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SINGULAR SOLUTIONS TO LINEAR ELLIPTIC SYSTEMS

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Abstract: Discontinuous solutions of linear elliptic systems with bounded measurable coefficients are studied. A construction of an elliptic system, which has solutions discontinuous on a dense set, is given. Further it is shown that generic solutions of De Giorgi type equations are singular.

Key words: Regularity of solutions, elliptic systems.

Classification: 35B65, 35D10

In 1968 De Giorgi [1] (see also [2]) constructed a linear elliptic system with bounded measurable coefficients, which has a discontinuous (and even unbounded) solution. His construction can be expressed in the following simple terms. Let B_1 be a unit ball in R^n, $n \geq 3$.

Lemma 1. Let $b = (b_i^k)$, $i,k = 1,\dots,n$ be a matrix of L^2-functions such that

(0.1) $$\int_{B_1} b_i^k \Phi_{x_i}^k = 0, \quad \forall \Phi \in C_0^\infty(B_1, R^n).$$

Let us denote (the summation convention is assumed)

(0.2) $$a_{ij}^{kh} = \delta_{ij}\delta^{kh} + \frac{d_i^k d_j^h}{u_{x_\rho}^s d_\rho^s}$$

where

(0.3) $$d_i^k = b_i^k - u_{x_i}^k, \quad u \in H^1(B_1, R^n).$$

Let us suppose, moreover, that

(0.4) $$u_x \cdot d = u_{x_i}^k \, d_i^k > 0,$$

(0.5) $$\frac{b \cdot d}{u_x \cdot d} \leq M.$$

Then u is a solution of the elliptic system

(0.6) $$\int_{B_1} a_{ij}^{kh} \, u_{x_i}^k \, \Phi_{x_j}^h \, dx = 0, \quad \forall \Phi \in C_0^\infty (B_1, R^n).$$

Proof. By computation. The ellipticity and boundedness of a_{ij}^{kh} follow from (0.5) and (0.4).

The example of Giusti and Miranda [2] corresponds to the choice

(0.7) $$u^k(x) = x_k |x|^{-1},$$

(0.8) $$b_i^k = |x|^{-1} \left(n \, \delta_{ik} + \frac{n}{n-2} x_k \, x_i |x|^{-2}\right).$$

The solution (0.7) is discontinuous at the origin of R^n.

In the first part of this paper we give a construction of an elliptic system, which has solutions discontinuous on a dense set of points. This implies, in particular, that the vector-valued quasi-minima for the Dirichlet integral [3, 7] need not be even "partially" regular. In the second part we show that almost every solution to the equation of De Giorgi type is singular, or more precisely, the generic solution is singular. In particular, we shall give a sufficient condition on the boundary data guaranteeing that the solution is singular. The first part of this paper was already partly reported in [3, 7]. We would like to thank M. Giaquinta and J. Nečas for discussions and inspirations in connection with this work.

1. We shall consider the following situation (and notation)

(1.1) $u^k(x) = n_k = \frac{x_k}{r}$, $r = |x|$, $u^k_{x_i} = r^{-1}(\delta_{ki} - n_k n_i)$,

(1.2) $b^k_i = r^{-1}(F + 1)(\delta_{ki} + \frac{1}{n-2} n_k n_i)$, $F > 0$.

Then we have

(1.3) $d^k_i = r^{-1}(F\delta_{ki} + G n_k n_i)$, $G = \frac{F+1}{n-2} + 1 > 0$,

(1.4) $d \cdot u_x = r^{-2} F(n - 1)$,

(1.5) $d \cdot d = r^{-2} H$, $H = F^2(n - 1) + (F + 1)^2(\frac{n-1}{n-2})^2$.

We see that the coefficients (0.2) are elliptic, since

(1.6) $\lambda_0 |\xi|^2 \leq \xi^k_i a^{kh}_{ij} \xi^h_j \leq \lambda_1 |\xi|^2$

with

(1.7) $\lambda_0 + 1$, $\lambda_1 = 1 + \frac{H}{F(n-1)}$.

