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No upper bound for cardinalities of Tychonoff C.C.C. spaces with a G_δ-diagonal exists (an answer to J. Ginsburg and R. G. Woods' question)

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NO UPPER BOUND FOR CARDINALITIES OF TYCHONOFF C.C.C. SPACES WITH A G_δ-DIAGONAL EXISTS (AN ANSWER TO J. GINSBURG AND R. G. WOODS' QUESTION)

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Abstract. It is proved that every ind-zero-dimensional Tychonoff space X with a G_δ-diagonal can be embedded as a closed subspace in an ind-zero-dimensional Tychonoff space Y with a G_δ-diagonal satisfying the countable chain condition. In particular, for any cardinal τ there exists a Tychonoff c.c.c. space Z with a G_δ-diagonal such that $|Z| \geq \tau$. This settles the question advanced by J.Ginsburg and R.G.Woods and repeated by A.V.Arhangel'skii as well.

Key words and phrases: Tychonoff space, countable chain condition (=c.c.c.), space with a G_δ-diagonal.

Classification: 54A25, 54C25.

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1. Introduction.

J.Ginsburg and R.G.Woods showed that the cardinality of a collectionwise Hausdorff topological space with a G_δ-diagonal satisfying the countable chain condition does not exceed $2^{\aleph_0}$ ([1], Corollary 2.3). They also constructed an example of a Hausdorff non regular space with a G_δ-diagonal satisfying the countable chain condition of cardinality 2^{c} ([1], Example 2.4) and raised the following

Question 1.1 ([1], Question 2.5). Is it true that the cardinality of a regular space with a G_δ-diagonal satisfying the countable chain condition does not exceed $2^{\aleph_0}$?

Question 1.1 was also mentioned in A.V.Arhangel'skii's survey ([2], open problem 16). In this paper we give a complete answer to Question 1.1 (see Corollary 3.3).

2. Notations and terminology.

Notations and terminology follow [3]. A space means topological space. All spaces are assumed to be Tychonoff (= completely regular + T_1). A space X is zero-dimensional iff $\operatorname{ind} X = 0$, i.e., X has a base consisting of closed-and-open sets. A space X is said to have a G_δ-diagonal iff the diagonal $\Delta = \{(x,x): x \in X\} \subset X \times X$ is a G_δ-set in $X \times X$. Symbols $|X|$, $w(X)$, $\psi(X)$ and $\chi(X)$ denote the cardinality, weight, pseudocharacter and character respectively. A space X is said to satisfy the countable chain condition iff the Souslin number $c(X) = \sup\{|\gamma| : \gamma$ is a family of pairwise disjoint nonempty open subsets of $X\}$ of the space X is countable. A space X is left-separated iff there exists a well-order $<$ on X such that every left interval $X_{\to x} = \{y \in X : y < x\}$ is closed in X. As usual cardinals are identified with initial ordinals. For a set X let $\exp X = \{F : F$ is a subset of $X\}$.

3. Main results.

Theorem 3.1. Every zero-dimensional Tychonoff space X with a G_δ-diagonal can be embedded as a closed subspace in a zero-dimensional Tychonoff space Y with a G_δ-diagonal satisfying the countable chain condition.

Theorem 3.2. If in addition to the assumptions of Theorem 3.1 the space X is left-separated, then so is the space Y.

Corollary 3.3. For any cardinal τ there exists a Tychonoff (left-separated) space Z with a G_δ-diagonal satisfying the countable chain condition such that $|Z| \geq \tau$.

Corollary 3.3 gives a complete answer to Question 1.1.

Theorem 3.4. Every zero-dimensional Tychonoff space X can be embedded as a closed subspace in a zero-dimensional Tychonoff space Y such that $\psi(Y) \leq \psi(X)$ and $c(Y) = \aleph_0$.

Theorem 3.5. If in addition to the assumptions of Theorem 3.4 the space X is left-separated, then so is the space Y.

Corollary 3.6. (M.J.Zeitlin [4]). There exists a Tychonoff space Z with a G_δ-diagonal without one-to-one continuous mapping onto a Hausdorff first-countable space.

Corollary 3.6 gives an answer to a question of A.V.Arhangel'skii. It is worth noticing that our space Z constructed in Corollary 3.6 satisfies the countable chain condition while the corresponding space of M.J.Zeitlin doesn't.

4. Proofs.

The constructions are similar to those described by the author in [5].

We need the following well-known

Proposition 4.1. ([6]). For any space X the following conditions are equivalent:

(i) X has a G_δ-diagonal,

(ii) there exists a sequence $\{\gamma_n : n \in \omega\}$ of open covers of X such that for any distinct points $x, y \in X$ one can find an $n \in \omega$ with $\{U \in \gamma_n : \{x, y\} \subset U\} = \emptyset$.

Proof of Theorem 3.1. For every $\alpha < \omega_2$ by transfinite induction we construct the structure $\Xi_\alpha = \langle X_\alpha, \mathcal{B}_\alpha, \overline{\mathcal{B}}_\alpha, \pi_\alpha, \mathcal{F}_\alpha, \theta_\alpha, \mathcal{E}_\alpha \rangle$ with the properties (1)-(9).

