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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
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EQUATIONAL THEORIES OF SOME ALMOST UNARY GROUPOIDS

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Abstract: A finite basis is found for the identities of a finite unary groupoid whose multiplication is changed so that one of its elements becomes a zero.

Key words: Term, equation, groupoid.

Classification: 08B05

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1. Introduction. For every $n \geq 3$ let us denote by A_n the groupoid with the underlying set $\{0,1,\dots,n-1\}$ and the binary operation ab defined as follows: if $b=0$ then $ab=0$; if $b \neq 0$ then $ab=\varphi(a)$ where $\varphi(0)=0$, $\varphi(1)=2$, $\varphi(2)=3,\dots,\varphi(n-2)=n-1$, $\varphi(n-1)=0$.

The aim of this paper is to find a finite basis for the identities of the groupoid A_n. More interesting perhaps than the result itself is the method used in its proof. A list of known universal algebras with finite bases for their identities is contained e.g. in W. Taylor's Appendix 4 in [1], which also contains an explanation of the fundamental notions and can be recommended to the reader as a summary of equational logic.

2. Terms and equations. Notation. The absolutely free groupoid over the countably infinite set of variables is denoted by W and its elements are called terms.

The set of variables occurring in a term t is denoted by var(t).

If $t_1,\dots,t_k$ is a finite sequence of terms then the term $(((t_1t_2)t_3)\dots)t_k$ is denoted by $[t_1,\dots,t_k]$ and the term $t_k(\dots(t_3(t_2t_1)))$ by $[t_k,\dots,t_1]^*$.

Let t, u be two terms. We write $t \leq u$ if there exists an endomorphism h of W such that h(t) is a subterm of u.

In order to be able to speak consistently about occurrences of subterms in a given term, we introduce the following definitions.

The free monoid over the set $\{1,2\}$ is denoted by E. Its operation is denoted multiplicatively and its unit element is denoted by $\emptyset$. If $e,f \in E$ and the word e is a beginning of the word f, we write $e \leq f$. Two elements e, f of E are incomparable if neither $e \leq f$ nor $f \leq e$.

Let t be a term. For every $e \in E$ define an element $t\langle e\rangle$ of $W \cup \{\emptyset\}$ by induction on the length of e as follows: $t\langle\emptyset\rangle=t$; if $t\langle e\rangle=pq$ for some terms p, q, put $t\langle e1\rangle=p$ and $t\langle e2\rangle=q$; if either $t\langle e\rangle$ is a variable or $t\langle e\rangle=\emptyset$, put $t\langle e1\rangle=t\langle e2\rangle=\emptyset$.

The set $\{t\langle e\rangle; e \in E\}$ is just the union of $\{\emptyset\}$ with the set of subterms of t. If $t\langle e\rangle=u$, we say that e is an occurrence of u in t.

Obviously, any two occurrences of variables in t are imcomparable.

Let t be a term, $k \geq 1$ and $e_1,\dots,e_k$ be pairwise incomparable occurrences of subterms $u_1,\dots,u_k$ in t. Let, moreover, $v_1,\dots,v_k$ be arbitrary terms. It is easy to see that there exists a unique term t' such that $t'\langle e_1\rangle=v_1,\dots,t'\langle e_k\rangle=v_k$ and $t'\langle e\rangle=t\langle e\rangle$ for any $e \in E$ incomparable with any of the occurrences $e_1,\dots,e_k$. This term t' is called the term obtained from t by replacing the occurrences $e_1,\dots,e_k$ of $u_1,\dots,u_k$ by $v_1,\dots,v_k$.

For any $e \in E$ denote by R(e) the largest number $i \geq 0$ such that e ends with 1^i.

If t is a term and $x \in var(t)$, denote by $R(x,t)$ the maximum of the numbers $R(e)$ where e ranges over all occurrences of x in t. In other words, $R(x,t)$ is the largest number i such that $[x,a_1,\dots,a_i]$ is a subterm of t for some terms $a_1,\dots,a_i$.

