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A RESULT ABOUT IMBEDDED EIGENVALUES IN THE OPERATOR VALUED FRIEDRICHS MODEL

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Abstract. It is shown that the operator (1) (describing the operator valued Friedrichs model) has only a finite number of eigenvalues belonging to the continuous spectrum.

Key words: Fredholm theory, analytic functions, operator valued Friedrichs model.

Classification: 45B05, 81C10

This paper is a continuation of the paper [1]. We use some of the notations and results of [1].

Let H be a self-adjoint operator acting on the Hilbert space $L_2([a,b],\mathcal{H})$ according to the following formula:

$$(1)\quad (H(f)(x)=u(x)f(x)+\int_a^b K(x,y)f(y)dy,\ f\in L_2([a,b],\mathcal{H})$$

Here, $\mathcal{H}$ is an n-dimensional complex Hilbert space and the matrices

$$u(x)=\begin{pmatrix} u_1(x) & 0\ldots 0 & 0\\ 0 & u_2(x)\ldots 0 & 0\\ \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots\\ 0 & 0\ldots u_{n-1}(x) & 0\\ 0 & 0\ldots 0 & u_n(x)\end{pmatrix}$$

and

$$K(x,y)=K(y,x)=\begin{pmatrix} K_{11}(x,y)\ldots K_{1n}(x,y)\\ \ldots\ldots\ldots\\ \ldots\ldots\ldots\\ K_{n1}(x,y)\ldots K_{nn}(x,y)\end{pmatrix}$$

are self-adjoint. We shall suppose that $u_j(x)$ and $K_{js}(x,y)$ =

$= K_{sj}(x,y)$, $j,s = 1,2,\dots,n$ are real-analytic functions on $[a,b]$ and $[a,b]\times[a,b]$, respectively.

The main result is the following.

Theorem 1. The operator (1) has only a finite number of eigenvalues belonging to the continuous spectrum.

The rest of the paper is devoted to the proof of this theorem. Denote by M the union of n disjoint copies of the segment $[a,b]$, i.e.

$$M = \bigcup_{j=1}^{n} [a,b]_j, \quad [a,b]_j = [a,b], \quad j = 1,2,\dots,n.$$

Define a measure on M such that its restriction to each $[a,b]_j = [a,b]$, $j = 1,2,\dots,n$ coincides with the Lebesgue measure. We define the function $u(\lambda)$ on M as

$$\hat{u}(\lambda) = u_j(x), \quad \lambda = x \in [a,b]_j, \quad j = 1,2,\dots,n$$

and also the function (kernel)

$$\hat{K}(\lambda,\mu) = K_{js}(x,y), \quad \lambda = x \in [a,b]_j, \quad \mu = y \in [a,b]_s, \quad j,s=1,2,\dots,n.$$

Proposition 1. The operator H is unitarily equivalent to some operator $\hat{H}$, acting on $L_2(M, \mathbb{C}^1)$ according to the formula

$$(\hat{H}f)(\lambda)=\hat{u}(\lambda)f(\lambda)+ \int_M \hat{K}(\lambda,\mu)f(\mu)d\mu, \quad f \in L_2(M, \mathbb{C}^1)$$

Here $L_2(M, \mathbb{C}^1)$ is the Hilbert space of all square integrable complex functions defined on M.

Proof. It is clear that the operator (1) is unitarily equivalent to the operator

$$H\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} =$$

$$= \begin{pmatrix} u_1(x)f_1(x)+ \int_a^b K_{11}(x,y)f_1(y)dy+\dots+ \int_a^b K_{1n}(x,y)f_n(y)dy \\ \vdots \\ u_n(x)f_n(x)+ \int_a^b K_{n1}(x,y)f_{11}(y)dy+\dots+ \int_a^b K_{nn}(x,y)f_n(y)dy \end{pmatrix}$$

acting on $L_2([a,b], \mathbb{C}^n)$, where $\mathbb{C}^n = \underbrace{\mathbb{C}^1 \times \dots \times \mathbb{C}^1}_{n}$.