(1) $X_\alpha, \mathcal{B}_\alpha, \overline{\mathcal{B}}_\alpha$ are sets, $\mathcal{B}_\alpha \cap \overline{\mathcal{B}}_\alpha = \emptyset$, $\pi_\alpha : \mathcal{B}_\alpha \to \overline{\mathcal{B}}_\alpha$ is a one-to-one mapping, $\mathcal{F}_\alpha = \{F : F \subset \mathcal{B}_\alpha \cup \overline{\mathcal{B}}_\alpha, F$ is finite and $F \neq \emptyset\}$,

(2) $\theta_\alpha : \mathcal{B}_\alpha \cup \overline{\mathcal{B}}_\alpha \to \exp X_\alpha$ is a mapping satisfying the following condition:

if $b \in \overline{\mathcal{B}}_\alpha$, then $\theta_\alpha(b) = X_\alpha \setminus \theta_\alpha(\pi_\alpha^{-1}(b))$,

(3) $\mathcal{E}_\alpha = \{\mathcal{E}_{\alpha,n} : n \in \omega\}$; the family $\mathcal{E}_\alpha$ must satisfy the following conditions:

(3a) $\mathcal{E}_{\alpha,n} \subset \mathcal{B}_\alpha$ for every $n \in \omega$,

(3b) $\bigcup\{\theta_\alpha(b) : b \in \mathcal{E}_{\alpha,n}\} = X_\alpha$ for any $n \in \omega$,

(3c) for every two distinct points $x, y \in X_\alpha$ there exists an $n \in \omega$ (which depends on x and y) such that $\{b \in \mathcal{E}_{\alpha,n} : \{x,y\} \subset \theta_\alpha(b)\} = \emptyset$,

(3d) $\mathcal{E}_{\alpha,n} \cap \mathcal{E}_{\alpha,m} = \emptyset$ as soon as $n \neq m$.

In our further constructions the properties (4)-(8) must hold in case $\beta < \alpha$.

(4) $X_\beta \subset X_\alpha$, $\mathcal{B}_\beta \subset \mathcal{B}_\alpha$, $\overline{\mathcal{B}}_\beta \subset \overline{\mathcal{B}}_\alpha$, $\pi_\alpha|_{\mathcal{B}_\beta} = \pi_\beta$,

(5) if $b \in \mathcal{B}_\beta$, then $\theta_\alpha(b) \cap X_\beta = \theta_\beta(b)$,

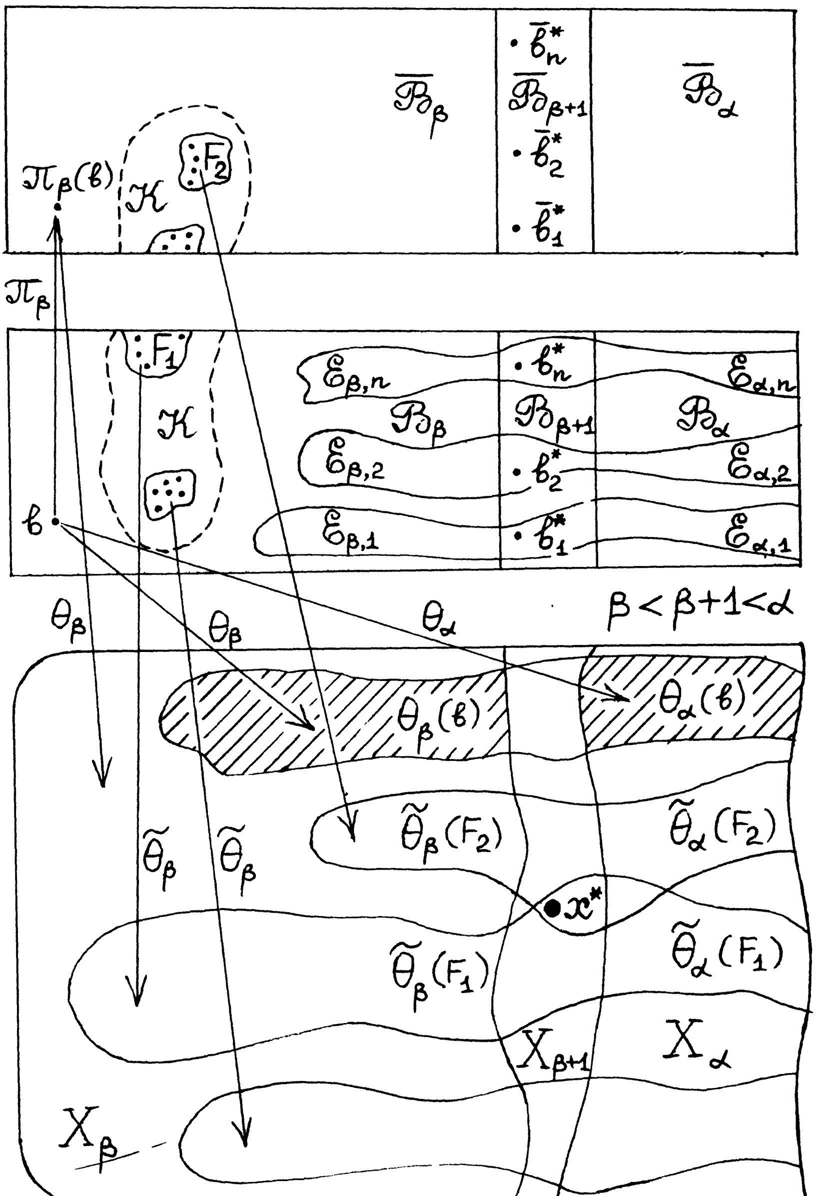
(6) $\theta_\alpha(b) \cap X_\beta = \emptyset$ for each $b \in \mathcal{B}_\alpha \setminus \mathcal{B}_\beta$,