If t is a term then t can be written uniquely in the form $t=[x,a_1,\dots,a_i]$ for some variable x, some $i \geq 0$ and some terms $a_1,\dots,a_i$. The variable x (the first, the most left variable in t) will be denoted by $L(t)$ and the number i by $R(t)$. Of course, $R(t)=R(1^i)$ where 1^i is an occurrence of $L(t)$ in t and so $R(t) \leq$ $\leq R(L(t),t)$.

By an equation we mean an ordered pair of terms. An equation (u,v) will be often denoted by u=v in spite of the danger involved in it. If we write $u \underset{\alpha}{=} v$, we mean that the equation (u,v) is a consequence of some other equation denoted by (α).

An equation (u,v) is satisfied in A_n if $h(u)=h(v)$ for any homomorphism $h:W \longrightarrow A_n$. An equation which is satisfied in A_n is also called an identity of A_n. The set of all identities of A_n is the equational theory of A_n.

3. The equational theory of A_n. A description of the equational theory of A_n is given in the following simple proposition.

3.1. Proposition. Let u, v be two terms. The equation (u,v) is satisfied in A_n iff either $u,v \geq [x_1,\dots,x_n]$ or the following five conditions are satisfied:

(i) $u \ngeq [x_1,\dots,x_n]$ and $v \ngeq [x_1,\dots,x_n]$;

(ii) $var(u)=var(v)$;

(iii) $R(u)=R(v)$;

(iv) $R(x,u)=R(x,v)$ for every variable $x \in var(u)$;

(v) if $L(u)=x$ and $L(v)=y$ then either $x=y$ or $R(x,u)=R(y,u)=n-2$.

Proof. Let (u,v) be satisfied in A_n and let either $u \not\geq [x_1,\dots,x_n]$ or $v \not\geq [x_1,\dots,x_n]$. For every variable x and every pair a, b of elements of A_n denote by $h_{x,a,b}$ the homomorphism of W into A_n such that $h_{x,a,b}(x)=a$ and $h_{x,a,b}(y)=b$ for all variables y different from x. Both (i) and (iii) follow from $h_{x_1,1,1}(u)=h_{x_1,1,1}(v)$. If $x \in var(u) \setminus var(v)$ then $h_{x,0,1}(u)=0$, and $h_{x,0,1}(v) \neq 0$. This proves (ii). We have $R(x,u) \leq n-2$ and $R(x,v) \leq$ $\leq n-2$ for all $x \in var(u)$. If $R(x,u) < R(x,v)$ for some x then $h_{x,n-R(x,v),1}(u) \neq 0$ and $h_{x,n-R(x,v),1}(v)=0$. This proves (iv). Let $L(u)=x$, $L(v)=y$ and $x \neq y$. If $R(x,u) < n-2$ then $h_{x,2,1}(u) \neq h_{x,2,1}(v)$. This proves (v).

If t is a term such that $t \geq [x_1,\dots,x_n]$ then evidently $h(t)=0$ for any homomorphism $h:W \to A_n$. It remains to prove that if the conditions (i),...,(v) are satisfied then (u,v) is an identity of A_n. Let $h:W \to A_n$ be a homomorphism. We are going to prove that $h(u)=h(v)$. Consider first the case $h(u)=0$ and let p be a minimal subterm of u with $h(p)=0$. We can write $p=[x,p_1,\dots,p_k]$ for some variable x and some terms $p_1,\dots,p_k$. If $p=x$, we get $h(v)=0$ from $h(x)=0$ by (ii). If $p \neq x$ then $k \geq 1$, $h(x)=n-k$ and $R(x,u) \geq k$; by (iv), $R(x,v) \geq k$ and so $h(x)=n-k$ implies $h(v)=0$. This finishes the proof in the case $h(u)=0$. In the case $h(v)=0$ the proof is similar. Now let $h(u) \neq 0$ and $h(v) \neq 0$. Put $L(u)=x$ and $L(v)=y$. If $x \neq y$ then it follows from (v) that $h(x)=h(y)=1$. So, we have $h(x)=h(y)$ in any case. Evidently $h(u)=h(x)+R(u)$ and $h(v)=h(y)+R(v)$. By (iii) we get $h(u)=h(v)$.

