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THE APPROXIMATION OF AN OPTIMAL SHAPE CONTROL PROBLEM GOVERNED BY A VARIATIONAL INEQUALITY WITH FLUX COST FUNCTIONAL

Jaroslav HASLINGER, Ján LOVÍŠEK

Abstract: The paper deals with the finite element approximation of an optimal shape design problem, when the state relation is given by a unilateral boundary value problem. Dual norm of the normal derivative of the solution on the boundary is taken as the cost functional. The relation between continuous model and its finite dimensional approximations is established.

Key words: Structural optimization, optimal shape, design problem.

Classification: 49A22

It is the aim of the present paper to continue the analysis from [1], where the existence of an optimal shape for a problem, governed by a variational inequality with the cost functional expressed as the dual norm of the normal derivative of the state along the unknown part of the boundary, has been proved. The present paper deals with the approximation of this problem, using approximate finite element spaces. The main effort is devoted to the study of the relation between discrete and continuous model. Using an equivalent expression for the dual norm, it is possible to give another form of the cost functional, more convenient for the practical computations.

In Section 1, the continuous problem is defined. In Section 2, the approximation of the optimal shape control problem is described, using piecewise linear functions in 1 and 2 variables, for the approximation of the shape, the state relations, respectively. In Section 3 it is proved that discrete optimal shapes are close in the appropriate sense to the continuous ones. In Section 4 we derive the exact form of the cost functional gradient in the discrete case. It is shown that the cost functional, as a function of design parameters is 1x continuously differentiable.

1. <u>Setting of the problem</u>. Let $\{\Omega(\alpha)\}$, $\alpha \in \mathcal{U}_{ad}$ be a family of bounded plane domains,

$$\mathcal{U}_{ad} = \{\alpha \in C^0(\langle 0,1\rangle) \mid 0 < \alpha_0 \leq \alpha(x_2) \leq \beta_0,$$
$$|\alpha'(x_2)| \leq c_1; \ \int_0^1 \alpha(x_2)\,dx_2 = c_2; \ x_2 \in (0,1)\},$$

where α_0, β_0, c_1, c_2 are given positive constants. With any $\alpha \in \mathcal{U}_{ad}$ we associate the following unilateral boundary value problem:

$$(1.1) \quad \begin{cases} -\Delta y(\alpha) + y(\alpha) = f & \text{in } \Omega(\alpha), \\ \dfrac{\partial y(\alpha)}{\partial \nu} = 0 & \text{on } \Gamma_1(\alpha), \\ y(\alpha) \geq 0; \dfrac{\partial y(\alpha)}{\partial \nu} \geq 0; \ y(\alpha)\dfrac{\partial y(\alpha)}{\partial \nu} = 0 & \text{on } \Gamma_2(\alpha), \end{cases}$$

where the decomposition of the boundary $\partial\Omega(\alpha)$ into $\Gamma_1(\alpha)$, $\Gamma_2(\alpha)$ is clear from fig. 1:

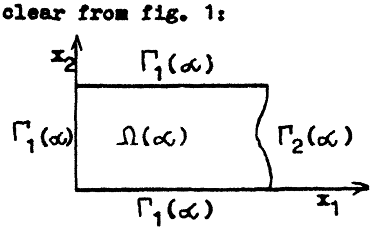

Fig. 1

Our aim is to determine such $\Omega(\alpha^*) \in \{\Omega(\alpha)\}$, $\alpha \in \mathcal{U}_{ad}$, i.e. such a function $\alpha^* \in \mathcal{U}_{ad}$, satisfying

$(\mathbb{P})'$ $\qquad \tilde{E}(\alpha^*) \leq \tilde{E}(\alpha) \qquad \forall \alpha \in \mathcal{U}_{ad},$

where

$$(1.2) \qquad \tilde{E}(\alpha) = 1/2 \left\| \frac{\partial y(\alpha)}{\partial \nu} \right\|^2_{-1/2,\partial\Omega(\nu)} \; {}^{*)}$$

and $y(\alpha)$ is a function of α through (1.1).

In order to give the rigorous mathematical formulation, we introduce some notations.

Let $\alpha \in \mathcal{U}_{ad}$. Set

$V(\alpha) = H^1(\Omega(\alpha))$

$K(\alpha) = \{\varphi \in V(\alpha) \mid \varphi \geq 0 \text{ on } \Gamma_2(\alpha)\}$

and

$$J_\alpha(\varphi) = \frac{1}{2}\|\varphi\|^2_{1,\Omega(\alpha)} - (f,\varphi)_{0;\Omega(\alpha)},$$

with

$$f \in L^2(\Omega_{\hat{\beta}});\ \Omega_{\hat{\beta}} = (0;\hat{\beta})x(0;1);\ \hat{\beta} \geq \beta_0.$$

We denote by $\|\cdot\|_{1,\Omega(\alpha)}$, $(\cdot;\cdot)_{0;\Omega(\alpha)}$ $H^1(\Omega(\alpha))$ - norm, $L^2(\Omega(\alpha))$ - scalar product, respectively.

By a <u>weak formulation</u> of the state inequality (1.1) we call the problem

$(\mathcal{P}(\alpha))'$ find $y = y(\alpha) \in K(\alpha)$ such that $J_\alpha(y(\alpha)) \leq J_\alpha(\varphi)$ $\forall \varphi \in K(\alpha)$

or equivalently

find $y = y(\alpha) \in K(\alpha)$ such that

$(\mathcal{P}(\alpha))$ $\quad (y(\alpha),\varphi - y(\alpha))_{1,\Omega(\alpha)} \geq (f;\varphi - y(\alpha))_{0,\Omega(\alpha)}$

$\forall \varphi \in K(\alpha)$.

As $H^{-1/2}(\partial\Omega)$ - norm, by means of which the cost functional $\tilde{E}$ is defined, is not suitable for the treatment, we reformulated in

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*) $\|\mu\|_{-1/2;\partial\Omega}$ denotes the norm of the linear functional $\mu \in (H^{\frac{1}{2}}(\partial\Omega))'$.

