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ISOMORPHISMS OF PRODUCTS OF INFINITE CONNECTED GRAPHS

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Abstract: We construct a connected countable simple graph G isomorphic to $G \times G \times G$ but not to $G \times G$, for $\times$ being the Cartesian product or the normal product.

Key words: Products of graphs, connected graphs.

Classification: 05C40

For simple graphs $G = (V,E)$, $G' = (V',E')$, the following three types of products are examined in the literature (see e.g. [1]):

$G \times G' = (V \times V', E_1)$,

$G + G' = (V \times V', E_2)$,

$G \cdot G' = (V \times V', E_3)$,

where E_1, E_2, E_3 are defined so that a pair $\xi = (x, x')$, $\zeta = (z,z')$ of distinct elements of $V \times V'$ belongs to

E_1 iff $\{x,z\} \in E$ and $\{x',z'\} \in E'$

E_2 iff either $x = z$ and $\{x',z'\} \in E'$ or $\{x,z\} \in E$ and $x' = z'$

$E_3 = E_1 \cup E_2$.

To be able to speak about all the three types of products simultaneously, let us denote $\times$ by $\overset{1}{\times}$, + by $\overset{2}{\times}$ and $\cdot$ by $\overset{3}{\times}$.

In the present paper, we investigate the following implication (called the Tarski cube property):

$$G \simeq G \overset{i}{\times} G \overset{i}{\times} G \Longrightarrow G \simeq G \overset{i}{\times} G.$$

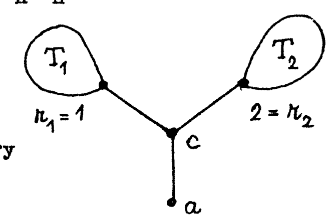
Its validity depends on the class of investigated graphs. It is fulfilled trivially in the class of finite graphs. On the other hand, it is fulfilled for none of the products $\overset{1}{\times}$, $\overset{2}{\times}$, $\overset{3}{\times}$ in the class of all countable simple graphs, see [7]. In the present paper, we investigate this implication within the class of all connected countable simple graphs. The connectedness has been chosen because it changes arithmetic properties of products of some close structures (see e.g. [4] for cardinal products of relational structures, [2] for products of partial orders), so it could influence also the validity of the above implication. Let us state shortly that this is not the case for $\overset{1}{\times}$ and $\overset{3}{\times}$. The proof of it is just the aim of the present paper. Let us denote by $\mathcal{C}_i$, $i = 1,2,3$, the class of all countable simple graphs G isomorphic to $G \overset{i}{\times} G \overset{i}{\times} G$ but not isomorphic to $G \overset{i}{\times} G$. In the parts I and II of the present paper, we construct a connected graph in $\mathcal{C}_3$ and a connected graph in $\mathcal{C}_1$. In the part III, we present some related results which either can be seen directly from the constructions in I and II or obtained from them by some modifications (we also present here a corrected proof of the theorem in [6] characterizing the chromatic number and the set of degrees of the graphs in $\mathcal{C}_1$). Finally, let us state explicitly that we do not know whether $\mathcal{C}_2$ also contains a connected graph.

I. Construction of a connected graph in $\mathcal{C}_3$

1. In this part, we investigate only the product $\overset{3}{\times}$, so we denote it only by $\times$ (or $\prod$ for infinite systems).

Let N be the set of all non-negative integers. Let us denote by T an infinitary tree, i.e. a graph $(\{r\} \cup \bigcup_{k=1}^{\infty} N^k, E)$, where E consists of all $\{r,n\}$ with $n \in N = N^1$ and of all $\{p,q\}$ with $p \in N^k$, $q \in N^{k+1}$ such that p is the initial segment of q (r is called the root of T).