4. A finite basis for the identities of A_n.

4.1. Theorem. Let $n \geq 3$. The equational theory of the groupoid A_n is generated by the following nine equations:

(1) $y[x_1,\dots,x_n] = [x_1,\dots,x_n]y = [x_1,\dots,x_n]$,

(2) $[x,[y,z_1,\dots,z_{n-2}],u_2,\dots,u_{n-2}] = [y,[x,z_1,\dots,z_{n-2}],u_2,\dots$
$\dots,u_{n-2}]$,

(3) $xy\cdot z = xz\cdot y$,

(4) $x(y\cdot zu) = x(z\cdot yu)$,

(5) $x\cdot xy = xy$,

(6) $xy\cdot zu = xu\cdot zy$,

(7) $xx\cdot y = xy\cdot y$,

(8) $x\cdot yy = x\cdot yx\cdot$

Proof. Denote by T the equational theory generated by these nine equations. It follows easily from 3.1 that any of the nine equations is satisfied in A_n and so T is contained in the equational theory of A_n. Conversely, it remains to prove that every equation which is satisfied in A_n belongs to T. The proof of this fact will be divided into lemmas.

4.2. Lemma. Let $m\geq 1$. Then the equation

$$[y_m,\dots,y_1,x]^* z = [y_m,\dots,y_1,z]^* x$$

belongs to T.

Proof. By induction on m. For m=1 it is just the equation (3). For m=2 we have

$$(y_2\cdot y_1x)z \underset{3}{=} y_2z\cdot y_1x \underset{6}{=} y_2x\cdot y_1z \underset{3}{=} (y_2\cdot y_1z)x.$$

For m > 2 we have

$$[y_m,\dots,y_1,x]^* z \underset{I}{=} [y_m,\dots,y_2,z]^*\cdot y_1x \underset{I}{=} [y_m,\dots,y_3,y_1,x]^*\cdot$$
$$\cdot y_2z \underset{I}{=}$$
$$[y_m,\dots,y_3,y_1,y_2,z]^* x \underset{4}{=} [y_m,\dots,y_3,y_2,y_1z]^* x$$

where $\underset{I}{=}$ means a use of the induction assumption.

4.3. Lemma. Let t be a term and e, f be two occurrences of two variables x, y in t both of them ending with 2. Let t´ be the term obtained from t by replacing the occurrence e of x by y and the occurrence f of y by x. Then $(t,t') \in T$.

Proof. By induction on t. Let $t=t_1t_2$. If both e and f start with 1 (or both with 2, resp.), we can apply the induction assumption to the term t_1 (or t_2, resp.). Since the last remaining situation is symmetric, it remains to consider the case when e starts with 1 and f with 2, so that e is an occurrence of x inside t_1 and f is an occurrence of y inside t_2. We have $t_1= [u_k,\dots,u_1,z]^*$ for some variable z and some terms $u_1,\dots,u_k$ where $k \geq 1$. If $e \neq 12^k$ then e is an occurrence inside one of the subterms u_i; we have $(t, [u_k,\dots,u_1,t_2]^* z) \in T$ by 4.2 and so we can apply the induction assumption on the term $[u_k,\dots,u_1,t_2]^*$ which is shorter than t. So, let $e=12^k$; we then have x=z. We can express t_2 in the form $t_2= [v_l,\dots,v_1,q]^*$ for some variable q and some terms $v_1,\dots,v_l$. Several applications of 4.2 give

$$[u_k,\dots,u_1,x]^* [v_l,\dots,v_1,q]^* = [u_k,\dots,u_1,v_l,\dots,v_1,q]^* x =$$
$$[u_k,\dots,u_1,v_l,\dots,v_1,x]^* q = [u_k,\dots,u_1,q]^* [v_l,\dots,v_1,x]^*.$$