The mapping

$$W: L_2([a,b], \mathbb{C}^n) \longrightarrow L_2(M, \mathbb{C}^1)$$

defined by the formula

$$W: (f_1(x), \dots, f_n(x)) \longrightarrow \hat{f}(\lambda),$$

where $\hat{f}(\lambda) = f(x)$, for $\lambda = x \in [a,b]_j$, $j = 1,2,\dots,n$ has a bounded inverse W^{-1}, defined on $L_2(M, \mathbb{C}^1)$ by

$$W^{-1}: \hat{f}(\lambda) \longrightarrow (f_1(x), \dots, f_n(x)),$$

where $f_j(x)$ is the restriction of $\hat{f}(\lambda)$, $\lambda \in M$ on the segment $[a,b]_j$, $j = 1,2,\dots,n$.

Obviously, W is a unitary operator and $WH = \hat{H}W$.

<u>Theorem 2</u>. The resolvent $R_z(H)$ of H exists. It can be expressed by the formula

$$(R_z f)(x) = [u(x)-zE]^{-1} f(x) - [u(x)-zE]^{-1} \int_a^b \frac{\mathfrak{D}(x,y;z)}{\Delta(z)} f(y)\,dy$$

for all $z \in \mathbb{C}^1$, $\operatorname{Im} z \neq 0$. Here

$$\mathfrak{D}(x,y;z) = \begin{pmatrix} \mathfrak{D}_{11}(x,y;z) \dots \mathfrak{D}_{1n}(x,y;z) \\ \vdots \\ \mathfrak{D}_{n1}(x,y;z) \dots \mathfrak{D}_{nn}(x,y;z) \end{pmatrix},$$

$$(2)\quad \mathfrak{D}_{js}(x,y;z) = \frac{K_{js}(x,y)z)}{u_s(y)-z} + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} d_\nu^{(js)}(x,y;z),$$

$$d_\nu^{(js)}(x,y;z) =$$

$$= \sum_{j_1, j_2, \dots, j_\nu = 1}^{n} \int_a^b \dots \int_a^b \begin{vmatrix} K_{js}(x,y) K_{jj_1}(x,t_1) \dots K_{jj_\nu}(x,t_\nu) \\ K_{j_1 s}(t_1,y) K_{j_1 j_1}(x_1,t_1) \dots K_{j_1 j_\nu}(t_1,t_\nu) \\ \vdots \\ K_{j_\nu s}(t_\nu,y) K_{j_\nu j_1}(t_\nu,t_1) \dots K_{j_\nu j_\nu}(t_\nu,t_\nu) \end{vmatrix} \times$$

$$\times \frac{dt_1\, dt_2 \dots dt_\nu}{(u_s(y)-z)(u_{j_1}(t_1)-z) \dots (u_{j_\nu}(t_\nu)-z)}$$

$$(3)\quad \Delta(z) = 1 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} d_\nu(z)$$

$$d_\nu(z) = \sum_{j_1, j_2, \ldots, j_\nu = 1}^{n} \int_a^b \cdots \int_a^b \begin{vmatrix} K_{j_1 j_1}(t_1,t_1) \ldots K_{j_1 j_\nu}(t_1,t_\nu) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ K_{j_\nu j_1}(t_\nu,t_1) \ldots K_{j_\nu j_\nu}(t_\nu,t_\nu) \end{vmatrix} \times$$

$$\times \frac{dt_1\, dt_2 \ldots dt_\nu}{(u_{j_1}(t_1)-z)\ldots(u_{j_\nu}(t_\nu)-z)} .$$

A more general version of this theorem is proved in [1].