(7) if $x \in X_\alpha \setminus X_\beta$, then there is a $b \in \mathcal{B}_\alpha \setminus \mathcal{B}_\beta$ with $x \in \theta_\alpha(b)$,

(8) $\mathcal{E}_{\beta,n} \subset \mathcal{E}_{\alpha,n}$ and $\mathcal{E}_{\alpha,n} \setminus \mathcal{E}_{\beta,n} \subset \mathcal{B}_\alpha \setminus \mathcal{B}_\beta$ as soon as $n \in \omega$.

<u>Convention 4.2</u>. From now on we let $\widetilde{\theta}_\alpha(F) = \bigcap\{\theta_\alpha(b) : b \in F\}$ for every $F \in \mathcal{F}_\alpha$.

(9) Let $\alpha < \omega_2$. Let $\mathcal{K}$ be a family with $\mathcal{K} \subset \mathcal{F}_\alpha$, $|\mathcal{K}| =$



$\overline{b}_n^*$
$\overline{\mathcal{B}}_\beta$
$\overline{\mathcal{B}}_{\beta+1}$
$\overline{\mathcal{B}}_\alpha$
$\overline{b}_2^*$
$\pi_\beta(b)$
$\mathcal{K}$
F_2
$\overline{b}_1^*$
π_β
F_1
b_n^*
$\mathcal{E}_{\beta,n}$
$\mathcal{E}_{\alpha,n}$
$\mathcal{K}$
$\mathcal{B}_\beta$
$\mathcal{B}_{\beta+1}$
$\mathcal{B}_\alpha$
$\mathcal{E}_{\beta,2}$
b_2^*
$\mathcal{E}_{\alpha,2}$
b
$\mathcal{E}_{\beta,1}$
b_1^*
$\mathcal{E}_{\alpha,1}$
θ_β
θ_β
θ_α
$\beta < \beta+1 < \alpha$
$\theta_\beta(b)$
$\theta_\alpha(b)$
$\tilde{\theta}_\beta(F_2)$
$\tilde{\theta}_\alpha(F_2)$
$\tilde{\theta}_\beta$
$\tilde{\theta}_\beta$
x^*
$\tilde{\theta}_\beta(F_1)$
$\tilde{\theta}_\alpha(F_1)$
$X_{\beta+1}$
X_α
X_β

$=\omega_1$ and $\widetilde{\theta}_\alpha(F)\neq\emptyset$ for each $F\in\mathcal{K}$. Then one can find two distinct elements $F_1, F_2\in\mathcal{K}$ with $\widetilde{\theta}_{\alpha+1}(F_1)\cap\widetilde{\theta}_{\alpha+1}(F_2)\neq\emptyset$.

Convention 4.3. Henceforth we fix a symbol Ξ_α for denoting the structure $\Xi_\alpha=\langle X_\alpha, \mathcal{B}_\alpha, \overline{\mathcal{B}}_\alpha, \pi_\alpha, \mathcal{F}_\alpha, \theta_\alpha, \mathcal{E}_\alpha\rangle$ and we will use it only in that meaning.

One can think of each X_α as being a piece of our future space Y, of a family $\{\theta_\alpha(b): b\in\mathcal{B}_\alpha\cup\overline{\mathcal{B}}_\alpha\}$ as being a subbase for a topology on X_α. Each $b\in\mathcal{B}_\alpha\cup\overline{\mathcal{B}}_\alpha$ is a name for the subbase set $\theta_\alpha(b)$. For every $b\in\mathcal{B}_\alpha$, $\pi_\alpha(b)$ is a name for the set $X_\alpha\setminus\theta_\alpha(b)$, so (2) makes each subbase set $\theta_\alpha(b)$ closed-and-open in X_α. (3c) assures T_1 of X_α; (3a)-(3c) guarantee the existence of a sequence of open covers of X_α satisfying the condition (ii) of Proposition 4.1, which provides a G_δ-diagonal of X_α; (6) makes each X_α closed in Y; (9) is responsible for c.c.c. of Y.

A basis of induction. Let X be an original space and let $\{U_b: b\in B\}$ be a base on X consisting of closed-and-open sets. Since X has a G_δ-diagonal, one can find a family $\{\gamma_n: n\in\omega\}$ satisfying the condition (ii) of Proposition 4.1. Since X is zero-dimensional, we can think of each γ_n as consisting of closed-and-open sets. Let $\gamma_n=\{V_{b,n}: b\in B_n\}$, where $B_n\cap B_m=\emptyset$ whenever $n\neq m$, and $B\cap(\cup\{B_n: n\in\omega\})=\emptyset$.