So, if $f=2^{l+1}$, we are through. If $f \neq 2^{l+1}$ then $f=2^i 1g$ for some $i \in \{1,\dots,l\}$ and some occurrence g of y in v_i. Denote by $\overline{v}_i$ the term obtained from v_i by replacing the occurrence g of y by x. By what has been proved above, $t= [u_k,\dots,u_1,q]^* [v_l,\dots,v_1,x]^*$. By induction, $[v_l,\dots,v_1,x]^* = [v_l,\dots,v_{i+1},\overline{v}_i,v_{i-1},\dots,v_1,y]^*$ and so several applications of 4.2 give

$$t = [u_k,\dots,u_1,q]^* [v_l,\dots,v_{i+1},\overline{v}_i,v_{i-1},\dots,v_1,y]^* =$$
$$[u_k,\dots,u_1,v_l,\dots,v_{i+1},\overline{v}_i,v_{i-1},\dots,v_1,y]^* q =$$
$$[u_k,\dots,u_1,v_l,\dots,v_{i+1},\overline{v}_i,v_{i-1},\dots,v_1,q]^* y =$$
$$[u_k,\dots,u_1,y]^* [v_l,\dots,v_{i+1},\overline{v}_i,v_{i-1},\dots,v_1,q]^* = t'.$$

In the proofs of the following lemmas $u \underset{L}{=} v$ expresses the fact that $(u,v) \in T$ follows from 4.3.

4.4. Lemma. Let t be a term and e, f be two occurrences of two variables z, x in t such that e ends with 1 and f ends with 2. Let t' be the term obtained from t by replacing the occurrence f of x by zx. Then $(t,t')\in T$.

Proof. By induction on t. It follows easily from 4.3 that it is enough to prove our assertion under the assumption that $f=2^k$ for some $k\geq 1$, so that $t=[t_k,\dots,t_1,x]^*$ for some terms $t_k,\dots,t_1$. By induction it is enough to consider the case when e is an occurrence of z inside t_k, i.e. $e=1g$ for some occurrence g of z in t_k. If $t_k=z$ then

$$t = [z,t_{k-1},\dots,t_1,x]^* \underset{5}{=} [z,z,t_{k-1},\dots,t_1,x]^* \underset{4}{=}$$
$$[z,t_{k-1},z,t_{k-2},\dots,t_1,x]^* \underset{4}{=} \dots \underset{4}{=} [z,t_{k-1},\dots,t_1,z,x]^* = t'.$$

So, let t_k be a composed term. Denote by i the positive integer such that 2^i is an occurrence of a variable in t_k; this variable, the last variable in t_k, denote by y. Let a be the term obtained from t_k by replacing the occurrence 2^i of y by x and let b be the term obtained from t_k by replacing the occurrence 2^i of y by zx. By induction, $(a,b)\in T$. By 4.3, $(t,[a,t_{k-1},\dots,t_1,y]^*)\in T$ and $(t',[b,t_{k-1},\dots,t_1,y]^*)\in T$. From this we get $(t,t')\in T$.

4.5. Lemma. The following equations belong to T:

(9) $(z(zx\cdot u))x = zx\cdot u$,

(10) $(z(zz\cdot u))x = zu\cdot x$,

(11) $uv\cdot(uu\cdot z) = uv\cdot z$,

(12) $uv\cdot((u\cdot zz)x) = uv\cdot zx$.

Proof.

(9) $(z(zx\cdot u))x \underset{3}{=} zx\cdot(zx\cdot u) \underset{5}{=} zx\cdot u$;

(10) $(z(zz\cdot u))x \underset{7}{=} (z(zu\cdot u))x \underset{3}{=} zx\cdot(zu\cdot u) \underset{6}{=} zu\cdot(zu\cdot x) \underset{5}{=} zu\cdot x$;

(11) $uv\cdot(uu\cdot z) \underset{7}{=} uv\cdot(uz\cdot z) \underset{6}{=} uz\cdot(uz\cdot v) \underset{5}{=} uz\cdot v \underset{3}{=} uv\cdot z$;

(12) $uv\cdot((u\cdot zz)x) \underset{8}{=} uv\cdot((u\cdot zu)x) \underset{3}{=} uv\cdot(ux\cdot zu) \underset{6}{=} uv\cdot(uu\cdot zx) \underset{11}{=} uv\cdot zx$.

4.6. Lemma. Let $m,k \geq 0$. Then the equation

$[y_m,\dots,y_1,x]^*[z_k,\dots,z_1,zx]^* = [y_m,\dots,y_1,x]^*[z_k,\dots,z_1,zz]^*$

belongs to T.