From Proposition 1 of [1]and from Lemma 1 it follows that the continuous spectra of H and $\hat{H}$ are the same. They equal to

$$\Sigma_{cont}(H) = \Sigma_{cont}(\hat{H}) = \bigcup_{j=1}^{n} [\min u_j(x), \max u_j(x)] .$$

Let $Q_j \subset \mathbb{C}^1$ be some complex neighborhood of the segment $[a,b]_j = [a,b]$, $j = 1,2,\ldots,n$ and $\Omega = \bigcup_{j=1}^{n} Q_j$; i.e. let Ω be some neighborhood of the set M. We define $C(\Omega)$ as the collection of all holomorphic ("regular") functions on Ω which are continuous on $\bar{\Omega}$. Taking the norm $\|\varphi\| = \max_{\lambda \in \bar{\Omega}} |\varphi(\lambda)|$, $C(\Omega)$ is a Banach space.

For any $z \in \mathbb{C}^1 \setminus \Sigma_{cont}(H)$ we define the operator $\hat{K}(z)$ according to the formula

$$[\hat{K}(z)\varphi](\lambda) = \int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-z} \varphi(\mu) d\mu, \quad \varphi \in C(\Omega).$$

Obviously, the operator $\hat{K}(z)$, $z \in \mathbb{C}^1 \setminus \Sigma_{cont}(H)$ acts in the space $C(\Omega)$.

Definition. A point $A' \in \Sigma_{cont}(H)$ is called a singular point of the continuous spectrum of an operator H if it is a value of some function $u_j(x)$, $j = 1,2,\ldots,n$ in some of its critical point.

Let Γ be the set of all singular points of the continuous spectrum of H, and $u_j^{-1}(A') \subset [a,b]$ be the pre-image of A' with respect to the mapping u_j.

Lemma 1. For any $A' \in \Sigma_{cont}(H) \setminus \Gamma$ and $\varphi \in C(\Omega)$ there exists a limit

$$[\hat{K}(A'\pm i0)\varphi](\lambda) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} [\hat{K}(A'\pm i\varepsilon)\varphi](\lambda)$$

which determines some operator $\hat{K}(A'\pm i0)$.

Proof. Let us denote by $u_j(\zeta)$ and $K_{j_1j_2}(\zeta_1,\zeta_2)$ the analytic continuations of $u_j(x)$ and $K_{j_1j_2}(x_1,x_2)$ into $Q\subset\mathbb{C}^1$ and $Q\times Q\subset\mathbb{C}^2$, respectively. Let $A'\in\Sigma_{cont}(H)\setminus\Gamma$ and let $u_j^{-1}(A') = \{x_{j1},x_{j2},\dots,x_{jP_j}\}$. It follows from the definition of Γ that $u_j'(x_{j\nu})\neq 0$ for any $\nu = 1,2,\dots,P_j$. Because $u_j(\zeta)$ is regular in $x = x_{j\nu}$, $\nu = 1,2,\dots,P_j$, there are some $\varepsilon>0$ and $\sigma>0$ (in the following we shall assume that these numbers are sufficiently small) such that for each $z\in V_\varepsilon(A') = \{z\in\mathbb{C}^1 : |z-A'|<\varepsilon\}$ the equation $u_j(\zeta) - z = 0$ has a unique solution in the disc $V_\sigma(x_{j\nu})$. This solution is regular in $V_\varepsilon(A')$ and can be expanded into the series

$$\zeta = \psi_{j\nu}(z) = x_{j\nu} + C_{j1}^{\nu}(z-A') + C_{j2}^{\nu}(z-A')^2 + \dots$$

where

$$C_{j1}^{\nu} = \frac{1}{u_j'(x_{j\nu})}$$

From (5) and from the smallness of $\varepsilon>0$ and $\sigma>0$ it follows that $\zeta = \psi_{j\nu}(z)\in\{\zeta\in\mathbb{C}^1 : |\zeta - x_{j\nu}|<\sigma,\ \mathrm{Im}\,\zeta>0\}$ for any $z\in V_\varepsilon(A')$, $\mathrm{Im}\,z>0$ and

$$u_j^{-1}(V_\varepsilon(A'))\subset\bigcup_{\nu=1}^{P_j} V_\sigma(x_{j\nu}).$$

Using this and also the "residuum theorem" we can represent the function $[\hat{K}(z)]\varphi(\lambda)$, $\mathrm{Im}\,z>0$ in the following way:

$$[\hat{K}(z)\varphi](\lambda) = \int_M \frac{\hat{K}(\lambda,\mu)}{u(\mu)-z}\varphi(\mu)d\mu =$$

$$= \sum_{j=1}^{n}\int_a^b \frac{\hat{K}_j(\lambda,\zeta)}{\hat{u}_j(\zeta)-z}\varphi_j(\zeta)d\zeta = \sum_{j=1}^{n}\sum_{\nu=1}^{P_j}\frac{\hat{K}_j(\lambda,\psi_{j\nu}(z))}{u_j'(\psi_{j\nu}(z))}\times$$

$$\times\varphi_j(\psi_{j\nu}(z)) + \sum_{j=1}^{n}\int_{\Gamma_\sigma}\frac{\hat{K}_j(\lambda,\zeta)}{u_j(\zeta)-z}\varphi_j(\zeta)d\zeta .$$

Here

$\hat{K}_j(\lambda,\zeta) = K_{sj}(x,\zeta)$, $\lambda = x\in[a,b]_s$, $\zeta\in[a,b]_j$,

$\varphi_j(x) = \varphi(\lambda)$, $x = \lambda \in [a,b]_j$, $j,s = 1,2,\ldots,n$,

and Γ_δ is the contour, coinciding with $[a,b]$ outside of all intervals

$$(x_{j1} - \delta,\ x_{j1} + \delta), \ldots, (x_{jP_j} - \delta,\ x_{jP_j} + \delta)$$

and containing all the half-circles

$\{\xi \in \mathbf{C}^1 : |\xi - x_{j\nu}| = \delta,\ \operatorname{Im} z \geq 0\}$.

Since $\xi \in \Gamma_\delta$, we conclude that $u_j(\xi) \not\in V_\varepsilon(A')$. Therefore, the function $\int_{\Gamma_\delta} \frac{\hat{K}_j(\lambda,\xi)}{u_j(\xi)-z}\, d\xi$

is regular in $V_\varepsilon(A')$. Putting $z = A' + i\varepsilon$ and taking the limit $\varepsilon \to 0$ we obtain

$$[\hat{K}(A' + i0)\varphi](\lambda) = \lim_{\varepsilon \to 0}[\hat{K}(A' + i\varepsilon)\varphi](\lambda) =$$

$$= \lim_{\varepsilon \to 0} \sum_{j=1}^{m} \sum_{\nu=1}^{P_j} \frac{\hat{K}_j(\lambda, \psi_{j\nu}(A' + i\varepsilon))}{u_j'(\psi_{j\nu}(A' + i\varepsilon))} \varphi_j(\psi(A' + i\varepsilon)) +$$

$$+ \lim_{\varepsilon \to 0} \sum_{j=1}^{m} \int_{\Gamma_\delta} \frac{\hat{K}_j(\lambda,\xi)}{u_j(\xi) - A' - i\varepsilon}\, d\xi =$$

$$= \sum_{j=1}^{m} \sum_{\nu=1}^{P_j} \frac{\hat{K}_j(\lambda, x_{j\nu})}{u_j'(x_{j\nu})} \varphi_j(x_{j\nu}) + \sum_{j=1}^{m} \int_{\Gamma_\delta} \frac{\hat{K}_j(\lambda,\xi)}{u_j(\xi) - A' - i0}\, d\xi\,.$$

Because the right hand side of this relation does not depend on $\delta > 0$, we can take the limit $\delta \to 0$:

$$[\hat{K}(A' + i0)\varphi](\lambda) = \sum_{j=1}^{m} \sum_{\nu=1}^{P_j} \frac{\hat{K}_j(\lambda, x_{j\nu})}{u_j'(x_{j\nu})} \varphi_j(x_{j\nu}) +$$

$$+ \int_a^b \frac{\hat{K}_j(\lambda,y)}{u_j(y) - A' - i0} \varphi_j(y)\,dy\,.$$

This integral can be understood in the sense of the principal (Cauchy) value.