Now define the structure Ξ_0. Let $X_0=X$, $\mathcal{B}_0=\cup\{B_n: n\in\omega\}\cup B$, $\mathcal{E}_{0,n}=B_n$, $\mathcal{E}_0=\{\mathcal{E}_{0,n}: n\in\omega\}$; if $b\in B$, then $\theta_0(b)=U_b$ and if $b\in B_n$, then $\theta_0(b)=V_{b,n}$. We choose sets $\overline{\mathcal{B}}_0, \mathcal{F}_0$ and a mapping π_0 in accordance with (1) and define a mapping θ_0 on $\overline{\mathcal{B}}_0$ by letting $\theta_0(b)=X_0\setminus\theta_0(\pi_0^{-1}(b))$ for every $b\in\overline{\mathcal{B}}_0$. One can easily verify that the $\boxminus_0$ so constructed satisfies the conditions (1)-(3).

Convention 4.4. Everywhere below we will identify the sets X and X_0.

Remark 4.5. By construction the family $\{\theta_0(b): b\in \in\mathcal{B}_0 \cup \overline{\mathcal{B}}_0\}$ consists of closed-and-open subsets of X and the topology generated by taking it as a subbase, coincides with the original topology on X.

An inductive step. (I) For limit ordinals an inductive step is carried out by

Lemma 4.6. Let α^* be a limit ordinal and suppose that the structures $\boxminus_\beta$ with the properties (1)-(8) have already been defined for every $\beta<\alpha^*$.

Let $X_{\alpha^*}=\bigcup\{X_\beta:\beta<\alpha^*\}$, $\mathcal{B}_{\alpha^*}=\bigcup\{\mathcal{B}_\beta:\beta<\alpha^*\}$, $\overline{\mathcal{B}}_{\alpha^*}= =\bigcup\{\overline{\mathcal{B}}_\beta:\beta<\alpha^*\}$, $\mathcal{F}_{\alpha^*}=\bigcup\{\mathcal{F}_\beta:\beta<\alpha^*\}$, $\mathcal{E}_{\alpha^*,n}=\bigcup\{\mathcal{E}_{\beta,n}:\beta<\alpha^*\}$, $\mathcal{E}_{\alpha^*}=\{\mathcal{E}_{\alpha^*,n}:n\in\omega\}$. Determine the map $\pi_{\alpha^*}:\mathcal{B}_{\alpha^*}\to\overline{\mathcal{B}}_{\alpha^*}$ by $\pi_{\alpha^*}(b)= =\pi_\beta(b)$, where β is any ordinal with $\beta<\alpha^*$ and $b\in\mathcal{B}_\beta$. Define the map $\theta_{\alpha^*}:\mathcal{B}_{\alpha^*}\cup\overline{\mathcal{B}}_{\alpha^*}\to\exp X_{\alpha^*}$ by letting $\theta_{\alpha^*}(b)= =\bigcup\{\theta_\beta(b):\beta<\alpha^*,\ b\in\mathcal{B}_\beta\cup\overline{\mathcal{B}}_\beta\}$ for each $b\in\mathcal{B}_{\alpha^*}\cup\overline{\mathcal{B}}_{\alpha^*}$.

Then the structure $\boxminus_{\alpha^*}$ satisfies the properties (1)-(8).

Proof of Lemma 4.6. A verification of the properties (1), (2), (4)-(8) is trivial and can be omitted. Let us verify (3).

(3a) is obvious.

(3b) Arbitrarily choose $x\in X_{\alpha^*}$ and $n\in\omega$. Then $x\in X_\beta$ for some $\beta<\alpha^*$. By (3b) for this β, one can find a $b\in\mathcal{E}_{\beta,n}$ with $x\in\theta_\beta(b)$. But $\mathcal{E}_{\beta,n}\subset\mathcal{E}_{\alpha^*,n}$ and $\theta_\beta(b)\subset\theta_{\alpha^*}(b)$ imply $x\in\bigcup\{\theta_{\alpha^*}(b): b\in\mathcal{E}_{\alpha^*,n}\}$.

(3c) Let $x,y\in X_{\alpha^*}$, $x\neq y$. Then $x,y\in X_\beta$ for some $\beta<$

$<\alpha^*$. By (3c) for β, there is an $n\in\omega$ such that $\{b\in\mathcal{E}_{\beta,n}:\{x,y\}\subset\theta_\beta(b)\}=\emptyset$. The properties (6) and (8) imply $\theta_{\alpha^*}(b)\cap X_\beta=\emptyset$ whenever $b\in\mathcal{E}_{\alpha^*,n}\setminus\mathcal{E}_{\beta,n}$. Now from (6) it follows that $\{b\in\mathcal{E}_{\alpha^*,n}:\{x,y\}\subset\theta_{\alpha^*}(b)\}\subset\{b\in\mathcal{E}_{\beta,n}:\{x,y\}\subset\theta_\beta(b)\}=\emptyset$.