Proof. For m=0,

$x[z_k,\dots,z_1,zx]^* \underset{5}{=} [x,x,z_k,\dots,z_1,zx]^* \underset{4}{=} \dots \underset{4}{=}$

$[x,z_k,\dots,z_1,x\cdot zx]^* \underset{8}{=} [x,z_k,\dots,z_1,x\cdot zz]^* \underset{4}{=} \dots \underset{4}{=}$

$[x,x,z_k,\dots,z_1,zz]^* \underset{5}{=} x[z_k,\dots,z_1,zz]^*.$

For $m \geq 1$,

$[y_m,\dots,y_1,x]^*[z_k,\dots,z_1,zx]^* \underset{4}{=} \dots \underset{4}{=} [y_m,\dots,y_1,x]^*[z,z_k,\dots$
$\dots,z_1,x]^*$

$\underset{12}{=} [y_m,\dots,y_1,x]^*[y_m\cdot zz,z_k,\dots,z_1,x]^* \underset{L}{=} [y_m,\dots,y_1,x]^*[y_mx,z_k,$
$\dots,$

$z_1,zz]^* \underset{L}{=} [y_m,\dots,y_1,z_k,\dots,z_1,zz]^*(y_mx\cdot x) \underset{7}{=} [y_m,\dots$

$\dots,y_1,z_k,\dots,$

$z_1,zz]^*(y_my_m\cdot x) \underset{L}{=} [y_m,\dots,y_1,x]^*[y_my_m,z_k,\dots,z_1,zz]^* \underset{11}{=}$
$[y_m,\dots,y_1,x]^*[z_k,\dots,z_1,zz]^*.$

4.7. Lemma. Let t be a term of the form $t=[t_k,\dots,t_1,zx]^*$ where x, z are variables and x has at least two occurrences in t. Then $(t,[t_k,\dots,t_1,zz]^*) \in T$.

Proof. By induction on t. If $x \in var(t_1) \cup \dots \cup var(t_{k-1})$, we can use the induction assumption. Hence, let $x \in var(t_k)$. If x is the last variable in t_k, the assertion follows from 4.6; if not, we can use 4.3 and the induction.

4.8. Lemma. Let t be a term of the form $t=[t_k,\dots,t_1,z]^*$ where z is a variable having an occurrence in t ending with 1. Then $(t,[t_k,\dots,t_1,zz]^*) \in T$.

Proof. If $z \notin var(t_k)$ then $z \in var(t_1) \cup \dots \cup var(t_k)$ and we can use the induction **assumption. Now, let** $z \in var(t_k)$. If $t_k=z$, the assertion follows from (4) and (5). If t_k is not a variable, we can use 4.3 and the induction.

4.9. Lemma. Let t be a term and e, f be two occurrences of two variables z, x in t such that e ends with 1 and f ends with 2; let x have at least two occurrences in t. Let t' be the term obtained from t by replacing the occurrence f of x by z. Then $(t,t') \in T$.

Proof. It follows from 4.3, 4.4, 4.7 and 4.8.

4.10. Lemma. Let t be a term and $x \in var(t)$. Then $(tt,tx) \in T$.

Proof. Let us fix a variable $z \notin var(t)$. By 4.9 we have $(z \cdot tx, z \cdot tz) \in T$, so that $(t \cdot tx, t \cdot tt) \in T$; by (5) we get $(tx,tt) \in T$.

4.11. Lemma. Let t be a term and $x,z \in var(t)$; let f be an occurrence of x in t, ending with 2, and let x have at least two occurrences in t. Let t' be the term obtained from t by replacing the occurrence f of x by z. Then $(t,t') \in T$.

Proof. By induction on t. By 4.3 we can assume that $f=2^k$ for some $k \geq 1$, so that $t=[t_k,\ldots,t_1,x]^*$ for some terms $t_1,\ldots,t_k$. By 4.9 we can suppose that all occurrences of z in t end with 2. Let $x \in var(t_i)$ and $z \in var(t_j)$.