Let $\{p_n \mid n \in N\}$ be an increasing sequence of primes with $p_0 \geq 2$. For each $n \in N$, denote by $H_n = (V_n, E_n)$ the following graph: we start with vertices $\{c, a, 1, \ldots, p_n - 1\}$ and edges $\{\{a,c\}\} \cup \{\{c,i\} \mid i = 1, \ldots, p_n - 1\}$ and glue a copy T_i of the infinitary tree T on the vertex i such that we identify it with the root r_i of T_i,



for all $i = 1, \ldots, p_n - 1$ (where we suppose that all the $T_1, \ldots$ $\ldots, T_{p_n - 1}$ are disjoint; H_3 is visualized on the picture). For any map $f \in N^N$ which is not the constant zero $\mathbb{O}$, we investigate the product

$$P(f) = \prod_{n \in N, f(n) \neq 0} H_n^{f(n)},$$

where $H_n^{f(n)} = H_n \times \ldots \times H_n$ f(n)-times. Let us denote by H(f) its full subgraph consisting of all vertices x with all its coordinates equal to a except possibly a finite number. Since $(\prod H_n^{f(n)}) \times (\prod H_n^{g(n)}) \simeq \prod H_n^{f(n)+g(n)}$, $H(f) \times H(g)$ is isomorphic to $H(f+g)$.

2. In the next constructions, we use <u>coproducts</u> of graphs. If $\{M_i \mid i \in I\}$ is a system of graphs with pairwise disjoint sets of vertices, then their coproduct, denoted by $\coprod_{i \in I} M_i$, is the graph with the set of vertices being the union of the sets of vertices of all the M_i's, all the M_i's are full subgraphs of it and it contains no other edge. If the sets of vertices of $\{M_i \mid i \in I\}$ are not pairwise disjoint, we replace them by isomorphic graphs $\{\bar{M}_i \mid i \in I\}$ which already have this property and then

we form the coproduct as before (hence $\coprod_{i \in I} M_i$ is defined up to isomorphism).

3. For $f, g \in N^N$, we already used in I.1 the addition $f + g$ defined by $(f + g)(n) = f(n) + g(n)$. Now, for any $B, C \subseteq N^N$, we define $B + C$ by

$$B + C = \{f + g \mid f \in B, g \in C\}.$$

By [5], there exists a countable set $A \subseteq N^N$ such that $\mathcal{O} \notin A$ and $A + A + A = A$ but $A + A \neq A$. We define a graph H as a coproduct of $\aleph_0$ copies of the graph

$$\coprod_{f \in A} H(f).$$

It can be seen easily that H is isomorphic to $H \times H \times H$. Since $A = A + A + A$ and $H(f_1) \times H(f_2) \times H(f_3)$ is isomorphic to $H(f_1 + f_2 + f_3)$, each component of H is isomorphic to a component of $H \times H \times H$ and vice versa. Since H contains each of its components in $\aleph_0$ copies, it must be isomorphic to $H \times H \times H$.

4. We show that H is not isomorphic to $H \times H$. Clearly, $H \times H$ is isomorphic to a coproduct of $\aleph_0$ copies of the graph $\coprod_{g \in A+A} H(g)$. $H(f)$, $f \in A$, are just the components of H and $H(g)$, $g \in A + A$, are just the components of $H \times H$ (each contained in the graph in $\aleph_0$ copies) and $A \neq A + A$, it is sufficient to prove the following implication:

$$H(f) \simeq H(g) \Longrightarrow f = g.$$

5. For an arbitrary countable graph M and its arbitrary vertex x, we denote by $c(M,x)$ the supremum of the sets C of vertices of M such that

a) each element of C is joined by an edge with x and

b) no two distinct elements of C are joined by an edge.

If we inspect the graphs $H(f)$, we can see the following:

$c(H(f),x) = 1$ iff x has all its coordinates equal to a;

$c(H(f)\ x) = p_n$ iff x has all its coordinates equal to a except precisely one, which is equal to c and this coordinate is on a place corresponding to H_n.

Hence $f(n)$ is precisely the number of the vertices x of $H(f)$ with $c(H(f),x) = p_n$. This is valid for each $n \in N$, hence f can be recognized from $H(f)$, the above implication follows.