(3d) Let $b\in\mathcal{B}_{\alpha^*}$ be chosen arbitrarily. Then $b\in\mathcal{B}_\beta$ for some $\beta<\alpha^*$. By (8), we have $\{n\in\omega: b\in\mathcal{E}_{\gamma,n}\}=\{n\in\omega: b\in\mathcal{E}_{\beta,n}\}$ provided $\beta<\gamma<\alpha^*$ and therefore $\{n\in\omega: b\in\mathcal{E}_{\alpha^*,n}\}=\{n\in\omega: b\in\mathcal{E}_{\beta,n}\}$. But the last set is either empty or consists of a single element since (3d) holds for β.

The proof of Lemma 4.6 is complete.

(II) Let $\alpha^*=\beta^*+1$. The step from β^* to α^* is done with the help of an auxiliary inductive construction.

<u>An auxiliary inductive construction</u>. Let $\mathcal{C}=\{\mathcal{K}:\mathcal{K}\subset\mathcal{F}_{\beta^*}, |\mathcal{K}|=\omega_1$ and $\tilde{\theta}_{\beta^*}(F)\neq\emptyset$ for all $F\in\mathcal{K}\}$. Enumerate elements of $\mathcal{C}$ by non-limit ordinals not exceeding some limit ordinal δ:

$(\bigstar)$ $\qquad \mathcal{C}=\{\mathcal{K}_\alpha: 0<\alpha<\delta,\ \alpha$ is a non-limit ordinal$\}$.

Let $X_0=X_{\beta^*}$, $\mathcal{B}_0=\mathcal{B}_{\beta^*}$, $\overline{\mathcal{B}}_0=\overline{\mathcal{B}}_{\beta^*}$, $\pi_0=\pi_{\beta^*}$, $\mathcal{F}_0=\mathcal{F}_{\beta^*}$, $\theta_0=\theta_{\beta^*}$, $\mathcal{E}_0=\mathcal{E}_{\beta^*}$. Starting from Ξ_0, by transfinite induction for every $\alpha<\delta$ define the structure Ξ_α with the properties (1)-(8) satisfying also the following condition:

$(*_\alpha)$ there are $F_1,F_2\in\mathcal{K}_\alpha$ with $\tilde{\theta}_\alpha(F_1)\cap\tilde{\theta}_\alpha(F_2)\neq\emptyset$ whenever α is a non-limit ordinal.

For limit ordinals an inductive step is carried out by Lemma 4.6. In case of non-limit ordinals we make use of the following

<u>Lemma 4.7</u>. Let $\alpha=\beta+1$ and suppose that the structure $\boxminus_\beta$ has the properties (1)-(8). Then there exists the structure $\boxminus_\alpha$ having not only the properties (1)-(8) but the property $(*_\alpha)$ as well.

<u>Proof of Lemma 4.7</u>. Here we must "grow out" "old" subbase sets $\mathcal{B}_\beta \cup \overline{\mathcal{B}}_\beta$ in such a way that some $\widetilde{\theta}_\alpha(F_1)$ and $\widetilde{\theta}_\alpha(F_2)$ with $F_1, F_2 \in \mathcal{K}_\alpha$ would be forced to meet. However, we are not free on it because of the property (2). Indeed, if $b \in \in F_1$ and $\pi_\beta(b) \in F_2$ for some $b \in \mathcal{B}_\beta$, then $\widetilde{\theta}_\alpha(F_1) \cap \cap \widetilde{\theta}_\alpha(F_2)=\emptyset$ no matter how "growing out" is done. So to choose F_1 and F_2 such as in $(*_\alpha)$, we must eliminate the case described above.

Now, let us turn to details. Let $X_\alpha = X_\beta \cup \{x^*\}$, where $x^* \notin X_\beta$ and let $\mathcal{B}_\alpha = \mathcal{B}_\beta \cup \{b_n^* : n \in \omega\}$, $\overline{\mathcal{B}}_\alpha = \overline{\mathcal{B}}_\beta \cup \cup \{\overline{b}_n^* : n \in \omega\}$, where $(\{b_n^* : n\in\omega\} \cup \{\overline{b}_n^* : n \in \omega\}) \cap (\mathcal{B}_\beta \cup \cup \overline{\mathcal{B}}_\beta) = \emptyset$, $\{b_n^* : n\in\omega\} \cap \{\overline{b}_n^* : n\in\omega\} = \emptyset$ and $b_n^* \neq b_m^*$, $\overline{b}_n^* \neq \neq \overline{b}_m^*$ whenever $n \neq m$. Let $\pi_\alpha(b) = \pi_\beta(b)$ as soon as $b \in \mathcal{B}_\beta$ and $\pi_\alpha(b) = \overline{b}_n^*$ in case $b = b_n^*$. Put also $\mathcal{F}_\alpha = = \{F : F \subset \mathcal{B}_\alpha \cup \overline{\mathcal{B}}_\alpha, F$ is finite and $F \neq \emptyset\}$, $\mathcal{E}_{\alpha,n} = = \mathcal{E}_{\beta,n} \cup \{b_n^*\}$ for all $n \in \omega$, $\mathcal{E}_\alpha = \{\mathcal{E}_{\alpha,n} : n \in \omega\}$.