It is obvious that $[\hat{K}(A'+i0)\varphi](\lambda) \in C(\Omega)$ for $\varphi(\lambda) \in C(\Omega)$.

<u>Lemma 2</u>. Let $\varphi \in C(\Omega)$ be a solution of the homogeneous equation

(6) $$\varphi(\lambda) + \hat{K}(z)\varphi(\lambda) = 0$$

for $z = A' + i0$ or $z = A' - i0$, where $A' \in \Sigma_{cont}(H) \setminus \Gamma$. Then the following relation holds:

$$\varphi(\omega)\Big|_{\omega \in \bigcup_{j=1}^{n} u_j^{-1}(A')} = 0 .$$

We call (see [2]) a point $A' \in \Sigma_{cont}(H) \setminus \Gamma$ a singular point of the operator $\hat{K}(z)$, if the equation (6) has a nonzero solution from $C(\Omega)$.

<u>Lemma 3</u>. A point $A' \in \Sigma_{cont}(H) \setminus \Gamma$ is a singular point of the operator $K(z)$ iff it belongs to the discrete spectrum of an operator $\hat{H}$.

These lemmas can be proved in the same way as Lemmas 3.7 and 3.8 of [2].

It follows from (2) that the continuous spectrum of the operator H consists of a finite number of nonintersecting segments

$[A_1,B_1]$, $[A_2,B_2], \dots, [A_m,B_m]$, $m \le n$.

Let $A_j = A_{j1} < A_{j2} < \dots < A_{jm_j} = B_j$ be singular points of continuous spectrum, belonging to the segments $[A_j,B_j]$, $j = 1,2,\dots,m$ and let $V_\varepsilon(A_{js},A_{js+1}) \subset \mathbb{C}^1$ be a complex ε-neighborhood of segments $[A_{js},A_{js+1}]$, $s = 1,2,\dots,m_j-1$.

Denote by $V_\varepsilon^+(A_{js},A_{js+1})$ the set

$$\{z \in \mathbb{C}^1 : \mathrm{Im}\ z \ge 0\} \cap \{V_\varepsilon(A_{js},A_{js+1}) \setminus \setminus ([A_{js} - \varepsilon, A_{js}] \cup [A_{js+1}, A_{js+1} + \varepsilon])\}$$

and denote by $V_\varepsilon^-(A_{js},A_{js+1})$ another set

$$\{z \in \mathbb{C}^1 : \mathrm{Im}\ z \le 0\} \cap \{V_\varepsilon(A_{js},A_{js+1}) \setminus \setminus ([A_{js} - \varepsilon, A_{js}] \cup [A_{js+1}, A_{js+1} + \varepsilon])\}.$$

<u>Lemma 4</u>. The restriction $\Delta(z)/_{\mathbb{C}^1_+}$ (resp. $\Delta(z)/_{\mathbb{C}^1_-}$) of the function $\Delta(z)$ which is defined by (3), on the upper half-plane $\mathbb{C}^1_+$ (resp. lower half-plane $\mathbb{C}^1_-$) has an analytic continuation in

$V_{\varepsilon}^{+}(A_{js},A_{js+1})$ (resp. $V_{\varepsilon}^{-}(A_{js},A_{js+1})$) across the interval (A_{js},A_{js+1}). This continuation $\Delta_{js}^{+}(z)$ (resp. $\Delta_{js}^{-}(z)$) is a regular function in the region $\mathbb{C}_{+}^{1} \cup V_{\varepsilon}^{1}(A_{js},A_{js+1})$ (resp. $\mathbb{C}_{-}^{1} \cup V_{\varepsilon}^{-}(A_{js},A_{js+1})$).

The proof of this lemma follows from the principle of an analytical continuation and from the following two lemmas which are proved in [1].

