdot k} h''\right]$. If $r=1$ then there exists $h \in P_m^{ij}$ such that $\frac{nkx}{r} \leq h \leq b_m^{ij}$ and $x \in \left[0, \frac{r}{n \cdot k} h\right] = \left[\frac{r-1}{n \cdot k} h, \frac{r}{n \cdot k} h\right]$. Let $x = \frac{r-1}{n \cdot k} a_m^{ij}$. Then there exists a sequence $\{h_s\}_{s \in N} \subset P_m^{ij}$ such that $\lim_{s \to \infty} h_s = a_m^{ij}$ and for all $s \in N$, $h_s \geq a_m^{ij}$. Therefore, for each s,

$$\frac{r-1}{n \cdot k} h_s \in \bigcup_{h \in P_m^{ij}} \left[\frac{r-1}{n \cdot k} h, \frac{r}{n \cdot k} h\right] \text{ and}$$

$$\frac{r-1}{n \cdot k} a_m^{ij} \in \mathrm{cl}\left(\bigcup_{h \in P_m^{ij}} \left[\frac{r-1}{n \cdot k} h, \frac{r}{n \cdot k} h\right]\right).$$

In a similar way we prove that $\frac{r}{n \cdot k} b_m^{ij} \in \mathrm{cl}\left(\bigcup_{h \in P_m^{ij}} \left[\frac{r-1}{n \cdot k} h, \frac{r}{n \cdot k} h\right]\right)$.

Since it is obvious that $cl(\bigcup_{h\in P_m^{ij}} [\frac{r-1}{n\cdot k} h, \frac{r}{n\cdot k} h]) \subset [\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}]$ the proof of (iii) is completed.

To prove the statement (iv) we observe that for all $h\in P_m^{ij}$, $[\frac{r-1}{n\cdot k} h, \frac{r}{n\cdot k} h] \subset F$ and $[\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}] \subset F$. By the above observatiob we have that $[\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}] \setminus I_0 \subset F \setminus I_0 \subset \{x: f(x) > \frac{1}{p}\}$. Thus $\{x: f(x) \leqq \frac{1}{p}\} \cap [\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}] \in I$. We suppose that there exists $x_1 \in [\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}]$ such that $f(x_1) < \frac{1}{p}$. Then $x_1 \in \mathcal{C}(\{x: f(x) < \frac{1}{p}\})$ and therefore $\{x: f(x) < \frac{1}{p}\} \cap [\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}] \notin I$ which is impossible and for each $x \in [\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}]$, $f(x) \geqq \frac{1}{p}$. So we have proved that $[\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}] \subset \subset \{x: f(x) \geqq \frac{1}{p}\} \subset \{x: f(x) > 0\}$.

Let $c_m^{ij} = \frac{r-1}{n\cdot k} a_m^{ij} + \frac{1}{3n\cdot k} a_m^{ij}$ and $d_m^{ij} = \frac{r}{n\cdot k} b_m^{ij} - \frac{1}{3n\cdot k} b_m^{ij}$.
Then $[c_m^{ij}, d_m^{ij}] \subset (\frac{r-1}{n\cdot k} a_m^{ij}, \frac{r}{n\cdot k} b_m^{ij}) \subset \operatorname{int}\{x: f(x) > 0\}$ and for each $m, m' \in N$, $m \neq m'$, $|m-m'| \neq 1$, $[c_m^{ij}, d_m^{ij}] \cap [c_{m'}^{ij}, d_{m'}^{ij}] = \emptyset$. For each $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,k\}$ let $F_{ij} = \bigcup_{m=1}^{\infty} [c_m^{ij}, d_m^{ij}]$ and $P^+ = \bigcup_{m=1}^{\infty} ([\frac{1}{m+1}, \frac{1}{m}] \cap \bigcup_{n=1}^{m} \bigcup_{i=1}^{n} \bigcup_{j=1}^{k} F_{ij}) \cup \{0\}$.