[1] our problem under the consideration, as follows:

($\mathbb{P}$) find $\alpha^* \in \mathcal{U}_{ad}$ such that $E(\alpha^*) \leq E(\alpha)$ $\forall \alpha \in \mathcal{U}_{ad}$

where

$$E(\alpha) = 1/2 \; \|z(y)\|^2_{1,\Omega(\alpha)}$$

$z(y) \in V(\alpha)$ is the unique solution of an auxiliary problem

$$(\mathcal{A}(\alpha)) \quad \begin{aligned} &\text{find } z(y) \in V(\alpha) \text{ such that } (z(y); \varphi)_{1,\Omega(\alpha)} = \\ &= (y(\alpha); \varphi)_{1,\Omega(\alpha)} - (f; \varphi)_{0,\Omega(\alpha)} \quad \varphi \in V(\alpha) \end{aligned}$$

and $y(\alpha) \in K(\alpha)$ is a solution of $(\mathcal{P}(\alpha))$. In [1] it has been proved that ($\mathbb{P}$) possesses at least one solution.

2. <u>Approximation of ($\mathbb{P}$)</u>. Let $D_h: 0 = x_2^{(0)} < x_2^{(1)} < \ldots < x_2^{(N)} = 1$ be a partition of $\langle 0;1 \rangle$, the norm of which tends to zero and set

$$S_h = \{\alpha_h \in C(\langle 0;1 \rangle) \mid \alpha_h|_{x_2^{(i-1)} x_2^{(i)}} \in P_1(\overline{x_2^{(i-1)} x_2^{(i)}})\}$$

$$\mathcal{U}^h_{ad} = \mathcal{U}_{ad} \cap S_h \; .$$

In other words, $\Omega(\alpha_h)$, $\alpha_h \in \mathcal{U}^h_{ad}$ is a domain with a piecewise-linear variable part $\Gamma_2(\alpha_h)$. As $\Omega(\alpha_h)$ is a polygonal domain, one can construct its triangulation $\mathcal{T}_h(\alpha_h)$. Next we shall assume only such families of $\{\mathcal{T}_h(\alpha_h)\}$ which are <u>uniformly regular</u> with respect to $\alpha_h \in \mathcal{U}^h_{ad}$; $h \to 0^+$, i.e. there exists $\vartheta_0 > 0$ such that

$$(2.1) \qquad \min_{\alpha_h \in \mathcal{U}^h_{ad}} \vartheta(h; \alpha_h) \geq \vartheta_0 \qquad \forall h \to 0_+,$$

where $\vartheta(h; \alpha_h)$ is the smallest interior angle of T_i, computed among all $T_i \in \mathcal{T}_h(\alpha_h)$.

With any $\mathcal{T}_h(\alpha_h)$, $\alpha_h \in \mathcal{U}^h_{ad}$ we associate the finite-dimensional space $V_h(\alpha_h)$, containing all continuous, piecewise-linear functions and its closed convex subset $K_h(\alpha_h)$:

$$V_h(\alpha_h) = \{\varphi_h \in C^0(\overline{\Omega(\alpha_h)}) \mid \varphi_h|_{T_i} \in P_1(T_i) \quad \forall T_i \in \mathcal{T}_h(\alpha_h)\}$$

$$K_h(\alpha_h) = \{\varphi_h \in V_h(\alpha_h) \mid \varphi_h \geq 0 \;\; \text{on} \;\; \Gamma_2(\alpha_h)\}.$$

The approximation of $(\mathcal{P}(\alpha_h))$, with $\alpha_h \in \mathcal{U}^h_{ad}$, will be defined by means of Ritz-Galerkin procedure on $K_h(\alpha_h)$. The approximate of state inequality (take fixed $\alpha_h \in \mathcal{U}^h_{ad}$) is defined by

$$(\mathcal{P}(\alpha_h))'_h \quad \begin{array}{l} \text{find } y_h = y_h(\alpha_h) \in K_h(\alpha_h) \text{ such that} \\ J_h(y_h(\alpha_h)) \leq J_h(\varphi_h) \qquad \forall \varphi_h \in K_h(\alpha_h) \end{array}$$

where

$$J_h(\varphi_h) = 1/2 \, \|\varphi_h\|^2_{1,\Omega(\alpha_h)} - (f;\varphi_h)_{0,\Omega(\alpha_h)}$$

or equivalently

$$(\mathcal{P}(\alpha_h))_h \quad \begin{array}{l} \text{find } y_h(\alpha_h) \in K_h(\alpha_h) \text{ such that} \\ (y_h(\alpha_h);\varphi_h - y_h(\alpha_h))_{1,\Omega(\alpha_h)} \geq \\ \geq (f;\varphi_h - y_h(\alpha_h))_{0,\Omega(\alpha_h)} \qquad \forall \varphi_h \in K_h(\alpha_h). \end{array}$$

Thus the optimal shape control problem $(\mathbb{P})$ can be stated as follows:

$(\mathbb{P})_h$ find $\alpha^*_h \in \mathcal{U}^h_{ad}$ such that $E_h(\alpha^*_h) \leq E_h(\alpha_h) \quad \forall \alpha_h \in \mathcal{U}^h_{ad}$

where

$$(2.2) \qquad E_h(\alpha_h) = 1/2 \, \|z_h(y_h)\|^2_{1,\Omega(\alpha_h)}$$

and $z_h \in V_h(\alpha_h)$ is a unique solution of

$$(\mathcal{A}(\alpha_h))_h \qquad (z_h;\varphi_h)_{1,\Omega(\alpha_h)} = (y_h(\alpha_h);\varphi_h)_{1,\Omega(\alpha_h)} - -(f;\varphi_h)_{0,\Omega(\alpha_h)} \qquad \forall \varphi_h \in V_h(\alpha_h).$$

$y_h(\alpha_h)$ appearing in the definition of $(\mathcal{A}(\alpha_h))_h$ denotes the solution of $(\mathcal{P}(\alpha_h))_h$.

Using the classical compactness arguments, one can easily prove

Theorem 2.1. For any $h > 0$ there exists at least one solution of $(\mathbb{P})_h$.