First, let $i \neq j$ and $j \neq k$. By the induction assumption it is enough to consider the case $i=k$; by (4) we can assume that $j=1$, so that $z \in var(t_1)$. It follows from 4.7 that $(t,[t_k,\ldots,t_1,t_1]^*) \in T$. By 4.10, $(t_1 t_1, t_1 z) \in T$ and so $(t,t') \in T$.

Next, let $i=j$. Then we can assume that $i=j=k$, since otherwise we could make use of the induction. This means $x,z \in var(t_k)$. By 4.3 we can suppose that $t_k=[u_1,\ldots,u_1,z]^*$ for some terms $u_1,\ldots,u_1$. We have $(t,[u_1,\ldots,u_1,t_{k-1},\ldots,t_1,x]^* z) \in T$ by 4.3 and so $(t,[u_1,\ldots,u_1,t_{k-1},\ldots,t_1,z]^* x) \in T$. From this we see that it is enough to prove the assertion under the assumption k=1.

If $x \in var(u_l)$ then

$t = [u_1,\dots,u_l,z]^* x \underset{L}{=} [u_1,\dots,u_l,x]^* z \underset{11}{=} ([u_1,\dots,u_l,x]^*(u_l u_l \cdot z)$
$\underset{7}{=} [u_1,\dots,u_l,x]^*(u_l z \cdot z) \underset{L}{=} [u_1,\dots,u_l,z]^*(u_l x \cdot z) = \text{(by 4.11)} =$
$[u_1,\dots,u_l,z]^*(u_l u_l \cdot z) \underset{11}{=} [u_1,\dots,u_l,z]^* z.$

If $x \notin var(u_l)$ then by (4) we can assume $x \in var(u_{l-1})$. Then

$t = [u_1,\dots,u_l,z]^* x \underset{3}{=} u_l x \cdot u_{l-1}[u_{l-2},\dots,u_1,z]^* \underset{5}{=}$
$u_l x \cdot (u_{l-1} \cdot u_{l-1}[u_{l-2},\dots,u_1,z]^*) \underset{6}{=} [u_1,\dots,u_l,z]^* \cdot u_{l-1} x \underset{L}{=}$
$[u_1,\dots,u_l,x]^* \cdot u_{l-1} z \underset{4}{=} [u_l,u_{l-2},\dots,u_1,u_{l-1}x]^* \cdot u_{l-1} z = \text{(by 4.10)} =$
$[u_l,u_{l-2},\dots,u_1,u_{l-1}u_{l-1}]^* \cdot u_{l-1} z \underset{L}{=} [u_l,u_{l-2},\dots,u_1,u_{l-1},u_{l-1},z]^* \cdot$
$\cdot u_{l-1} \underset{5}{=} [u_l,u_{l-2},\dots,u_1,u_{l-1},z]^* \cdot u_{l-1} \underset{L}{=} [u_l,u_{l-2},\dots,u_1,$
$u_{l-1}u_{l-1}]^* z = \text{(by 4.9)} = [u_l,u_{l-2},\dots,u_1,u_{l-1}z]^* z \underset{4}{=} [u_1,\dots$
$\dots,u_l,z]^* z.$

It remains to consider the case when $x \in var(t_1)$ and $z \in var(t_k)$. If $t_1 \neq x$ then by interchanging the last (occurrence of) variable in t_1 with z we get either the case considered earlier (the case $x \in var(t_k)$, $z \in var(t_1)$) or the case settled down by induction. So let $t_1 = x$. We can, moreover, assume that z is the last variable in t_k and so $t = [u_1,\dots,u_l,z]^* [t_{k-1},\dots,t_2,xx]^*$ for some terms $u_1,\dots,u_l$. Then

$t = [u_1,\dots,u_l,xx]^* [t_{k-1},\dots,t_2,z]^* \underset{L}{=} [u_1,\dots,u_l,xz]^* [t_{k-1},\dots$
$\dots,t_2,x]^*.$

From the already investigated case when both the variables belonged to $var(t_k)$ we conclude that

$t = [u_1,\dots,u_l,xz]^* [t_{k-1},\dots,t_2,z]^* \underset{L}{=}$
$[u_1,\dots,u_l,z]^* [t_{k-1},\dots,t_2,x,z]^* = t'.$

By a slender term we mean a term t such that whenever a, b are two terms and ab is a subterm of t then either a or b is a variable.