Since $\mathcal{K}_\alpha \subset \mathcal{F}_0$, $|\mathcal{K}_\alpha| = \omega_1$, applying the standard Δ-system arguments, one can find a $J \in \mathcal{F}_0 \cup \{\emptyset\}$ and a $\mathcal{K}'_\alpha \subset \mathcal{K}_\alpha$ with $|\mathcal{K}'_\alpha| = \omega_1$ and $F' \cap F'' = J$ for all pairs $F', F'' \in \in \mathcal{K}'_\alpha$. Choose an $F_1 \in \mathcal{K}'_\alpha$. Suppose that $F_1 = \{b_1, \dots, b_k, \pi_0(b_{k+1}), \dots, \pi_0(b_m)\}$, where $b_1, \dots, b_m \in \mathcal{B}_0$ and $\{b_1, \dots, b_k\} \cap \{b_{k+1}, \dots, b_m\} = \emptyset$ (the last follows from (2)). Let $P = \{b_1, \dots, b_m, \pi_0(b_1), \dots, \pi_0(b_m)\}$. Then there is a finite set $\mathcal{M} \subset \mathcal{K}'_\alpha$ such that $F \cap P = J$ for every $F \in \mathcal{K}'_\alpha \setminus \mathcal{M}$. Pick an $F_2 \in \mathcal{K}'_\alpha \setminus \mathcal{M}$. Suppose that $F_2 = \{a_1, \ldots, a_s, \pi_0(a_{s+1}), \ldots, \pi_0(a_t)\}$, where $a_1, \ldots, a_t \in \mathcal{B}_0$ and $\{a_1, \ldots, a_s\} \cap \{a_{s+1}, \ldots, a_t\} = \emptyset$. Then we have $\{a_1, \ldots, a_t\} \cap \{b_1, \ldots, b_m\} = J'$, where $J' = \emptyset$ if $J = \emptyset$, and $J' = \{c_1, \ldots, c_\ell, c_{\ell+1}, \ldots, c_r\}$ if $J = \{c_1, \ldots, c_\ell, \pi_0(c_{\ell+1}), \ldots, \pi_0(c_r)\}$ (in the last case $c_i \neq c_j$ as soon as $i \neq j, i \leq r, j \leq r$). Define a map $\theta_\alpha : \mathcal{B}_\alpha \to \exp X_\alpha$ by letting

$$\theta_\alpha(b) = \begin{cases} \theta_\beta(b) \cup \{x^*\} & \text{if } b \in \{a_1, \ldots, a_s, b_1, \ldots, b_k\}, \\ \theta_\beta(b) & \text{if } b \in \mathcal{B}_\beta \setminus \{a_1, \ldots, a_s, b_1, \ldots, b_k\}, \\ \{x^*\} & \text{if } b \in \{b_n^* : n \in \omega\}. \end{cases}$$

Extend θ_α over $\overline{\mathcal{B}}_\alpha$ by letting $\theta_\alpha(b) = X_\alpha \setminus \theta_\alpha(\pi_\alpha^{-1}(b))$ for every $b \in \overline{\mathcal{B}}_\alpha$.

The structure Ξ_α is thus completely defined. The properties (1), (2), and (4)-(8) follow directly from our constructions. Besides $x^* \in \widetilde{\theta}_\alpha(F_1) \cap \widetilde{\theta}_\alpha(F_2) \neq \emptyset$ and hence $(*_\alpha)$ holds. We need only to verify (3). Items (a), (b) and (d) are obvious. To verify (c), consider two cases.

(i) $\{x, y\} \subset X_\beta$. The property (3c) for β implies $\{b \in \mathcal{E}_{\beta,n} : \{x, y\} \subset \theta_\beta(b)\} = \emptyset$ for some $n \in \omega$. But $\mathcal{E}_{\alpha,n} \setminus \mathcal{E}_{\beta,n} = \{b_n^*\}$ and $\theta_\alpha(b_n^*) = \{x^*\}$, $x^* \notin X_\beta$. Now it follows from (5) that $\{b \in \mathcal{E}_{\alpha,n} : \{x, y\} \subset \theta_\alpha(b)\} = \emptyset$.

(ii) One of points x and y is x^*, say $x = x^*$. (3d) for β implies $\{a_1, \ldots, a_s, b_1, \ldots, b_k\} \cap \mathcal{E}_{\beta,n} = \emptyset$ for some $n \in \omega$. But $(\mathcal{E}_{\alpha,n} \setminus \mathcal{E}_{\beta,n}) \cap \mathcal{B}_\beta = \emptyset$ by (8), and $\{a_1, \ldots, a_s, b_1, \ldots, b_k\} \subset \mathcal{B}_0 \subset \mathcal{B}_\beta$. Therefore $\{a_1, \ldots, a_s,$

$b_1, \dots, b_k\} \cap \mathcal{E}_{\alpha,n} = \emptyset$. It follows from the definition of θ_α and the property (6) for α that $\{b \in \mathcal{B}_\alpha : \{x^*, y\} \subset \subset \theta_\alpha(b)\} = \{a_1, \dots, a_s, b_1, \dots, b_k\}$ and $\{b \in \mathcal{E}_{\alpha,n} : \{x^*, y\} \subset \theta_\alpha(b)\} = \emptyset$ as required.