6. The constructed graph H is not connected. Now, we embed it in a connected graph. First, we choose a fix isomorphism φ of H onto $H \times H \times H$. Let us denote by G_0 a graph obtained from H by adding one new vertex, say ξ , and this new vertex is joined by an edge with every vertex of H. We extend the isomorphism φ to $\varphi_0 : G_0 \longrightarrow G_0 \times G_0 \times G_0$ by putting $\varphi_0(\xi) = (\xi, \xi, \xi)$, so that φ_0 is an embedding of G_0 onto a full subgraph of $G_1 = G_0 \times G_0 \times$ $\times G_0$. We investigate the sequence

$$G_0 \xrightarrow{\varphi_0} G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\varphi} G_3 \xrightarrow{\varphi} \dots$$

with $G_{k+1} = G_k \times G_k \times G_k$ and $\varphi_{k+1} = \varphi_k \times \varphi_k \times \varphi_k$ for all $k \in N$. Let $G = (V,E)$ be its colimit, i.e.

$$V = \bigcup_{k=0}^{\infty} \psi_k(W_k), \quad E = \bigcup_{k=0}^{\infty} (\psi_k \times \psi_k)(F_k),$$

where $(W_k, F_k) = G_k$ and $\psi_k : G_k \longrightarrow G$ are maps such that $\psi_k =$ $= \psi_{k+1} \circ \varphi_k$ for all $k \in N$.

It can be verified easily that G is a connected countable graph such that $G \simeq G \times G \times G$.

7. It remains to prove that G is not isomorphic to $G \times G$. If x is a vertex of $G_k = G_0 \times \dots \times G_0$ (3^k-times) such that at least one coordinate of x is equal to ξ then $c(G_k,x) = \aleph_0$

(because H has infinitely many components). Hence, for any vertex x of $G \setminus \psi_0(H)$, we have $c(G,x) = \aleph_0$. If x is a vertex of G_k such that none of its coordinates is equal to ξ , i.e. $x = \varphi_{k,o}(y)$ for some vertex y of H (where $\varphi_{k,o} = \varphi_k \circ \dots \circ \varphi_1 \circ \varphi_0$), we can see that $c(G_k,x) = c(H,y)$. (In fact, since ξ is joined by an edge with any vertex of H, any vertex $z = (z_1,\dots,z_{3^k})$ of $G \setminus \varphi_{k,o}(H)$ is joined by an edge with any vertex $z' = (z_1',\dots,z_{3^k}')$ obtained from z by replacing each its ξ - coordinate by any vertex of H (all the other coordinates remaining unchanged). Hence such vertices cannot influence the value of c.) We conclude that for any vertex x of $\psi_0(H)$, the equation $c(G,x) = c(\psi_0(H),x)$ is fulfilled. If we proceed analogously with $G \times G$, we see that

$$c(G \times G,x) = c(\psi_0(G) \times \psi_0(G),x) \text{ for any vertex x of } \psi_0(G_0) \times \psi_0(G_0),$$

$$c(G \times G,x) = \aleph_0 \text{ otherwise.}$$

8. Now, we "recognize" the set A from G and the set A + A from $G \times G$ by the following procedure.

For a graph M, let us denote by $\mathcal{J}(M)$ the set of all vertices x of M such that $c(M,x) = 1$ and for any $x \in \mathcal{J}(M)$ and each $n \in N$, by $f_x(n)$ the number of all the vertices y which are joined by an edge with x and $c(M,y) = p_n$. Finally, let us denote by $\mathbb{F}(M)$ the set of all f_x with $x \in \mathcal{J}(M)$. If we use the conclusion of I.7 and repeat the reasoning of I.5, we see that

$$\mathbb{F}(G) = A, \quad \mathbb{F}(G \times G) = A + A,$$

hence G and $G \times G$ cannot be isomorphic.

II. Construction of a connected graph in $\mathcal{C}_1$

1. In this part, we investigate only the product $\overset{1}{\times}$, so

we denote it only by $\times$ or $\prod$.

Let N and T be as in I.1. Let K be the graph $(N \cup \{p,q\},E)$, where

$E = \{\{p,i\}, \{i,q\} \mid i \in N\}$.

For every $n \in N$, $n \geq 2$, we denote by (V_n,E_n) the following graph:

$V_n = \{a,b,c,d_n\} \cup \{1,2,\dots,n\}$

$E_n = \{\{a,b\}, \{b,c\}\} \cup \{\{c,i\}, \{i,d_n\} \mid i = 1,1,\dots,n\}$.