We shall prove that 0 is a right-hand I-density point of P^+. Let $n \in N$. We choose k and $\delta > 0$ for the set F by Lemma 1. Let $\delta_1 = \min(\frac{1}{n}, \delta)$ and $k_0 = 3k$. Then for each $h \in (0, \delta_1)$, $i \in \{1,\dots,n\}$ and for $j = j(h,i)$ there exists $m \in N$ such that

$$\left(\frac{(i-1)k+j-1}{(i-1)k+j}\right)^m \cdot \delta \leqq h < \left(\frac{(i-1)k+j-1}{(i-1)k+j}\right)^{m-1} \cdot \delta .$$

Then $h \in P_m^{ij}$ and

$$\left[\frac{(i-1)k+j-1}{n\cdot k} \cdot h, \frac{(i-1)k+j}{n\cdot k} \cdot h\right] \subset \left[\frac{(i-1)k+j-1}{n\cdot k} a_m^{ij}, \frac{(i-1)k+j}{n\cdot k} b_m^{ij}\right]$$

where $a_m^{ij} = \inf P_m^{ij}$ and $b_m^{ij} = \sup P_m^{ij}$. Therefore

$$\frac{(i-1)k+j-1}{n\cdot k} a_m^{ij} < \frac{(i-1)3k+3j-2}{3n\cdot k} a_m^{ij} \leqq \frac{(i-1)3k+3j-2}{3n\cdot k} \cdot h$$

and

$$\frac{(i-1)k+j}{n\cdot k} b_m^{ij} > \frac{(i-1)3k+3j-1}{3n\cdot k} b_m^{ij} \geq \frac{(i-1)3k+3j-1}{3n\cdot k}\cdot h.$$

Thus

$$\left[\frac{(i-1)3k+3j-2}{3n\cdot k}\cdot h, \frac{(i-1)3k+3j-1}{3n\cdot k}\cdot h\right] \subset [c_m^{ij}, d_m^{ij}] \subset F_{ij}.$$

We have shown that for each natural n, there exist $\delta_1 > 0$ and $k_0=3k$ such that, for each $h \in (0, \delta_1)$ and for each $i \in \{1,\dots,n\}$, there exists $j \in \{1,\dots,k_0\}$ such that

$$\left[\frac{(i-1)k_0+j-1}{n\cdot k_0}\cdot h, \frac{(i-1)k_0+j}{n\cdot k_0}\cdot h\right] \subset P^+.$$

So 0 is a right-hand I-density point of P^+.

In a similar way we can find a set P^- such that 0 is a left-hand I-density point of P^-. Let $P=P^+\cup P^-$.

We shall show that there exists $P_1 \subset P$ such that $P_1 \in \mathcal{P}(0)$ and $P_1 \subset \operatorname{int}\{x:f(x)>0\}\cup\{0\}$. Since $F_{ij}=\bigcup_{m=1}^{\infty}[c_m^{ij}, d_m^{ij}]$ and for all m, $0<c_{m+1}^{ij}<d_{m+1}^{ij}<c_{m-1}^{ij}$, then for every natural number m the set $A=\bigcup_{m=1}^{m}\bigcup_{i=1}^{n}\bigcup_{j=1}^{k} F_{ij}\cap[\frac{1}{m+1}, \frac{1}{m}]$ includes a finite number of closed intervals. Therefore P^+ is a closed interval or there exists the natural number n_0 such that $P^+\cap[0,\frac{1}{n_0}]$ is a closed interval or $P^+=\bigcup_{s=1}^{\infty}[c_s^*, d_s^*]\cup\{0\}$ where $0<c_s^*<d_s^*<c_{s-1}^*$ for each $s\in N$. Since the set P^- satisfies a similar property we can define a set P_1 as follows:

1. if P^- and P^+ are closed intervals, we define $P_1=P$,
2. if $P^+=\bigcup_{s=1}^{\infty}[c_s^*, d_s^*]\cup\{0\}$ and $P^-=\bigcup_{s=1}^{\infty}[a_s^*, b_s^*]\cup\{0\}$ where for each $s\in N$, $b_{s-1}^*<a_s^*<b_s^*<0$ we define $P_1=P$,
3. if P^- is a closed interval and $P^+=\bigcup_{s=1}^{\infty}[c_s^*, d_s^*]\cup\{0\}$, we define $P_1=P^+\cup(P^-\cap P^*)$ where P^* is an arbitrary set belonging to $\mathcal{P}(0)$,
4. if P^+ is an interval and $P^-=\bigcup_{s=1}^{\infty}[a_s^*, d_s^*]\cup\{0\}$ we define $P_1=P^-\cup(P^+\cap P^{**})$ where P^{**} is an arbitrary set belonging to $\mathcal{P}(0)$.