Lemma 4.7 is completely proved.

The auxiliary inductive construction having been done, applying Lemma 4.6, we should define the structure Ξ_δ and should determine $X_{\alpha^*} = X_\delta$, $\mathcal{B}_{\alpha^*} = \mathcal{B}_\delta$, $\overline{\mathcal{B}}_{\alpha^*} = \overline{\mathcal{B}}_\delta$, $\pi_{\alpha^*} = \pi_\delta$, $\mathcal{F}_{\alpha^*} = \mathcal{F}_\delta$, $\theta_{\alpha^*} = \theta_\delta$, $\mathcal{E}_{\alpha^*} = \mathcal{E}_\delta$. The structure Ξ_{α^*} is as required. The properties (1)-(8) hold by Lemma 4.6. Since for every non-limit ordinal $\alpha < \delta$ the property $(*_\alpha)$ holds, $(\bigstar)$ implies that the property (9) is fulfilled for the ordinal β^*. This completes the inductive construction.

The inductive construction having been done, apply Lemma 4.6 to obtain Ξ_{ω_2}

<u>Convention 4.8</u>. Let $Y = X_{\omega_2}$. Henceforth for the sake of simplicity we omit the index ω_2 in $\mathcal{B}_{\omega_2}$, $\overline{\mathcal{B}}_{\omega_2}$, π_{ω_2}, $\mathcal{F}_{\omega_2}$, θ_{ω_2}, $\mathcal{E}_{\omega_2,n}$ and $\mathcal{E}_{\omega_2}$.

Consider the family $\eta = \{\theta(b) : b \in \mathcal{B} \cup \overline{\mathcal{B}}\}$ and the topology $\mathcal{T}$ generated by it as a subbase. The space $(Y, \mathcal{T})$ is that we need.

<u>Proposition 4.9</u>. The space $(Y, \mathcal{T})$ is zero-dimensional and Tychonoff.

<u>Proof</u>. All elements of η are closed-and-open in $(Y, \mathcal{T})$ since (2) and (5) imply that $Y \setminus \theta(b) = \theta(\pi(b))$ for any $b \in \mathcal{B}$. Since η is a subbase for the topology $\mathcal{T}$, the space $(Y, \mathcal{T})$ is zero-dimensional. By (3c), the family η separates points of Y, so $(Y, \mathcal{T})$ is a Tychonoff space.

Proposition 4.10. The set X_0 is closed in $(Y,\mathcal{T})$. Moreover, this is true for all X_β with $\beta<\omega_2$.

Proof. Suppose $\beta<\omega_2$ and $x\in Y\setminus X_\beta$. Then $x\in X_\alpha\setminus X_\beta$ for some $\alpha<\omega_2$. Applying (7), one can find a $b\in\mathcal{B}_\alpha\setminus\mathcal{B}_\beta$ with $x\in\theta_\alpha(b)$. Now (6) implies $\theta_\alpha(b)\cap X_\beta=\emptyset$ and hence $\theta(b)\cap X_\beta=\emptyset$ by (5).

Proposition 4.11. The topology $\mathcal{T}$ induces on $X_0=X$ the original topology of X.

Proof. If $b\in\mathcal{B}\setminus\mathcal{B}_0$, then by (2), (5), (6), $\theta(b)\cap X=\emptyset$ and $\theta(\pi(b))\cap X=(Y\setminus\theta(b))\cap X=X$. Therefore, $\theta(b)\cap$ $\cap X\in\{\emptyset,X\}$ provided $b\in\mathcal{B}\cup\overline{\mathcal{B}}\setminus(\mathcal{B}_0\cup\overline{\mathcal{B}}_0)$. Now suppose that $b\in\mathcal{B}_0\cup\overline{\mathcal{B}}_0$. By (5), $\theta(b)\cap X_0=\theta_0(b)$, and by Remark 4.5, the topology, generated on X by taking the family $\{\theta_0(b):$ $b\in\mathcal{B}_0\cup\overline{\mathcal{B}}_0\}$ as a subbase, coincides with the original topology of X. So $\mathcal{T}$ induces the original topology of X.

Proposition 4.12. The space $(Y,\mathcal{T})$ has a G_δ-diagonal.

Proof. By our construction, each $\gamma_n=\{\theta(b):b\in\mathcal{E}_n\}$ is an open cover of Y. Let $x,y\in Y, x\neq y$. Then $x,y\in X_\alpha$ for some $\alpha<\omega_2$. (3c) implies that $\{b\in\mathcal{E}_{\alpha,n}:\{x,y\}\subset\theta_\alpha(b)\}=$ $=\emptyset$ for some $n\in\omega$. By (6) and (8), we have $\theta(b)\cap X_\alpha=\emptyset$ whenever $b\in\mathcal{E}_\alpha\setminus\mathcal{E}_{\alpha,n}$. From (5) it follows that $\{b\in\mathcal{E}_n:$ $\{x,y\}\subset\theta(b)\}=\emptyset$. Therefore, the family $\{\gamma_n: n\in\omega\}$ satisfies the property (ii) of Proposition 4.1 by which we conclude that the space $(Y,\mathcal{T})$ has a G_δ-diagonal.