Denote by $\overline{H}_n$ a countable simple graph satisfying all the conditions α) – ζ) below.

α) (V_n,E_n) is its full subgraph;

β) a, b, c is the unique path of the length 2 from a to c in $\overline{H}_n$;

γ) c, i, d_n, where $i = 1,\dots,n$, are the only paths of the length 2 from c to d_n in $\overline{H}_n$;

δ) $\overline{H}_n$ is bipartite;

ε) for every pair x, y of vertices of $\overline{H}_n$ with $d(x,y) = 2$ (where $d(x,y)$ denotes the length of the shortest path from x to y) such that $\{x,y\} \neq \{a,c\}$ and $\{x,y\} \neq \{c,d_n\}$ there are infinitely many paths of the length 2 from x to y in $\overline{H}_n$;

ζ) the degree of each vertex of $\overline{H}_n$ is equal to $\aleph_0$.

The graph $\overline{H}_n$ can be constructed so that we start from (V_n,E_n) and glue a copy of the infinitary tree T on each its vertex (identifying it with the root of the copy of T) and then glue a copy of K on each path y_0, y_1, y_2 with $y_0 \neq y_2$, which is distinct from the paths a, b, c and c, i, d_n for all $i = 1,\dots,n$ (by the identification of the path y_0, y_1, y_2 with the path p, 0, q of the copy of K); we repeat this procedure over all natural numbers.

2. Let $\{p_n \mid n \in N\}$ be an increasing sequence of primes, $p_0 \geq 2$.

We denote $H_n = \widetilde{H}_{p_n}$ and we construct $H \in \mathcal{C}_1$ by means of the system $\{H_n | n \in N\}$ rather analogously to I. If $f \in N^N$, $f \neq \mathbb{O}$, we denote

$$P(f) = \prod_{n\in N, f(n)\neq 0} H_n^{f(n)}.$$

Let $A \subseteq N^N \setminus \{\mathbb{O}\}$ be as in I.3, i.e. $A = A + A + A$ and $A \neq A + A$. Let us denote by $P(A)$ a coproduct of $\aleph_0$ copies of each $P(f)$ with $f \in A$, say,

$$P(A) = \coprod_{(f,k)\in A\times N} (P(f))_k.$$

It can be seen easily that $P(A)\times P(A)\times P(A)$ is isomorphic to $P(A)$, but for the next reasoning we need an isomorphism with some special properties. Since A is countable and $A = A + A + A$, the set $B(f) = \{(f_1,f_2,f_3) | f_i \in A$ for $i = 1,2,3$ and $f_1 + f_2 + f_3 = f\}$ is non-empty and countable for each $f \in A$. Thus the sets $B(f)\times N\times N\times N$ and $\{f\}\times N$ have the same cardinality so that we can find a bijection

$$\varphi : (A\times N)\times(A\times N)\times(A\times N) \longrightarrow A\times N$$

with the following properties:

(a) for every $k_1,k_2,k_3 \in N$ and $f_1,f_2,f_3 \in A$ there exists $m \in N$ such that $\varphi((f_1,k_1),(f_2,k_2),(f_3,k_3)) = (f_1 + f_2 + f_3, m)$;

(b) for every $f \in A$ there exists $(f_1,f_2,f_3) \in B(f)$ such that $\varphi((f_1,1),(f_2,1),(f_3,1)) = (f,1)$ and $\varphi((f_1,2),(f_2,2),(f_3,2)) = (f,2)$.

The bijection φ determines an isomorphism

$$\sigma : P(A)\times P(A)\times P(A) \longrightarrow P(A)$$

such that $(P(f_1))_{k_1}\times(P(f_2))_{k_2}\times(P(f_3))_{k_3}$ is sent to $(P(f_1 + f_2 + f_3))_m$ by the collecting of coordinates only.