Since $P_1\subset P$ therefore $P_1\subset \operatorname{int}\{x:f(x)>0\}\cup\{0\}$.

Let $x_0\in\{x:f(x)>a\}$ where $a\in R$. Then

$$0\in\{x:f(x+x_0)-a>0\}.$$

Since the function $h(x)=f(x+x_0)-a$ is I-approximately continuous at 0 then there exists a set $P \in \mathcal{P}(0)$ such that $P \subset \operatorname{int}\{x: f(x+x_0)-a>0\} \cup \{0\}$. It is obvious that the set $P+x_0 \in \mathcal{P}(x_0)$ and $P+x_0 \subset \operatorname{int}\{x: f(x)>a\} \cup \{x_0\}$. Therefore the proof of the theorem is completed.

Definition of $\Delta(M)$. Let M be a subset of R. Then $x \in \Delta(M)$, if and only if for every set $P \in \mathcal{P}(x)$, $\emptyset \neq P \cap M \neq \{x\}$.

Lemma 2. Let $M \subset R$. If $U \in \tau$ and $\Delta(M) \cap U \neq \emptyset$ then $\operatorname{int} U \cap M \neq \emptyset$.

Proof. Let $x_0 \in U \cap \Delta(M)$. Then there exists a set $P \in \mathcal{P}(x_0)$ such that $P \subset \operatorname{int} U \cup \{x_0\}$. Since $x_0 \in \Delta(M)$ therefore $\emptyset \neq P \cap M \neq \{x_0\}$ and $P \cap M \subset (\operatorname{int} U \cup \{x_0\}) \cap M = (\operatorname{int} U \cap M) \cup (\{x_0\} \cap M)$. Thus $\operatorname{int} U \cap M \neq \emptyset$.

Proposition 1. If M is a closed set in the natural topology, then $\Delta(M) \subset M$.

Proof follows from the fact that for each $x \in R$ the family $\mathcal{P}(x)$ includes all closed intervals such that x belongs to their interior.

Theorem 3. Let $X \subset R$. Then $\tau\text{-cl } X = X \cup \Delta(\text{cl } X) \subset \text{cl } X$. Moreover, x is a limit point of X in the τ-topology if and only if $x \in \Delta(\text{cl } X)$.

Proof. Let $x_0 \in \Delta(\text{cl } X)$ and $U \in \tau$ such that $x_0 \in U$. Then there exists a set $P \in \mathcal{P}(x_0)$ such that $P \subset \operatorname{int} U \cup \{x_0\}$. By the definition of $\Delta(\text{cl } X)$ we have $\emptyset \neq P \cap \text{cl } X \neq \{x_0\}$. Let $x_1 \in P \cap \text{cl } X \subset (\operatorname{int} U \cup \{x_0\}) \cap \text{cl } X$ and $x_1 \neq x_0$. Then there exists $x_2 \in \operatorname{int} U \cap X \subset U \cap X \neq \emptyset$ and $x_2 \neq x_0$. Thus $x_0 \in \tau\text{-cl } X$ and $X \cup \Delta(\text{cl } X) \subset \tau\text{-cl}(X)$. Now, we assume that $x_0 \notin X$ and $x_0 \notin \Delta(\text{cl } X)$. We have that there exists a set $P \in \mathcal{P}(x_0)$ such that $P \cap \text{cl } X = \emptyset$ or $P \cap \text{cl } X = \{x_0\}$. If $P \cap \text{cl } X = \emptyset$ then there exists an open set G (in the natural topology) such that $P \subset G$ and $G \cap \text{cl } X = \emptyset$. Therefore $G \cap X = \emptyset$. Since $x_0 \in P \subset G$ and $\mathcal{T} \subset \tau$ we have $x_0 \notin \tau\text{-cl } X$. Let $P \cap \text{cl } X = \{x_0\}$ and, for each $n \in N$, $S_{n+1} = \{x \in R: \frac{1}{n+1} \leq |x-x_0| \leq \frac{1}{n}\}$ and $S_1 = \{x \in R: |x-x_0| \geq 1\}$. Then for each $n \in N$, $S_n \cap P$ is the closed set in the natural topology and $S_n \cap P \cap \text{cl } X = \emptyset$. Let G_n, for each $n \in N$, be an open set such that $S_n \cap P \subset G_n$, $x_0 \notin G_n$ and $G_n \cap \text{cl } X = \emptyset$. Then for $U = \bigcup_n G_n \cup \{x_0\}$, $P \subset U$. Therefore $U \in \tau$ and $U \cap \text{cl } X = \{x_0\}$. Hence