3. Relation between $(\mathbb{P})_h$ and $(\mathbb{P})$. In this section we shall analyse the mutual relation between a family of $(\mathbb{P})_h$ and the continuous problem $(\mathbb{P})$ if $h \to 0^+$.

Let $\alpha \in \mathcal{U}_{ad}$. By P we denote the mapping from $H^1(\Omega(\alpha))$ into $L^2((0;1))$ defined through the relation

(3.1) $\quad (P(y);\xi)_\alpha \overset{\text{def.}}{=} - \int_0^1 y^-(\alpha(x_2),x_2)\,\xi(\alpha(x_2),x_2)\,dx_2,$

where $y^- = (|y| - y)/2$ is the negative part of y. It is easy to see that

$$y \in K(\alpha) \iff (P(y);\xi)_\alpha = 0 \qquad \forall \xi \in \mathcal{D}(R_2).$$

First we introduce 2 auxiliary results, useful in what follows.

Lemma 3.1. Let $\alpha_h \rightrightarrows \alpha$ (uniformly) in $\langle 0;1\rangle$, $\alpha_h \in \mathcal{U}^h_{ad}$, $\alpha \in \mathcal{U}_{ad}$. Let $y_h \in V_h(\alpha_h)$, $y \in V(\alpha)$ be such that

(3.2) $\quad y_h \rightharpoonup y$ (weakly) in $H^1(G_m(\alpha))$ for any m integer, where

(3.3) $\quad G_m(\alpha) = \{[x_1;x_2] \in \mathbb{R}_2 \mid x_1 \in (0;\alpha(x_2)-1/m),\ x_2(0;1)\}$. Then

(3.4) $\quad (P(y_h);\xi)_{\alpha_h} \to (P(y);\xi)_\alpha \qquad \forall \xi \in \mathcal{D}(R_2).$

For the proof we refer to [2].

Lemma 3.2. Let $\alpha \in \mathcal{U}_{ad}$ and $\varphi \in K(\alpha)$. Then there exists an extension φ^* of φ from $\Omega(\alpha)$ on $\Omega_{\hat\beta}$ and functions $\{\varphi^*_j\}_{j=1}^{\infty}$, $\varphi^*_j \in H^1(\Omega_{\hat\beta})$ such that

(i) $\quad \varphi^*_j \to \varphi^*$ in $H^1(\Omega_{\hat\beta})$

(ii) $\quad \varphi^*_j = \psi + \eta_j$; where $\psi \ge 0$ in $\Omega_{\hat\beta}$, $\eta_j \in C^\infty(\overline{\Omega_{\hat\beta}})$ and $\eta_j(x_1;x_2) = 0 \quad \forall [x_1,x_2] \in G^{m(j)}$, where $G^{m(j)} = \{[x_1;x_2] \in \mathbb{R}_2 \mid x_2 \in (0,1),\ x_1 \in (\alpha(x_2) - \frac{1}{m(j)};\hat\beta)\}$ and $m(j) \to \infty$; $j \to \infty$.

Proof can be found in [1].

Now we prove a fundamental result, by means of which we establish the mutual relation between $(\mathbb{P})_h$ and $(\mathbb{P})$.

Lemma 3.3. Let $\alpha_h \rightrightarrows \alpha$ in $\langle 0,1\rangle$, $\alpha_h \in \mathcal{U}_{ad}^h$, $\alpha \in \mathcal{U}_{ad}$ and let $y_h = y_h(\alpha_h) \in K(\alpha_h)$, $z_h \in V_h(\alpha_h)$ be solutions of $(\mathcal{P}(\alpha_h))_h$, $(\mathcal{A}(\alpha_h))_h$, respectively. Then there exist subsequences $\{y_{h_j}\} \subset \{y_h\}$, $\{z_{h_j}\} \subset \{z_h\}$ and elements $y \in K(\alpha)$, $z \in V(\alpha)$ such that

$$y_{h_j} \rightharpoonup y$$

$$z_{h_j} \rightharpoonup z$$

in $H^1(G_m(\alpha))$ for any m, where $G_m(\alpha)$ is given by (3.3) and y, z are solutions of $(\mathcal{P}(\alpha))$, $(\mathcal{A}(\alpha))$, respectively. Moreover,

$$\|y_{h_j}\|_{1,\Omega(\alpha_{h_j})} \to \|y\|_{1,\Omega(\alpha)}, \quad \|z_{h_j}\|_{1,\Omega(\alpha_{h_j})} \to \|z\|_{1,\Omega(\alpha)}, \quad h_j \to 0^+. \tag{3.5}$$

Proof. Sequences y_h, z_h are bounded in the following sense:

$$\exists\, c>0 \text{ independently on } h, \alpha_h \in \mathcal{U}_{ad}^h \text{ and such that } \|y_h(\alpha_h)\|_{1,\Omega(\alpha_h)} \le c, \quad \|z_h\|_{1,\Omega(\alpha_h)} \le c. \tag{3.6}$$

Indeed, substituting $\varphi_h = 0{,}2y_h$ into $(\mathcal{P}(\alpha_h))_h$ and using the fact $f \in L^2(\Omega_{\hat{\beta}})$, we immediately obtain the boundedness of y_h. From this and $(\mathcal{A}(\alpha_h))_h$, the boundedness of z_h follows.

Let m be fixed. Then there exists $h_o = h_o(m)$ such that $\overline{G_m(\alpha)} \subset \Omega(\alpha_h)$ for any $h \le h_o$ and (3.6) yields

$$\|y_h\|_{1,G_m} \le c, \quad \|z_h\|_{1,G_m} \le c \quad \forall h \le h_o. \tag{3.7}$$

Thus there exist subsequences $\{y_h(m)\} \subset \{y_h\}$, $\{z_h^{(m)}\} \subset \{z_h\}$ and functions $y^{(m)}$, $z^{(m)} \in H^1(G_m(\alpha))$ such that

$$y_h^{(m)} \rightharpoonup y^{(m)}$$

$$z_h^{(m)} \rightharpoonup z^{(m)} \quad \text{in } H^1(G_m(\alpha)).$$

Proceeding in the same way on $G_{m+1}(\alpha)$ with $\{y_h^{(m)}\}$, $\{z_h^{(m)}\}$ one can choose $\{y_h^{(m+1)}\} \subset \{y_h^{(m)}\}$, $\{z_h^{(m+1)}\} \subset \{z_h^{(m)}\}$ such that

$$y_h^{(m+1)} \rightharpoonup y^{(m+1)}, \ z_h^{(m+1)} \rightharpoonup z^{(m+1)} \text{ in } H^1(G_{m+1}(\alpha)).$$