3. For each $f \in A$, let us denote by $P_o(f)$ the full subgraph of $P(f)$ consisting of all the vertices x with all the coordina-

tes equal to the same vertex $z \in \{a,b,c\}$ except possibly for a finite number of coordinates and put

$$P_0(A) = \coprod_{(f,k)\in A\times N} (P_0(f))_k,$$

so that $P_0(A)$ is a full subgraph of $P(A)$. Let H be the smallest full subgraph of $P(A)$ such that

(1) $H = \coprod_{(f,k)\in A\times N} (H(f))_k$, where $H(f)$ is a countable full subgraph of $P(f)$ containing $P_0(f)$;

(2) the domain- range-restriction of $\sigma : P(A)\times P(A)\times P(A) \to P(A)$ is an isomorphism of $H\times H\times H$ onto H.

(The graph H can be constructed by the following enlarging procedure: $P_1(A)$ is the smallest full subgraph of $P(A)$ of the form $\coprod_{(f,k)\in A\times N}(P_1(f))_k$ containing $P_0(A) \cup \sigma(P_0(A)\times P_0(A)\times P_0(A)) \cup Q_0$, where Q_0 is the smallest full subgraph of $P(A)$ such that $Q_0\times Q_0\times Q_0 \supseteq \sigma^{-1}(P_0(A))$; we repeat this over all natural numbers and $H = \bigcup_{j=0}^{\infty} P_j(A)$. In [6] and [7], this enlarging procedure is described more in detail.)

4. To prove that $H \in \mathcal{C}_1$, it is sufficient to show that H is not isomorphic to $H\times H$.

For an arbitrary graph M, let us denote by $\mathcal{J}(M)$ the set of all vertices x of M for which there exists a vertex $\bar{x}$ such that

(i) $d(x,\bar{x}) = 2$ and there is a unique path of the length 2 from x to $\bar{x}$ and

(ii) if y is a vertex of M with $d(x,y) = 2$ and $y \neq \bar{x}$, then there are infinitely many paths of the length 2 from x to y.

For each $x \in \mathcal{J}(M)$ and each $n \in N$, let us denote by $f_x(n)$ the number of all vertices z of M which fulfil the following:

(iii) $d(x,z) = 2$;

(iv) there exists a vertex $\bar{z}$ such that

a) $d(z,\bar{z}) = 2$ and there are precisely p_n paths of the length 2 from z to $\bar{z}$;

b) if y is a vertex of M such that $d(z,y) = 2$ and $y \neq \bar{z}$ then there are either one or infinitely many paths of the length 2 from z to y.

Let us denote by $\mathbb{F}(M)$ the set $\{f_x \mid x \in \mathcal{J}(M)\}$. We show that

$$\mathbb{F}(H) = A \quad \text{and} \quad \mathbb{F}(H \times H) = A + A.$$

If $f \in A \cup (A + A)$ and we inspect the graph $H(f)$, we can see that a vertex x of $H(f)$ fulfils (i) and (ii) iff all its coordinates are equal to a. And a vertex z of $H(f)$ fulfils (iii) and (iv) with respect to this x iff all the coordinates of z are equal to a except precisely one which is equal to c; and this coordinate is on a place corresponding to the graph H_n. Thus, there are precisely $f(n)$ such vertices in $H(f)$. Since this is true for each $x \in \mathcal{J}(H)$ and each $x \in \mathcal{J}(H \times H)$ and each $n \in N$, we conclude that

$$\mathbb{F}(H) = A \quad \text{and} \quad \mathbb{F}(H \times H) = A + A.$$

5. Now, we embed H in a connected graph by a procedure analogous to I.6. We denote by G_0 a graph obtained from H by adding a new vertex ξ and this new vertex is joined by an edge with each vertex of H. We extend the isomorphism σ^{-1} to $\varphi_0 : G_0 \longrightarrow G_0 \times G_0 \times G_0$ by putting $\varphi_0(\xi) = (\xi, \xi, \xi)$ and investigate the sequence

$$G_0 \xrightarrow{\varphi_0} G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3 \xrightarrow{\varphi_3} \cdots$$

with $G_{k+1} = G_k \times G_k \times G_k$ and $\varphi_{k+1} = \varphi_k \times \varphi_k \times \varphi_k$. We denote its colimit by $G = (V,E)$, i.e. $V = \bigcup_{k=0}^{\infty} \Psi_k(W_k)$ and $E = \bigcup_{k=0}^{\infty} (\varphi_k \times \varphi_k)(F_k)$ as in I.6. Then G is a connected countable simple graph and G is isomorphic to $G \times G \times G$.