<u>Lemma 5</u>. Let $A' \in \Gamma$ and $u_j^{-1}(A') = \{x_{j1},x_{j2},\dots,x_{jP_j}\}$, $j = 1,2,\dots,n$. Then there is an ε-neighborhood $V_{\varepsilon}'(A') = \{z \in \mathbb{C}^1 : 0 < |z - A'| < \varepsilon\}$ of $z = A'$ such that the restriction $\Delta(z)/_{\mathbb{C}_{+}^{1}}$ of the function has an analytic continuation onto the $V_{\varepsilon}'(A')$. This analytic continuation $\Delta^{*}(z)$ is a multivalued function with the branching point $z = A'$ and can be in $V_{\varepsilon}'(A')$ expanded into the series

$$\Delta^{*}(z) = \sum_{s=-\hat{q}}^{\infty} F_{A',s}(K)(z - A')^{s/P}.$$

Here

$$\hat{q} = P \sum_{j=1}^{n} \sum_{s=1}^{P_j} \frac{R_{js}-1}{R_{js}}$$

and $R_{js} - 1 = R(x_{js}) - 1$ denotes the multiplicity of the root $x = x_{js}$ of the function $u_j'(x)$, $j = 1,2,\dots,n$ is the lowest common multiple of the numbers

$$\{R_{11},\dots,R_{1P_1},\dots,R_{n1},\dots,R_{nP_n}\}.$$

<u>Lemma 6</u>. Let $A' \in \Sigma_{cont}(H) \setminus \Gamma$. Then there is an ε-neighborhood $V_{\varepsilon}(A')$ of $z = A'$ such that the restriction $\Delta(z)/_{\mathbb{C}_{+}^{1}}$ of the $\Delta(z)$ has an analytic continuation onto $V_{\varepsilon}(A')$. This analytic continuation $\Delta^{*}(z)$ is regular in $V_{\varepsilon}(A')$.

For any $g \in C(\Omega)$ denote by

$$\mathcal{G}(\lambda,z;g) = \int_{\Omega} \mathcal{L}(\lambda,\mu;z)g(\mu)d\mu$$

Here

$\hat{\mathfrak{D}}(\lambda,\mu;z) = \mathfrak{D}_{js}(x,y;z)$, $\lambda = x \in [a,b]_j$, $\mu = y \in [a,b]_s$, where $\mathfrak{D}_{js}(x,y;z)$, $j,s = 1,2,\dots,n$ is defined by (2). It follows from the definition of $\hat{\mathfrak{D}}(\lambda,\mu;z)$ that for any $\lambda \in M$ and $g \in C(\Omega)$ the function $\hat{\mathfrak{D}}(\lambda,z;g)$ is regular in $\mathbb{C}^1 \setminus \Sigma_{cont}(H)$.

Lemma 7. The function $\hat{\mathfrak{D}}(\lambda,z;g)$, which is regular in $\mathbb{C}^1_+$ (resp. $\mathbb{C}^1_-$), has an analytical continuation in $V^+_\varepsilon(A_{js},A_{js+1})$ (resp. $V^-_\varepsilon(A_{js},A_{js+1})$) across the interval (A_{js},A_{js+1}) whenever $\lambda \in M$, $g \in C(\Omega)$ and $j = 1,2,\dots,m$, $s = 1,2,\dots,m_j$. This analytic continuation $\mathfrak{D}^+_{js}(\lambda,z;g)$ ($\mathfrak{D}^-_{js}(\lambda,z;g)$) is regular in the region $\mathbb{C}^1_+ \cup V^+_\varepsilon(A_{js},A_{js+1})$ (resp. $\mathbb{C}^1_- \cup V^-_\varepsilon(A_{js},A_{js+1})$).

The proof of this lemma is analogous to the proof of Lemma 4, and therefore will be omitted.

Theorem 2. Let $\omega \in (A_{js},A_{js+1})$ be an eigenvalue of an operator H. Then $\Delta^+_{js}(\omega) = 0$ and $\Delta^-_{js}(\omega) = 0$, whenever $j = 1,2,\dots,m$, $s = 1,2,\dots,m_j$.