$x_0 \notin X$ implies $U \cap X=\emptyset$. We have shown that $x_0 \notin \tau$-cl X and the first part of the theorem is proved.

If $x_0 \in R \setminus \Delta(\mathrm{cl}\ X)$ then $(R\setminus \mathrm{cl}\ X) \cup \{x_0\} \in \tau$ and $X \cap ((R \setminus \mathrm{cl}\ X) \cup \{x_0\}) \subset \{x_0\}$. Hence each point of $R \setminus \Delta(\mathrm{cl}\ X)$ is not a limit point of X in the τ-topology. Let $x_0 \in \Delta(\mathrm{cl}\ X)$. Then, by the first part of the proof, we know that x_0 is a limit point in the τ-topology.

Corollary. If $X \subset R$ then τ-cl X is a perfect set in the τ-topology if and only if $X \subset \Delta(\mathrm{cl}\ X)$.

Proof. If τ-cl X is perfect in the τ-topology, then by Theorem 3 we have

$$X \subset \tau\text{-cl } X= \Delta(\mathrm{cl}(\tau\text{-cl } X)) \subset \Delta(\mathrm{cl}(\mathrm{cl}\ X))= \Delta(\mathrm{cl}\ X).$$

If $X \subset \Delta(\mathrm{cl}\ X)$ then by Theorem 3 we have

$$\tau\text{-cl } X=X \cup \Delta(\mathrm{cl}\ X)= \Delta(\mathrm{cl}\ X) \subset \Delta(\mathrm{cl}(\tau\text{-cl } X)).$$

Since τ-cl X is a closed set in the τ-topology, the proof of the corollary is completed.

Let $Z_0= \{A \subset R: \Delta(\mathrm{cl}\ A)=\emptyset\}$.

Proposition 2. The family Z_0 is an ideal and $Z_0 \subsetneq I$.

Proposition 3. There exists a sequence $\{x_n\}_{n\in N}$ such that for all n, $0< x_{n+1} < x_n$, $\lim_{n\to\infty} x_n=0$ and $\{x_n\}_{n\in N} \cup \{0\} \notin Z_0$.

Proof. Let $W= \{w_1,w_2,\ldots\}$ be a set of all rational numbers from $(\frac{1}{2},1)$. For every natural n we define a sequence $\{z_p^n\}_{p\geq 1}$ such that for each $p \in N$, $z_p^n= \frac{1}{2^p} w_n$. Then we observe that for each $p \in N$ and for each $n \in N$, $\frac{1}{2^{p+1}} < z_p^n < \frac{1}{2^p}$. Let $A=\bigcup_{i=1}^{\infty} \{ z_p^i\}_{p\geq i}$. Since the set A is countable, we can define $\{x_n\}_{n\in N}$ such that $A= \{x_n\}_{n\in N}$. It is obvious that $\lim_{n\to\infty} x_n=0$.