Proposition 4.13. The space $(Y,\mathcal{T})$ satisfies c.c.c.

Proof. Put $\tilde{\theta}(F)=\bigcap\{\theta(b):b\in F\}$ for any $F\in\mathcal{F}$. The family $\lambda=\{\tilde{\theta}(F):F\in\mathcal{F}\}$ is a base for the topology $\mathcal{T}$.

To prove 4.13, all we need is to show that any family $\xi \subset \lambda$ of cardinality ω_1 fails to be disjoint. Pick a $\xi = \{\widetilde{\theta}(F): F \in \mathcal{K}\} \subset \lambda$ such that $\mathcal{K} \subset \mathcal{F}, |\mathcal{K}| = \omega_1$ and $\widetilde{\theta}(F) \neq \emptyset$ for every $F \in \mathcal{K}$. Since $|\mathcal{K}| = \omega_1 < \omega_2$, there is an $\alpha < \omega_2$ such that $\mathcal{K} \subset \mathcal{F}_\alpha$ and $\widetilde{\theta}_\alpha(F) = X_\alpha \cap \widetilde{\theta}(F) \neq \emptyset$ for all $F \in \mathcal{K}$. Applying (9) to $\mathcal{K}$, pick $F_1, F_2 \in \mathcal{K}$ with $\emptyset \neq \widetilde{\theta}_{\alpha+1}(F_1) \cap \widetilde{\theta}_{\alpha+1}(F_2) \subset \widetilde{\theta}(F_1) \cap \widetilde{\theta}(F_2)$. Thus the family ξ is not disjoint.

Proof of Theorem 3.2. Let $<_X$ be a left well-order on X. Define a left well-order $<_Y$ on Y. On $X \times X$ $<_Y$ coincides with $<_X$. Examining attentively the proof of Theorem 3.1, one can see that in our auxiliary inductive construction we add a single point passing from α to $\alpha+1$ with the help of Lemma 4.7. The sequence in which we add new points to $X_0 = X$ gives us the desired left well-order $<_Y$ on Y.

Proof of Corollary 3.3. Apply Theorem 3.1 to the discrete space of cardinality τ.

Proofs of Theorems 3.4 and 3.5 are similar to those of 3.1 and 3.2 respectively and will be omitted.

Proof of Corollary 3.6. Consider a Tychonoff c.c.c. space Z with a G_δ-diagonal and $|Z| > 2^{\aleph_0}$ the existence of which is guaranteed by Corrollary 3.3. Assume that there exists a one-to-one continuous mapping of Z onto a Hausdorff first-countable space Y. Then $c(Y) = \aleph_0$ and from the well-known A.Hajnal and I.Juhász's result [7] it follows that $|Y| \leq \exp(\chi(Y) \cdot c(Y)) \leq 2^{\aleph_0}$. But $2^{\aleph_0} \geq |Y| = |Z| > 2^{\aleph_0}$, which is a contradiction.

5. Some positive results and final remarks.

In connection with the negative answer to Question 1.1 it is worth looking for classes of spaces in the realm of which Question 1.1 is settled positively. According to J.Ginsburg and R.G.Woods' result (see Introduction) in the realm of collectionwise Hausdorff spaces an answer to Question 1.1 is "yes". The following easy result is of the same kind.

Proposition 5.1. In the realm of Hausdorff spaces of pointwise-countable type an answer to Question 1.1 is "yes".

Proof. For spaces of pointwise-countable type $\psi(X) = \chi(X)$ (see [3], Exercise 3.1.F). Hence a space of pointwise-countable type with a G_δ-diagonal is first-countable. Now it suffices to apply A.Hajnal and I.Juhász's result $|X| \leq \exp(\chi(X)\cdot c(X))$ [7].

Corollary 5.2. The cardinality of a Čech-complete c.c.c. Hausdorff space with a G_δ-diagonal does not exceed $2^{\aleph_0}$.

With the help of a method, different from described above, the author obtained the following general result:

Theorem 5.3. Let us consider the following properties:

1) having a G_δ-diagonal,

2) being σ-discrete (this is to be a sum of countable family of its discrete closed subspaces),

3) normality,

4) metacompactness,

5) hereditary metacompactness.

Every Tychonoff space X can be embedded as a closed G_δ-subspace in a Tychonoff c.c.c. space Y in such a way that the space Y has any of the above properties whenever X has.

Corollary 5.4. For any cardinal τ there exists a normal hereditarily metacompact σ-discrete c.c.c. space Z such that $|Z| \geq \tau$.

Remark 5.5. Every σ-discrete space has a G_δ-diagonal.

Remark 5.6. This work had already been finished when I found out from the thesis of Toshiji Terada [8] that he had also given an answer to Question 1.1. I don't know his proofs, for his paper submitted to "Canadian Mathematical Journal" is not published yet. V.V. Uspenskii, after having learned arguments of the present paper, gave his own solution in [9].

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