Proof. From Lemma 3 it follows that $\omega \in (A_{js},A_{js+1})$ is a singular point of the operator $\hat{K}(z)$ i.e. for $z = \omega + i0$ and for $z = \omega - i0$ the equation (6) has a nonzero solution $\varphi \in C(\Omega)$. What is needed, is the proof of the fact that $\Delta^+_{js}(\omega) = 0$ and $\Delta^-_{js}(\omega) = 0$. We will show that $\Delta^+_{js}(\omega) = 0$. The relation $\Delta^-_{js}(\omega) = 0$ is proved in an analogous way.

Suppose, on the contrary, that $\Delta^+_{js}(\omega) \neq 0$. Then, as it will be shown below, the nonhomogeneous equation

$$(7)\qquad \varphi(\lambda) + (\hat{K}(\omega + i0)\varphi)(\lambda) = g(\lambda),$$

has, for any $g \in C(\Omega)$, a unique solution which is of the type

$$(8)\qquad \varphi(\lambda) = g(\lambda) - \frac{\mathfrak{D}^+_{js}(\lambda,\omega;g)}{\Delta^+_{js}(\omega)}$$

where $\mathfrak{D}^+_{js}(\lambda,\omega;g)$ is defined in Lemma 7. But then the homogeneous equation (6) has only a trivial solution at $z = \omega + i0$. First we will show that the function defined by the formula (8) is a solution of the equation (7). Substituting (8) into (7) and collecting all the terms on the left hand side, we obtain

$$g(\lambda)-R^{+}_{js}(\lambda,\omega;g)-g(\lambda)+\int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-\omega-i0}\,[g(\mu)-R^{+}_{js}(\mu,\omega;g)]\,d\mu = 0$$

or

$$(9)\qquad -R^{+}_{js}(\lambda,\omega;g)+$$

$$+\int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-\omega-i0}\,g(\mu)d\mu-\int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-\omega-i0}\,R^{+}_{js}(\mu,\omega;g)d\mu = 0$$

where we denote by

$$(10)\qquad R^{+}_{js}(\lambda,z;\,g) = \frac{\mathcal{D}^{+}_{js}(\lambda,z;g)}{\Delta^{+}_{js}(z)},\ j = 1,2,\dots,m,\ s = 1,2,\dots,m_j.$$

We will show that (9) really takes place. To this end we will consider the equation ("first fundamental Fredholm relation")

$$(11)\qquad \hat{R}(\lambda,\mu;z) = \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-z}+\int_M \frac{\hat{K}(\lambda,\mu')}{\hat{u}(\mu')-z}\,\hat{R}(\mu',\mu;z)d\mu',$$

where

$$\hat{R}(\lambda,\mu;z) = \frac{\hat{\mathcal{D}}(\lambda,\mu;z)}{\Delta(z)},\ \operatorname{Im} z>0$$

We multiply both parts of the equation (11) by $g\in C(\Omega)$ and integrate over μ. We obtain

$$\int_M \hat{R}(\lambda,\mu;z)g(\mu)d\mu = \int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-z}\,g(\mu)d\mu +$$

$$+\int_M \Big[\int_M \frac{\hat{K}(\lambda,\mu')}{\hat{u}(\mu')-z}\,\hat{R}(\mu',\mu;z)d\mu'\Big]\,g(\mu)d\mu .$$

Using the formula about the integration by parts in the last integral and taking $z \longrightarrow \omega$, we obtain (9), q.e.d.