Further, it is clear that

$$y^{(m)} = y^{(m+1)}, \ z^{(m)} = z^{(m+1)} \text{ on } G_m(\alpha).$$

The diagonal sequence, constructed by means of $\{y_h^{(q)}\}$, $\{z_h^{(q)}\}$, $q = m, m+1, \dots$ (and denoted by $\{y_{h_j}\}$, $\{z_{h_j}\}$ for sake of simplicity) have the following property:

$$(3.8) \qquad \begin{matrix} y_{h_j} \rightharpoonup y \\ z_{h_j} \rightharpoonup z \end{matrix} \qquad \text{in } H^1(G_m(\alpha))$$

for any m, with y and z defined by

$$(3.9) \qquad y \equiv y^{(m)}, \ z \equiv z^{(m)} \text{ on } G_m(\alpha).$$

Moreover, it is clear that $y, z \in V(\alpha)$. We next show that $y \in K(\alpha)$. For this we need the lemma 3.1. As $y_{h_j} \equiv y_{h_j}(\alpha_{h_j}) \in K(\alpha_{h_j})$, we have

$$(3.10) \qquad (P(y_{h_j}), \xi)_{\alpha_{h_j}} = 0 \qquad \forall \xi \in \mathcal{D}(R_2).$$

On the other hand, (3.4) yields

$$(3.11) \qquad (P(y_{h_j}), \xi)_{\alpha_{h_j}} \rightarrow (P(y), \xi)_{\alpha}, \qquad h_j \rightarrow 0^+.$$

Combining (3.10) and (3.11) we have

$$(P(y), \xi)_{\alpha} = 0 \qquad \forall \xi \in \mathcal{D}(R_2),$$

i.e. $y \in K(\alpha)$. Now we prove that y, z solve $(\mathcal{P}(\alpha))$, $(\mathcal{A}(\alpha))$, respectively.

We do it for the function y. Let $\xi \in K(\alpha)$ be given. According to Lemma 3.1 one can construct its extension ξ^* from $\Omega(\alpha)$ onto $\Omega_{\hat{\beta}}$ and functions $\xi^*_j \in H^1(\Omega_{\hat{\beta}})$, $\xi^*_j = \psi + \eta_j$, $\psi \geq 0$ in $\Omega_{\hat{\beta}}$; $\eta_j \in C^{\infty}(\overline{\Omega_{\hat{\beta}}})$, $\eta_j = 0$ in $G^{m(j)}$ (defined in Lemma 3.1) and such that

$$(3.12) \qquad \xi^*_j \rightarrow \xi^* \quad \text{in } H^1(\Omega_{\hat{\beta}}).$$

Without loss of generality, one can assume $\psi \in C^{\infty}(\overline{\Omega_{\hat{\beta}}})$, $\psi \geq 0$

(if not, ψ can be replaced by regularizations $\psi_j \in C^{\infty}(\overline{\Omega_{\hat\beta}})$, $\psi_j \geq 0$ in $\Omega_{\hat\beta}$ and $\psi_j \to \psi$ in $H^1(\Omega_{\hat\beta})$).

Let

$$\xi_{jh} = \pi_h(\xi^*_j|_{\Omega(\alpha_h)}) = \pi_h(\psi|_{\Omega(\alpha_h)}) + \pi_h(\eta_j|_{\Omega(\alpha_h)}) \in V_h(\alpha_h)$$

be a piecewise linear Lagrange interpolate of $\xi^*_j|_{\Omega(\alpha_h)}$ over $\mathcal{T}_h(\alpha_h)$. Let j be fixed and $h \to 0^+$. Then from the construction of the sequence ξ^*_j it follows that $\xi_{jh} \in K(\alpha_h)$, provided h is small enough and moreover:

$$(3.13) \qquad \| \xi_{j_h} - \xi_j \|_{1,\Omega(\alpha_h)} \leq ch\, |\xi_j|_{2,\Omega(\alpha_h)}$$

Let h_t be a filter of indices, for which (3.8) holds. If h_t is sufficiently small, $\xi_{jh_t} \in K(\alpha_{h_t})$ and one can substitute ξ_{jh_t} into $(\mathcal{P}(\alpha_{h_t}))$:

$$(3.14) \qquad (y_{h_t}, \xi_{jh_t} - y_{h_t})_{1,\Omega(\alpha_{h_t})} \geq (f, \xi_{jh_t} - y_{h_t})_{0,\Omega(\alpha_{h_t})}.$$

Let m be fixed. Then one can write:

$$\begin{aligned}
(y_{h_t}; \xi_{jh_t} - y_{h_t})_{1,\Omega(\alpha_{h_t})} &= (y_{h_t}; \xi_{jh_t} - y_{h_t})_{1,G_m(\alpha)} + \\
&+ (y_{h_t}; \xi_{jh_t} - y_{h_t})_{1,\Omega(\alpha_{h_t})\setminus\Omega(\alpha)} + \\
&+ (y_{h_t}; \xi_{jh_t} - y_{h_t})_{1,(\Omega(\alpha)-G_m(\alpha))\cap\Omega(\alpha_{h_t})} \leq \\
&\leq (y_{h_t}; \xi_{jh_t} - y_{h_t})_{1,G_m(\alpha)} + (y_{h_t}; \xi_{jh_t})_{1,\Omega(\alpha_{h_t})\setminus\Omega(\alpha)} + \\
&+ (y_{h_t}; \xi_{jh_t})_{1,(\Omega(\alpha)-G_m(\alpha))\cap\Omega(\alpha_{h_t})}.
\end{aligned}$$

From (3.8) and (3.13) we conclude that

$$(3.15) \qquad \limsup_{h_t \to 0^+} (y_{h_t}; \xi_{jh_t} - y_{h_t})_{1,G_m(\alpha)} \leq (y; \xi^*_j - y)_{1,G_m(\alpha)}.$$