4.12. Lemma. For every term t there exists a slender term a such that $(t,a) \in T$ and $L(t)=L(a)$.

Proof. By induction on t. Let $t=t_1 t_2$. We have $t_1 = [u_k, \ldots \ldots, u_1, x]^*$ for some variable x and terms $u_1, \ldots, u_k$. If $t_1 = x$, we can use the induction. Let $t_1 \neq x$. By 4.2, $(t, [u_k, \ldots, u_1, t_2]^* x) \in T$. By induction, $([u_k, \ldots, u_1, t_2]^*, b) \in T$ for some slender term b such that $L(b)=L(t)$. Hence $(t, bx) \in T$ where bx is slender.

Let $x_1, \ldots, x_k$ be a finite sequence of variables and let $m_1, \ldots, m_k$ be positive integers. We denote by $H(x_1, m_1; \ldots; x_k, m_k)$ the set of terms defined in this way:

$H(x_1, m_1)$ is the set of terms $[x_1, y_1, y_2, \ldots, y_{m_1}]$ where $y_1, \ldots \ldots, y_{m_1}$ are arbitrary variables;

if $k \geq 2$ then $H(x_1, m_1; \ldots; x_k, m_k)$ is the set of terms $[x_1, u, y_2, \ldots, y_{m_1}]$ where $u \in H(x_2, m_2; \ldots; x_k, m_k)$ and $y_2, \ldots, y_{m_1}$ are arbitrary variables.

4.13. Lemma. Let $1 \leq k \leq m$. The equation

$$[x, y_1, \ldots, y_m] = [x\,[x, y_1, \ldots, y_k], y_2, \ldots, y_m]$$

belongs to T.

Proof. For $m=1$ this is the equation (5). Let $m > 1$. We have

$$[x[x, y_1, \ldots, y_k], y_2, \ldots, y_m] \underset{L}{=}$$
$$[[x, y_k, y_2, \ldots, y_{k-1}] \cdot [x, y_1, \ldots, y_k], y_{k+1}, \ldots, y_m] \underset{L}{=}$$
$$[[x, y_k, y_2, \ldots, y_{k-1}] \cdot [x, y_k, y_2, \ldots, y_{k-1}] y_1, y_{k+1}, \ldots, y_m] \underset{5}{=}$$
$$[x, y_k, y_2, \ldots, y_{k-1}, y_1, y_{k+1}, \ldots, y_m] \underset{L}{=} [x, y_1, \ldots, y_m].$$

4.14. Lemma. Let $t \in H(x_1, m_1; \ldots; x_k, m_k)$, $i \in \{1, \ldots, k\}$ and $1 \leq j \leq m_i$. Then there is a term $t' \in H(x_1, m_1; \ldots; x_{i-1}, m_{i-1}; x_i, m_i; x_i, j; x_{i+1}, m_{i+1}; \ldots; x_k, m_k)$ with $(t, t') \in T$.

Proof. It follows easily from 4.13.

4.15. Lemma. Let $i,j \geq 1$. The equation

$$z[x[y,u_1,\dots,u_i],v_2,\dots,v_j] = z[y[x,u_1,v_2,\dots,v_j],u_2,\dots,u_i]$$

belongs to T.

Proof. $z[x[y,u_1,\dots,u_i],v_2,\dots,v_j] \underset{3}{=} z([x,v_2,\dots,v_j]$ $[y,u_1,\dots,u_i]) \underset{4}{=} z([y,u_1,\dots,u_{i-1}][x,v_2,\dots,v_j,u_i]) \underset{L}{=}$ $z[y[x,v_2,\dots,v_j,u_i],u_2,\dots,u_{i-1},u_i] \underset{L}{=} z[y[x,u_1,v_2,\dots,v_j],u_2,\dots,u_i]$.

4.16. Lemma. Let $t \in H(x_1,m_1;\dots;x_k,m_k)$ and let $i \in \{2,\dots,k-1\}$. Then there is a term $t' \in H(x_1,m_1;\dots;x_{i-1},m_{i-1};x_{i+1},m_{i+1};x_i,m_i;$ $x_{i+2},m_{i+2};\dots;x_k,m_k)$ with $(t,t') \in T$.