6. It remains to prove that G is not isomorphic to $G \times G$.

Both G and $G \times G$ have the following property:

$$(*) \quad \begin{cases} \text{any two distinct vertices can be joined} \\ \text{by a path of the length 2.} \end{cases}$$

For an arbitrary graph M, let us denote by D(M) the set of all vertices x such that $M \setminus \{x\}$ fails to have the property (*). It can be seen that

$$D(G) = \{\eta\} \text{ and } D(G \times G) = \{(\eta, \eta)\},$$

where $\eta = \psi_0(\xi)$. (In fact, if a_1 is chosen in $(H(f))_1$ and a_2 in $(H(f))_2$ for some $f \in A$, then, by II.2(b), $\varphi_{k,o}(a_1)$ cannot be joined with $\varphi_{k,o}(a_2)$ by a path of the length 2 in $G_k \setminus \varphi_{k,o}(\xi)$ [where $\varphi_{k,o} = \varphi_k \circ \dots \circ \varphi_0$] so that $\psi_0(a_1)$ cannot be joined with $\psi_0(a_2)$ by a path of the length 2 in $G \setminus \{\eta\}$; and analogously for $G \times G$.)

Now, the full subgraph of G (or $G \times G$) consisting of all the vertices joined by an edge with the unique element of D(G) (or $D(G \times G)$) is isomorphic to H (or $H \times H$, respectively). Since H is not isomorphic to $H \times H$, G is not isomorphic to $G \times G$.

III. Concluding remarks

1. Let (S,+) be a commutative semigroup. We say that a system $\{G(s) \mid s \in S\}$ of countable simple graphs is its representation by the product $\overset{i}{\times}$ ($i = 1,2,3$), if

(a) $G(s + s')$ is always isomorphic to $G(s) \overset{i}{\times} G(s')$ and

(b) if $s \neq s'$, then G(s) is not isomorphic to $G(s')$.

(If $G \in \mathscr{C}_i$, then $\{G(0), G(1)\}$ with $G(1) = G$ and $G(0) = G \overset{i}{\times} G$ form a representation of the cycklic group $c_2 = \{0,1\}$ of order 2 by the product $\overset{i}{\times}$.) Every countable commutative semigroup has a representation by each of the products $\overset{1}{\times}$, $\overset{2}{\times}$, $\overset{3}{\times}$, see [7].

Moreover, it can be required that each of the representing graphs has a given countable simple graph as its full subgraph. The techniques developed in [3],[5],[6],[7] admit to strengthen also the previous constructions and to obtain e.g. the following results:

- every countable simple graph can be embedded as a full subgraph in $2^{\aleph_0}$ non-isomorphic connected graphs from $\mathcal{C}_1$ (or $\mathcal{C}_3$, respectively);
- every semigroup embeddable in a countable direct product of finite cyclic groups (particularly each finitely generated Abelian group) has a representation by the products $\overset{1}{\times}$ and $\overset{3}{\times}$ by connected graphs (there are $2^{\aleph_0}$ non-isomorphic such representations, all the representing graphs contain a given graph as a full subgraph, they have the prescribed chromatic number ≥ 3 and some other properties).

On the other hand, a characterization of the semigroups which can be represented by $\overset{1}{\times}$ or $\overset{2}{\times}$ or $\overset{3}{\times}$ by <u>connected</u> graphs is not known (for any of these products).

2. Let us denote by $\chi(G)$ the chromatic number of a graph G and by $\mathfrak{D}(G)$ the set of the degrees of all its vertices. In [6], the following theorem is presented.

<u>Theorem</u>: Let $c \in N \cup \{\aleph_0\}$ and $D \subseteq N \cup \{\aleph_0\}$ be given. Then there exists $G \in \mathcal{C}_1$ such that $\chi(G) = c$ and $\mathfrak{D}(G) = D$ iff $c \geq 2$ and D fulfils the following condition (+).

(+) $\aleph_0 \in D$; if $D \setminus \{0, \aleph_0\} \neq \emptyset$, then $1 \in D$;
if $d_1, d_2 \in D \cap N$, then $d_1 \cdot d_2 \in N$.