In view of (3.8) and the fact that $\alpha_{h_t} \rightrightarrows \alpha$ in $\langle 0,1 \rangle$ it holds

$$(3.16) \qquad (y_{h_t}; \xi_{jh_t})_{1,\Omega(\alpha_{h_t})\setminus\Omega(\alpha)} \to 0, \qquad h_t \to 0^+.$$

As

$$|(y_{h_t};\ \xi_{jh_t}-\xi_j^*)_{1,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_t})}| \leq$$

$$\leq c\,\|\xi_{jh_t}-\xi_j^*\|_{1,\Omega(\alpha_{h_t})} \leq c\,h_t\,|\xi_j^*|_{2,\Omega(\alpha_{h_t})} \to 0, \quad h_t \to 0^+,$$

we have:

$$\limsup_{h_t\to 0^+} (y_{h_t};\ \xi_{jh_t})_{1,(\Omega(\alpha)\setminus G_m(\Omega))\setminus\Omega(\alpha_{h_t})} \leq$$

$$\leq \limsup_{h_t\to 0^+} (y_{h_t};\ \xi_j^*)_{1,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_t})} +$$

$$+ \limsup_{h_t\to 0^+} (y_{h_t};\ \xi_{jh_t}-\xi_j^*)_{1,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_t})} =$$

$$= \limsup_{h_t\to 0^+} (y_{h_t};\ \xi_j^*)_{1,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_t})} \leq$$

$$\leq c\,\|\xi_j^*\|_{1,\Omega(\alpha)\setminus G_m(\alpha)},$$

which along with (3.15) and (3.16) yields

$$(3.17)\quad \limsup_{h_t\to 0^+} (y_{h_t};\ \xi_{jh_t}-y_{h_t})_{1,\Omega(\alpha_{h_t})} \leq$$

$$\leq (y;\ \xi_j^*-y)_{1,G_m(\alpha)} + c\,\|\xi_j^*\|_{1,\Omega(\alpha)\setminus G_m(\alpha)}.$$

Also we have

$$(f;\ \xi_{jh_t}-y_{h_t})_{0,\Omega(\alpha_{h_t})} = (f;\ \xi_{jh_t}-y_{h_t})_{0,G_m(\alpha)} +$$

$$+ (f;\ \xi_{jh_t}-y_{h_t})_{0,\Omega(\alpha_{h_t})\setminus\Omega(\alpha)} +$$

$$+ (f;\ \xi_{jh_t}-y_{h_t})_{0,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_t})}.$$

Hence

$$(3.18)\quad \liminf_{h_t\to 0^+} (f;\ \xi_{jh_t}-y_{h_t})_{0,\Omega(\alpha_{h_t})} \geq (f;\ \xi_j^*-y)_{0,G_m(\alpha)} -$$

$$- c\,\{\|f\|_{0,\Omega(\alpha)\setminus G_m(\alpha)} + \|\xi_j^*\|_{1,\Omega(\alpha)\setminus G_m(\alpha)}\}.$$

Taking into account (3.14), (3.17) and (3.18) we see that

$$(3.19)\quad (y;\ \xi_j^*-y)_{1,G_m(\alpha)} + c\,\|\xi_j^*\|_{1,\Omega(\alpha)\setminus G_m(\alpha)} \geq$$

$$\geq (f;\ \xi_j^*-y)_{0,G_m(\alpha)} - c\,\{\|f\|_{0,\Omega(\alpha)\setminus G_m(\alpha)} +$$

$$+ \|\xi_j^*\|_{1,\Omega(\alpha)\setminus G_m(\alpha)}\} .$$

Letting $m \longrightarrow \infty$ in (3.19) we have

$$(y; \xi_j^*-y)_{1,\Omega(\alpha)} \geq (f; \xi_j^*-y)_{0,\Omega(\alpha)}.$$

Finally if $j \longrightarrow \infty$, then

$$(y; \xi-y)_{1,\Omega(\alpha)} \geq (f; \xi-y)_{0,\Omega(\alpha)} \qquad \forall \xi \in K(\alpha),$$

i.e. y is a solution of $(\mathcal{P}(\alpha))$. In a similar way one can prove that z, defined by (3.9) is a solution of $(\mathcal{R}(\alpha))$, taking into account $(3.8)_1$. Let us prove (3.5). As $K(\alpha_{h_j})$ is a convex cone containing zero, one has:

$$(3.20) \qquad (y_{h_j}; y_{h_j})_{1,\Omega(\alpha_{h_j})} = (f; y_{h_j})_{0,\Omega(\alpha_{h_j})}$$

so that

$$(3.21) \qquad \limsup_{h_j \to 0^+} \|y_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} = \limsup_{h_j \to 0^+} (f; y_{h_j})_{0,\Omega(\alpha_{h_j})} =$$

$$= \limsup_{h_j \to 0^+} \{(f; y_{h_j})_{0,G_m(\alpha)} + (f; y_{h_j})_{0,\Omega(\alpha_{h_j})\setminus\Omega(\alpha)} +$$

$$+ (f; y_{h_j})_{0,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_j})}\} \leq (f; y)_{0,G_m(\alpha)} + c(m);$$

$$c(m) = \limsup_{h_j \to 0^+} (f; y_{h_j})_{0,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_j})}$$

holds for any m. It is readily seen that $c(m) \longrightarrow 0$, if $m \longrightarrow \infty$. Indeed, we observe

$$|c(m)| \leq c \|f\|_{\Omega(\alpha)\setminus G_m(\alpha)} \longrightarrow 0; \qquad m \longrightarrow \infty,$$

which, using (3.21), implies

$$(3.22) \qquad \limsup_{h_j \to 0^+} \|y_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} \leq (f; y)_{0,\Omega(\alpha)} = \|y\|^2_{1,\Omega(\alpha)}$$

On the other hand

$$\liminf_{h_j \to 0^+} \|y_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} \geq \liminf_{h_j \to 0^+} \|y_{h_j}\|^2_{1,G_m(\alpha)} \geq \|y\|^2_{1,G_m(\alpha)}$$

holds for $\forall m$, so that

$$\liminf_{h_j \to 0^+} \|y_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} \geq \|y\|^2_{1,\Omega(\alpha)}.$$