Proof. It follows easily from 4.15.

4.17. Lemma. Let t,u be two terms such that $L(t)=L(u)$ and (t,u) is satisfied in A_n. Then $(t,u) \in T$.

Proof. By 4.12 it is enough to suppose that t, u are both slender. Then $t \in H(x_1,m_1;\dots;x_k,m_k)$ and $u \in H(y_1,c_1;\dots;y_l,c_l)$ for some x_i,m_i,y_i,c_i. If one of the terms t, u is $\geq [x_1,\dots,x_n]$ then by 3.1 both of them are and (t,u) is a consequence of (1). So, let this be not the case. The numbers $m_1,\dots,m_k,c_1,\dots,c_l$ are then all $\leq n-2$. We have $x_1=y_1$. Since $x_1,\dots,x_k$ are just the variables $x \in var(t)$ with $R(x,t) \neq 0$, by (ii) and (iv) we get $\{x_1,\dots,x_k\}=\{y_1,\dots,y_l\}$. Moreover, for every $x \in \{x_1,\dots,x_k\}$ the maximal i such that $(x,i) \in$ $\in \{(x_1,m_1),\dots,(x_k,m_k)\}$ coincides with the maximal i such that $(x,i) \in \{(y_1,c_1),\dots,(y_l,c_l)\}$.

Suppose first that $m_1=\dots=m_k=1$, so that $c_1=\dots=c_l=1$. Then $t =$ $= [x_1,\dots,x_k,y]^*$ and $u=[y_1,\dots,y_c,z]^*$ for some variables y, z such that either $y,z \in \{x_1,\dots,x_k\}$ or $y=z$. Since $x_1=y_1$, we get $(t,u) \in T$ by (4),(5) and 4.11.

Now let $m_i \geq 2$ for some i and $c_j \geq 2$ for some j. Put $\{w_1,\dots,w_d\} = var(t) \setminus \{x_1,\dots,x_k\} = var(u) \setminus \{y_1,\dots,y_l\}$. It follows from 4.14 and 4.16 that there exists a sequence $z_1,\dots,z_p$,

$r_1,\dots,r_p$ and two terms $t' \in H(z_1,r_1;\dots;z_p,r_p)$, $u' \in H(z_1,r_1;\dots$ $\dots;z_p,r_p)$ such that $(t,t')\in T$, $(u,u')\in T$ and $r_1+\dots+r_p-(p-1)\geq d$. Denote by $e_1,\dots,e_s$ all the (pairwise different) occurrences of variables in the term t', or in any term from $H(z_1,r_1;\dots;z_p,r_p)$ (since these are the same) that are ending with 2. We have $s=r_1+\dots+r_p-p+1\geq d$. Denote by t'' (by u'', resp.) the term obtained from t' (from u', resp.) by replacing the occurrences e_i of variables by w_i for $i\leq d$, and by x_1 for $i>d$. It follows from 4.11 and 4.3 that $(t',t'')\in T$ and $(u',u'')\in T$. However, evidently $t'' = u''$ and so $(t,u)\in T$.

4.18. Lemma. Let t, u be two terms such that $L(t)\neq L(u)$ and (t,u) is satisfied in A_n. Then $(t,u)\in T$.

Proof. Put $x=L(t)$ and $y=L(u)$. We shall consider only the case when neither t nor u is $\geq [x_1,\dots,x_n]$. By 3.1 we have $R(x,t)=$ $=R(x,u)=R(y,t)=R(y,u)=n-2$ and it is easy to see that there is a term v such that the equations

$$(t,[x,[y,v,x_2,\dots,x_{n-2}],x_2,\dots,x_{n-2}]),$$
$$(u,[y,[x,v,x_2,\dots,x_{n-2}],x_2,\dots,x_{n-2}])$$

where $x_2=\dots=x_{n-2}=x$ both belong to T. By (2) we get $(t,u)\in T$.

Now, Lemmas 4.17 and 4.18 finish the proof of Theorem 4.1.

Reference

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