Now, we suppose that $\{x_n\}_{n\in N} \cup \{0\} \in Z_0$. Then, by definition of Z_0, we have $\Delta(\{x_n\}_{n\in N} \cup \{0\})=\emptyset$. Therefore there exists a set $P \in \mathcal{P}(0)$ such that $P \cap (\{x_n\}_{n\in N} \cup \{0\})= \{0\}$. Let $\{t_{k_m}\}_{m\geq 1}$ be an arbitrary subsequence of the sequence $\{t_k\}_{k\geq 1}$ where, for each $k \in N$, $t_k=2^k$ and $G=R\setminus P$. We shall show that for each $m \in N$,

$\bigcup_{s=m}^{\infty}(t_{k_s}\cdot G\cap(\frac{1}{2},1))$ is a residual set in $(\frac{1}{2},1)$. Let $(a,b)\cap(\frac{1}{2},1)\neq\emptyset$. Then there exists $r\in N$ such that $w_r\in(a,b)\cap(\frac{1}{2},1)$. Therefore, for each $p\geq r$, $z_p^r=\frac{1}{2^p}w_r\in\{x_n\}_{n\in N}$. Let $s_0\geq m$ be a natural number such that $k_{s_0}\geq r$. Then $z_{k_{s_0}}^r=\frac{1}{2^{k_{s_0}}}w_r\in\{x_n\}_{n\in N}\subset G$ and $\frac{1}{2^{k_{s_0}+1}}<z_{k_{s_0}}^r<\frac{1}{2^{k_{s_0}}}$.

Thus $w_r=2^{k_{s_0}}z_{k_{s_0}}^r\in 2^{k_{s_0}}\cdot G\cap(\frac{1}{2},1)=t_{k_{s_0}}\cdot G\cap(\frac{1}{2},1)$ and

$(a,b)\cap\bigcup_{s=m}^{\infty}(t_{k_s}\cdot G\cap(\frac{1}{2},1))\neq\emptyset$. Since G is open, the set $\bigcap_{m=1}^{\infty}\bigcup_{s=m}^{\infty}(t_{k_s}\cdot G\cap(\frac{1}{2},1))$ is residual in $(\frac{1}{2},1)$. Then, by definition of I-dispersion point of an open set we know that 0 is not I-dispersion point of the set G. Therefore 0 is not I-density point of the set P, which is a contradiction.

<u>Theoren 4</u>. $\tau=\{U\in\mathcal{T}_I:U=G\cup M$ where $G\in\mathcal{T}$, $M\cap\Delta(R\setminus G)=\emptyset)$.

<u>Proof.</u> Let $U\in\mathcal{T}_I$ and $U=G\cup M$ where $G\in\mathcal{T}$ and $M\cap\Delta(R\setminus G)=\emptyset$. We suppose that there exists $x_0\in M$ such that $x_0\notin\tau$-int U. Then, for each $P\in\mathcal{P}(x_0)$, $P\not\subset$ int $U\cup\{x_0\}$. Therefore $P\not\subset G\cup\{x_0\}$. Thus $\emptyset\neq P\cap(R\setminus G)\neq\{x_0\}$ and $x_0\in\Delta(R\setminus G)$, which is a contradiction.

Now, let $U\in\tau$, G=int U and $M=U\setminus$ int U. We suppose that there exists $x_0\in M\cap\Delta(R\setminus G)$. Since $x_0\in\tau$-int U then there exists $P\in\mathcal{P}(x_0)$ such that $P\subset$ int $U\cup\{x_0\}$. Therefore $P\cap(R\setminus G)=\{x_0\}$, which is a contradiction.

<u>Theorem 5</u>. τ is a completely regular Hausdorff topology on R.

<u>Proof</u>. Since $\mathcal{T}\subset\tau$, τ is a Hausdorff topology. Let F be a closed set in the τ-topology and $x_0\notin F$. Since $R\setminus F\in\tau$ then there exists the set $P\in\mathcal{P}(x_0)$ such that $P\subset$ int$(R\setminus F)\cup\{x_0\}$. Let G=int$(R\setminus F)$ and

$$f(x)=\begin{cases}1 & x=x_0,\\ \dfrac{d(x,R\setminus G)}{d(x,R\setminus G)+d(x,P)} & x\neq x_0\end{cases}$$

where d(x,A) is the distance from x to the set A. It is easily seen that f is continuous at each $x \neq x_0$ and I-approximately continuous at $x=x_0$. Also $f(x_0)=1$ and $f(x)=0$ for all $x \in F$. Therefore the proof of the theorem is completed.

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