Now let us show that any solution of (7) has a form (8) for $z = \omega + i0$. Let $\varphi \in C(\Omega)$ be some solution of (7). Consider the equation

$$\varphi(\lambda) = g(\lambda) - \int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-z}\,\varphi(\mu)d\mu,\ \operatorname{Im} z>0$$

Multiplying both parts of this equation by $R(\mu',\mu;z)$ and integrating over μ', we get

$$(12)\quad \int_M \hat{R}(\mu',\lambda;z)\,\varphi(\lambda)d\lambda = \int_M \hat{R}(\mu',\lambda;z)g(\lambda)d\lambda - $$
$$-\int_M \left[\int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-z}\varphi(\mu)d\mu\right]\hat{R}(\mu',\lambda;z)d\lambda .$$

Because of the relation ("the second fundamental relation of Fredholm")

$$\hat{R}(\mu',\mu;z) = \frac{\hat{K}(\mu',\mu;z)}{\hat{u}(\mu)-z} - \int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-z}\hat{R}(\mu',\lambda;z)d\lambda\ ,\quad \mathrm{Im}\ z>0 .$$

We conclude from (12)

$$\int_M \hat{R}(\mu',\lambda;z)\,\varphi(\lambda)d\lambda = \int_M \hat{R}(\mu',\lambda;z)g(\lambda)d\lambda +$$
$$+\int_M \hat{R}(\mu',\mu;z)\,\varphi(\mu)d\mu - \int_M \frac{\hat{K}(\mu',\mu)}{\hat{u}(\mu)-z}\varphi(\mu)d\mu$$

or

$$\int_M \hat{R}(\mu',\lambda;z)g(\lambda)d\lambda - \int_M \frac{\hat{K}(\mu',\mu)}{\hat{u}(\mu)-z}\varphi(\mu)d\mu = 0.$$

Taking the limit $z \longrightarrow \omega + i0$ we get

$$R^{+}_{js}(\lambda,\mu;z) = \int_M \frac{\hat{K}(\lambda,\mu)}{\hat{u}(\mu)-\omega-i0}\ \varphi(\mu)d\mu .$$

Because $\varphi(\lambda)$ is a solution of (7), we conclude that

$$\varphi(\lambda) = g(\lambda) - R^{+}_{js}(\lambda,\omega;g).$$

The theorem is proved.

Proof of Theorem 1. Because of Theorem 2, it suffices to show that for any $j = 1,2,\dots,m$ and $s = 1,2,\dots,m_j-1$ the function $\Delta_{js}(z)$ has only a finite number of zeros in the interval (A_{js},A_{js+1}). The function $\Delta^{+}_{js}(z)$ is regular in $\mathbb{C}^1_+ \cup V^+_\varepsilon(A_{js}.A_{js+1})$ (see Lemma 4), therefore, for any $\varepsilon > 0$, it has only a finite number of zeros in $(A_{js}+\varepsilon,\ A_{js+1}-\varepsilon)$.

We will show that the zeros of $\Delta^{+}_{js}(z)$ cannot converge to A_{js} and A_{js+1}. It follows from Lemma 4 that in the region $V'_\varepsilon(A_{js}) \setminus (A_{js}-\varepsilon, A_{js})$ the function $\Delta^{+}_{js'}(z)$ can be expressed in the Puisseux series (see Lemma 4)

$$\Delta^{+}_{js}(z) = \sum_{\alpha=-\hat{q}}^{\infty} F_{A_{js},\alpha}(K)(z-A_{js})^{\alpha/p}, \quad z \in V'_{\varepsilon}(A_{js}) \setminus (A_{js}-\varepsilon, A_{js}).$$

Here, $F_{A_{js},\alpha}(K) \neq 0$ for some $\alpha = -\hat{q}, -\hat{q}+1, \dots$. In the opposite case, from the uniqueness theorem, $\Delta(z) \equiv 0$. Now let

$F_{A_{js}, -\hat{q}}(K) = 0, \dots, F_{A_{js},\alpha_0-1}(K) = 0$, and $F_{A_{js},\alpha_0}(K) \neq 0$. Then the equation $\Delta^{+}_{js}(z) = 0$ is equivalent to

$$F_{A_{js},\alpha_0}(K) + F_{A_{js},\alpha_0+1}(K)\,(z-A_{js})^{1/p} + \dots = 0.$$

It is easy to deduce from this relation that the zeros of the function cannot converge to A_{js}.

In the same way it can be proved that A_{js+1} is not a limit point of zeros of the function $\Delta^{+}_{js}(z)$.

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