Combining (3.22) and this inequality we obtain $(3.5)_1$. Now it will be shown below $(3.5)_2$. Inequality

$$\liminf_{h_j \to 0+} \|z_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} \geq \|z\|^2_{1,\Omega(\alpha)}$$

is obvious. Let $\tilde{y}_{h_j}, \tilde{z}_{h_j} \in L^2(\Omega_{\hat{\beta}})$ be functions, defined by means of

$$\tilde{y}_{h_j} = \begin{cases} y_{h_j} & \text{on } \Omega(\alpha_{h_j}) \\ 0 & \text{on } \Omega_{\hat{\beta}} \setminus \Omega(\alpha_{h_j}) \end{cases} \quad \text{(analogously } \tilde{z}_{h_j}\text{)}.$$

As

$$\|y_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} = \|\tilde{y}_{h_j}\|^2_{0,\Omega_{\hat{\beta}}} + \|\widetilde{\nabla y}_{h_j}\|^2_{0,\Omega_{\hat{\beta}}}$$

$$\|z_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} = \|\tilde{z}_{h_j}\|^2_{0,\Omega_{\hat{\beta}}} + \|\widetilde{\nabla z}_{h_j}\|^2_{0,\Omega_{\hat{\beta}}}$$

elements $\theta_{h_j} = (\tilde{y}_{h_j}, \widetilde{\nabla y}_{h_j})$, $\vartheta_{h_j} = (\tilde{z}_{h_j}, \widetilde{\nabla z}_{h_j}) \in (L^2(\Omega_{\hat{\beta}}))^3$ are bounded as follows from (3.6). Thus there exist subsequences of $\{\theta_{h_j}\}$, $\{\vartheta_{h_j}\}$ (denoted again by the same symbol) and elements $\theta, \vartheta \in (L^2(\Omega_{\hat{\beta}}))^3$ such that

(3.23)
$$\theta_{h_j} \rightharpoonup \theta = (y_1, y_2, y_3) \in (L^2(\Omega_{\hat{\beta}}))^3$$
$$\vartheta_{h_j} \rightharpoonup \vartheta = (z_1, z_2, z_3) \in (L^2(\Omega_{\hat{\beta}}))^3.$$

It is clear that $y_1 \equiv z_1 \equiv 0$ in $\Omega_{\hat{\beta}} \setminus \Omega(\alpha)$,

$$\{y_2, y_3\}|_{\Omega(\alpha)} = \nabla(y_1|_{\Omega(\alpha)})$$

$$\{z_2, z_3\}|_{\Omega(\alpha)} = \nabla(z_1|_{\Omega(\alpha)})$$

and .

$$y_1 = y(\alpha), \quad z_1 = z(\alpha).$$

Furthermore

(3.24)
$$\|\theta_{h_j}\|^2_{0,\Omega_{\hat{\beta}}} = \|y_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} \longrightarrow \|y\|^2_{1,\Omega(\alpha)} = \|\theta\|^2_{0,\Omega_{\hat{\beta}}}.$$

Taking into account (3.24) and (3.23), we get

$$\theta_{h_j} \longrightarrow \theta \text{ in } (L^2(\Omega_{\hat{\beta}}))^3.$$

This, in turn, implies

$$(3.25)\qquad (y_{h_j};z_{h_j})_{1,\Omega(\alpha_{h_j})} = (\theta_{h_j};\vartheta_{h_j})_{0,\Omega_{\hat\beta}} \longrightarrow (\theta,\vartheta)_{0,\Omega_{\hat\beta}} = (y,z)_{1,\Omega(\alpha)}.$$

From (3.25) and $(3.8)_2$ we obtain

$$(3.26)\qquad \limsup_{h_j\to 0+} \|z_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} = \limsup_{h_j\to 0+} \{(y_{h_j};z_{h_j})_{1,\Omega(\alpha_{h_j})} - (f;z_{h_j})_{0,G_m(\alpha)} - (f;z_{h_j})_{0,\Omega(\alpha_{h_j})\setminus\Omega(\alpha)} - (f;z_{h_j})_{0,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_j})}\} \le (y,z)_{1,\Omega(\alpha)} - (f;z)_{0,G_m(\alpha)} + c_1(m),$$

where

$$c_1(m) = \limsup_{h_j\to 0+}\{-(f;z_{h_j})_{0,(\Omega(\alpha)\setminus G_m(\alpha))\cap\Omega(\alpha_{h_j})}\}.$$

Moreover, one can easily verify that $\lim_{m\to\infty} c_1(m) = 0$, which, using (3.26) implies

$$\limsup_{h_j\to 0+} \|z_{h_j}\|^2_{1,\Omega(\alpha_{h_j})} \le (y;z)_{1,\Omega(\alpha)} - (f;z)_{0,\Omega(\alpha)} = \|z\|^2_{1,\Omega(\alpha)}.$$

This completes the proof.

The main result of this section is:

<u>Theorem 3.1</u>. Let $\alpha^*_h \in \mathcal{U}^h_{ad}$ be a solution of $(\mathbb{P})_h$ and let $y^*_h = y^*_h(\alpha^*_h)$ be the corresponding solution of $(\mathcal{P}(\alpha^*_h))_h$. Then there exists a subsequence $\{\alpha^*_{h_j}\} \subset \{\alpha^*_h\}$, and elements $\alpha^* \in \mathcal{U}_{ad}$, $y^* = y(\alpha^*) \in K(\alpha^*)$ such that $\alpha^*_{h_j} \rightrightarrows \alpha^*$ in $\langle 0,1\rangle$; $y^*_{h_j}(\alpha^*_{h_j}) \rightharpoonup y^*(\alpha^*)$ in $H^1(G_m(\alpha^*))$ for any m, where α^* is a solution of $(\mathbb{P})$ and $y^* = y^*(\alpha^*)$ is the solution of the corresponding state inequality $(\mathcal{P}(\alpha^*))$.