V. Puš found a mistake in the proof of this Theorem given in [6]. However, the Theorem is correct, let us present here a correction of the proof. If $G \in \mathcal{C}_1$, then necessarily $\chi(G) \geq 2$ (this is evident) and $\mathcal{D}(G)$ fulfils the condition (+) - this is proved correctly in [6] and we do not repeat it here. Conversely, let c and D with the above properties be given. We have to construct $G \in \mathcal{C}_1$ with $\chi(G) = c$ and $\mathcal{D}(G) = D$. Let us mention that if $c = 2$ and $D = \{\aleph_0\}$, then the graph H constructed in II.3 has the required properties. All the next cases will be modifications of this construction (therefore the constructed graph with the required properties will be denoted by H). For an arbitrary $c \geq 2$, we proceed as follows: we choose a countable simple graph $\widetilde{H}$ such that $\chi(\widetilde{H}) = c$, the degree of each its vertex is equal to $\aleph_0$ and for each pair x, y of vertices of $\widetilde{H}$ such that $d(x,y) = 2$ there is an infinite number of paths of the length 2 from x to y in $\widetilde{H}$ (it can be constructed so that we start from an arbitrary graph with the chromatic number c and glue a copy of K on each path of the length 2 of it and repeat this procedure infinitely many times).

(a) Let us suppose that $D = \{\aleph_0\}$. For every $n \in N$, $n \geq 2$, let $\overline{H}_n$ be as in II.1. Let $\{p_n \mid n \in N\}$ be an increasing sequence of primes, $p_0 \geq 0$. We denote

$$H_n = \widetilde{H} \amalg \overline{H}_{p_n}$$

and for each $f \in N^N$, $f \neq 0$, we put

$$P(f) = \prod_{n \in N, f(n) \neq 0} H_n^{f(n)}.$$

Let $A \subseteq N^N$ be as in I.3. For each $f \in A$, let us denote by $P_0(f)$ the full subgraph of $P(f)$ consisting of all vertices x with all the coordinates equal to the same vertex $z \in \{a,b,c\} \cup \widetilde{H}$ except possibly for a finite number of coordinates and put

$$P_o(A) = \coprod_{(f,k)\in A\times N} (P_o(f))_k$$

Proceeding as in II.3, we obtain a graph H isomorphic to $H\times H\times H$ with $\chi(H) = \chi(\widetilde{H})$ and $\mathcal{D}(H) = \{\aleph_o\}$. The proof that H is not isomorphic to $H\times H$ is the same as in II.4.

(b) <u>Let us suppose</u> $D\subseteq (N\cup\{\aleph_o\})\setminus\{0\}$ and $1\in D$. Let $\widetilde{H}$ be as above. For every $t\in D^+ = D\setminus\{1,\aleph_o\}$ denote by M_t the graph obtained from the graph $(\{0,1,\ldots,t\}, \{\{0,i\} \mid i = 1,\ldots,t\})$ by the glueing of a copy of the infinitary tree T on each vertex $i = 1,\ldots,t$. We denote

$$\widetilde{H}' = H \amalg \coprod_{t\in D^+} M_t.$$

Let $\overline{H}_n'$ be a countable simple graph satisfying all the conditions $\alpha) - \varepsilon)$ in II.1 and $\zeta)$ is replaced by

$\zeta')$ $\deg(a) = 1$ and $\deg(x) = \aleph_o$ for each vertex $x\neq a$.

(To obtain $\overline{H}_n'$, an evident modification of the construction of II.1 can be used.) Let $\{p_n \mid n\in N\}$ be an increasing sequence of primes, $p_o\geq 2$. For each $n\in N$, put

$$H_n = \widetilde{H}' \amalg \overline{H}_{p_n}'.$$

The construction of $H\in\mathcal{C}_1$ is now quite analogous to (a). (The proof that H is not isomorphic to $H\times H$ is easier, we can use the vertices with the degree equal to 1. These are precisely the vertices with all coordinates equal to a.)

c) <u>If</u> D <u>contains zero</u>, we use the case (a) or (b) for the set $D\setminus\{0\}$ and then add an infinite number of isolated vertices to the constructed graph.

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