Proof. As $\mathcal{U}^h_{ad} \subset \mathcal{U}_{ad}$ and $\mathcal{U}_{ad}$ is a compact set in $C^0(\langle 0,1\rangle)$ - norm, there exists a subsequence of $\{\alpha^*_h\}$ (denoted again by $\{\alpha^*_h\}$) and an element $\alpha^* \in \mathcal{U}_{ad}$ such that

(3.27) $\quad \alpha^*_h \rightrightarrows \alpha^*$ in $\langle 0,1\rangle$.

From Lemma 3.3 it follows the existence of subsequences $\{y^*_{h_j}(\alpha^*_{h_j})\} \subset \{y^*_h(\alpha^*_h)\}$, $\{z^*_{h_j}(y^*_{h_j})\} \subset \{z^*_h(y^*_h)\}$ and of elements $y^* \in K(\alpha^*)$, $z^* \in V(\alpha^*)$ such that

$$(3.28) \qquad \begin{array}{l} y^*_{h_j} \rightharpoonup y^* \\ z^*_{h_j}(y^*_{h_j}) \rightharpoonup z^*(y^*) \end{array} \qquad \text{in } H^1(G_m(\alpha^*)) \text{ for any } m,$$

$$(3.29) \qquad \begin{array}{l} \|y^*_{h_j}\|_{1,\Omega(\alpha^*_{h_j})} \longrightarrow \|y^*\|_{1,\Omega(\alpha^*)} \\ \|z^*_{h_j}\|_{1,\Omega(\alpha^*_{h_j})} \longrightarrow \|z^*(y^*)\|_{1,\Omega(\alpha^*)} \end{array},$$

and moreover $y^*(\alpha^*)$, $z^*(y^*)$ are solutions of $(\mathcal{P}(\alpha^*)),(\mathcal{A}(\alpha^*))$, respectively. By the definition of $(\mathbb{P})_h$ we have

$$(3.30) \qquad \tfrac{1}{2}\|z^*_{h_j}\|^2_{1,\Omega(\alpha^*_{h_j})} = E_{h_j}(\alpha^*_{h_j}) \le E_{h_j}(\alpha_{h_j}) = = \tfrac{1}{2}\|z_{h_j}(y_{h_j})\|^2_{1,\Omega(\alpha_{h_j})} \quad \forall \alpha_{h_j} \in \mathcal{U}^{h_j}_{ad}.$$

Let $\alpha \in \mathcal{U}_{ad}$ be given. Then (see [3]) one can find $\alpha_h \in \mathcal{U}^h_{ad}$ satisfying

$$\alpha_h \rightrightarrows \alpha \qquad \text{in } \langle 0,1\rangle .$$

Let $y_h(\alpha_h)$, $z_h(y_h)$ be solutions of $(\mathcal{P}(\alpha_h))_h$, $(\mathcal{A}(\alpha_h))_h$, respectively, with properties analogous to (3.28) and (3.29). Using (3.29) and similar results for $\{z_{h_j}(y_{h_j})\}$ and passing to the limit with $h_j \longrightarrow 0^+$ in (3.30) we obtain

$$\tfrac{1}{2}\|z^*(y^*)\|^2_{1,\Omega(\alpha^*)} = E(\alpha^*) \le E(\alpha) = \tfrac{1}{2}\|z(y)\|^2_{1,\Omega(\alpha)} \qquad \forall \alpha \in \mathcal{U}_{ad},$$

i.e. $\alpha^* \in \mathcal{U}_{ad}$ is a solution of $(\mathbb{P})$.

4. <u>Numerical realization of</u> $(\mathbb{P})_h$. Let $h, \alpha_h \in \mathcal{U}_{ad}^h$ be fixed. The state inequality $(\mathcal{P}(\alpha_h))_h$, expressed in the matrix form can be written as follows:

(4.1) find $x(\alpha) \in K$ such that $L(x(\alpha)) \leq L(x) \quad \forall x \in K$,

with

$$L(x) = 1/2(x, \mathbf{C}(\alpha)x) - (\mathcal{F}, x)$$

$$K = \{x \in R_n \mid x_i \geq 0 \quad \forall i \in I\},$$

where $\mathbf{C}(\alpha)$ is a stiffness matrix, depending on design parameters

$$\alpha \in R^q;$$

$\mathcal{F}(\alpha)$ ... linear term;

I ... set of indices, corresponding to constraint components of the nodal displacement field x.

Analogously, the matrix form of $(\mathcal{A}(\alpha_h))_h$ is the following:

(4.2) find $z(\alpha) = z(x(\alpha)) \in R^n$ such that
$$\mathbf{C}(\alpha)\, z = \mathbf{C}\, x(\alpha) - \mathcal{F}.$$

Finally, the matrix form of $(\mathbb{P})_h$ can be stated as follows:

(4.3) find $\alpha^* \in U$ such that
$$\varepsilon(\alpha^*) \leq \varepsilon(\alpha) \quad \forall \alpha \in U,$$

where $U = \{\alpha \in R^q \mid l_i(\alpha) \leq d_i \quad \forall i = 1, \dots, s;\ l_{s+1}(\alpha) = 0\}$,

$$\varepsilon(\alpha) = \tfrac{1}{2}(z(\alpha), \mathbf{C}(\alpha)z(\alpha)),$$

l_i, $i=1,\dots,s+1$ are linear forms of α,

$d_i \in R_1$ are given real numbers.

The last equality constraint corresponds to the constant constraint volume. Our aim will be to determine the gradient of ε.

It is known that the mapping $\alpha \longrightarrow x(\alpha)$ is not Fréchet differentiable and the same holds for the mapping $\alpha \longrightarrow \varepsilon(\alpha)$, in general. In our case, however, we prove that the function ε is, due to its special choice, Fréchet differentiable.

Let $\alpha, \tilde{\alpha} \in R^q$ be given and let us denote

$$\varepsilon'(\alpha)\tilde{\alpha} = \lim_{t\to 0^+} \frac{\varepsilon(\alpha+t\tilde{\alpha}) - \varepsilon(\alpha)}{t} .$$

It has been proved that such limit exists as well as

$$x'(\alpha)\tilde{\alpha} = \lim_{t\to 0^+} \frac{x(\alpha+t\tilde{\alpha}) - x(\alpha)}{t} \text{ (see [4],[5])}.$$

Now

(4.4) $$\varepsilon(\alpha) = \tfrac{1}{2}(\mathbf{C}(\alpha)\, z(\alpha), z(\alpha)) = -\tfrac{1}{2}(\mathbf{C}(\alpha)\, z(\alpha), z(\alpha)) + $$
$$+ (\mathbf{C}(\alpha)\, x(\alpha), z(\alpha)) - (\mathcal{F}(\alpha), z(\alpha)),$$

when (4.2) has been taken into account. Let us denote

$$\mathbf{C}'(\alpha)\tilde{\alpha} = ((\nabla_\alpha c_{ij}(\alpha) \cdot \tilde{\alpha}))_{i,j=1}^n$$
$$\mathcal{F}'(\alpha)\tilde{\alpha} = ((\nabla_\alpha \mathcal{F}_i(\alpha) \cdot \tilde{\alpha}))_{i=1}^n$$

Elements of $\mathbf{C}'(\alpha)\tilde{\alpha}$ are given by derivatives of elements of $\mathbf{C}$ at the point α and the direction $\tilde{\alpha}$ (analogously $\mathcal{F}'(\alpha)\tilde{\alpha}$).

Starting from (4.4) and using the previous notation, we can write:

$$\varepsilon'(\alpha)\tilde{\alpha} = -(z'(\alpha)\tilde{\alpha}, C(\alpha)z(\alpha)) -$$
$$- 1/2(C'(\alpha)\tilde{\alpha}\, z(\alpha), z(\alpha)) + (\mathbf{C}'(\alpha)\tilde{\alpha} x(\alpha), z(\alpha)) +$$
$$+ (\mathbf{C}(\alpha) x'(\alpha)\tilde{\alpha}, z(\alpha)) + (\mathbf{C}(\alpha) x(\alpha), z'(\alpha)\tilde{\alpha}) -$$
$$- (\mathcal{F}'(\alpha)\tilde{\alpha}, z(\alpha)) - (\mathcal{F}(\alpha), z'(\alpha)\tilde{\alpha}) =$$
$$= -(z'(\alpha)\tilde{\alpha}, \mathbf{C}(\alpha)z(\alpha) - \mathbf{C}(\alpha)x(\alpha) + \mathcal{F}(\alpha)) +$$
$$+ 1/2(\mathbf{C}'(\alpha)\tilde{\alpha}\, z(\alpha), z(\alpha)) + (\mathbf{C}'(\alpha)\tilde{\alpha}\, x(\alpha), z(\alpha)) +$$
$$+ (\mathbf{C}(\alpha) x'(\alpha)\tilde{\alpha}, z(\alpha)) - (\mathcal{F}'(\alpha)\tilde{\alpha}, z(\alpha)) =$$
$$= -1/2(\mathbf{C}'(\alpha)\tilde{\alpha}\, z(\alpha), z(\alpha)) + (\mathbf{C}'(\alpha)\tilde{\alpha}\, x(\alpha), z(\alpha)) +$$
$$+ (x'(\alpha)\tilde{\alpha}, \mathbf{C}(\alpha) z(\alpha)) - (\mathcal{F}'(\alpha), z(\alpha)),$$

making use of (4.2).

Now we derive another equivalent form of the term $(x'(\alpha)\tilde{\alpha}, \mathbf{C}(\alpha)z(\alpha))$, where $x'(\alpha)\tilde{\alpha}$ will not appear explicitly. To this end we present an equivalent formulation of (4.1), using Lagrange multipliers. It is known that $x(\alpha)$ is a solution of (4.1) if and only if $\exists \lambda \geq 0$, such that

$$(4.5)\qquad \begin{aligned} c_{ij}(\alpha)x_j(\alpha) &= \mathcal{F}_i(\alpha) \qquad \forall i\notin I \\ c_{ij}(\alpha)x_j(\alpha) &= \mathcal{F}_i(\alpha) + \lambda_i(\alpha) \qquad \forall i\in I \end{aligned}$$

and

$$(4.6)\qquad \sum_{j\in I} \lambda_j(\alpha)x_{i_j}(\alpha) = 0,$$

λ_j are multipliers, associated with the constraint $x(\alpha)\in K$.

From (4.2),(4.5) and (4.6) we see that

$$(x'(\alpha)\tilde{\alpha}, C(\alpha)z(\alpha)) = (x'(\alpha)\tilde{\alpha}\ ,\ C(\alpha)x(\alpha)-\mathcal{F}(\alpha)) = \sum_{j\in I} x'_{i_j}(\alpha)\tilde{\alpha}_{i_j}\lambda_j\ .$$

We prove that the last sum is equal to zero. Indeed:

$$\sum_{j\in I} x'_{i_j}(\alpha)\tilde{\alpha}_{i_j}\lambda_j = \sum_{j\in I_0} x'_{i_j}(\alpha)\tilde{\alpha}_{i_j}\lambda_j,$$

where $I_0\subseteq I$ and such that $j\in I_0 \Longleftrightarrow \lambda_j>0$. Let λ_j^t be Lagrange multipliers associated with the design parameters $\alpha+t\tilde{\alpha}$, $t>0$, i.e. λ_j^t satisfy the same relations (4.5),(4.6), only with α replaced by $\alpha+t\tilde{\alpha}$. As λ_j^t are continuous functions of t, then if

$$j\in I_0 \quad \text{also} \quad \lambda_j^t>0$$

and the corresponding constraint is active, i.e. $x_{i_j}(\alpha+t\tilde{\alpha}) = 0$ so that $x'_{i_j}(\alpha)\tilde{\alpha} = 0$.

Summing up all our considerations we see that

$$\mathcal{E}'(\alpha)\tilde{\alpha} = -1/2(C'(\alpha)\tilde{\alpha}\, z(\alpha), z(\alpha)) + (C'(\alpha)\tilde{\alpha}\ x(\alpha), z(\alpha)) - (\mathcal{F}'(\alpha)\tilde{\alpha}, z(\alpha))\ .$$

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