We shall use the following notation for translated functions

(1.8) $u^k_{(y)}(x) = n_{k(y)}(x) = u^k(x - y)$, $r_{(y)}(x) = |x - y|$

and similarly for $b^k_{i(y)}$, $d^k_{i(y)}$.

Let μ be a positive bounded Borel measure on B_1. We can form the superpositions using

(1.9) $\bar{u}^k = \int_{B_1} u^k_{(y)}\, d\mu(y)$, $\bar{b}^k_i = \int_{B_1} b^k_{i(y)}\, d\mu(y)$,

and similarly for $\bar{d}^k_i$. Finally, let us denote by

$$(1.10)\quad \bar{a}_{ij}^{kh} = \delta_{ij}\delta^{kh} + \frac{\bar{d}_i^k \, \bar{d}_j^h}{\bar{u}_x^s \, \bar{d}^s}$$

__Theorem 1.__ __The vector-function $\bar{u}^k$ is a solution to the elliptic system__

$$(1.11)\qquad \int_{B_1} \bar{a}_{ij}^{kh} \bar{u}_{x_i}^k \Phi_{x_j}^h \, dx = 0, \quad \forall \Phi \in C_0^\infty .$$

__The ellipticity constants $\bar{\lambda}_0, \bar{\lambda}_1$ of this system are related to λ_0, λ_1 from (1.7) by__

$$(1.12)\qquad \bar{\lambda}_0 = 1, \quad \bar{\lambda}_1 \le \lambda_1$$

__Let us suppose that__

$$(1.13)\qquad \mu = \sum_{k=1}^{\infty} \varepsilon_k \, \delta_{(y_k)}, \qquad \varepsilon_k > 0, \; y_k \in B_1,$$

__where $\delta_{(y)}$ is the n-dimensional Dirac measure concentrated at the point y.__
__Then $\bar{u}^k$ is discontinuous at points y_k and we can clearly choose the set $\{y_k\}$ to be dense in B_1.__

__Proof.__ Let us define functions $\sigma_{yz}(x)$, $y, z \in B_1$ by

$$(1.14)\qquad 1 - \sigma_{yz}(x) = | n_{k(y)}(x) \, n_{k(z)}(x) |^2 .$$

Clearly, $0 \le \sigma_{yz} \le 1$. Then we have

$$(1.15)\qquad \begin{aligned} u_{x(y)} \cdot d_{(z)} &= r_{(y)}^{-1} \, r_{(z)}^{-1} (F(n-1) + G\sigma_{yz}), \\ d_{(y)} \cdot d_{(z)} &= r_{(y)}^{-1} \, r_{(z)}^{-1} (H - G^2 \sigma_{yz}) \end{aligned}$$

and this implies

$$(1.16)\qquad \bar{u}_x \cdot \bar{d} = F(n-1)A + GB, \quad \bar{d} \cdot \bar{d} = HA - G^2 B,$$

where the functions A(x) and B(x) are defined by

$$(1.17)\quad A = \int r_{(y)}^{-1} r_{(z)}^{-1} d\mu(y) d\mu(z), \quad B = \int r_{(y)}^{-1} r_{(z)}^{-1} \sigma_{yz} d\mu(y) d\mu(z).$$

The best ellipticity constant $\bar{\lambda}_1$ can be expressed by

$$(1.18)\quad \bar{\lambda}_1 = 1 + \sup_{x \in B_1} \frac{H - G^2\Sigma}{F(n-1) + G\Sigma}, \quad \Sigma(x) = \frac{B(x)}{A(x)}.$$

The inequality $\Sigma \geq 0$ implies that $\bar{\lambda}_1 \leq \lambda_1$.

Remark. The optimal choice of F gives the smallest possible ellipticity constant

$$(1.19)\qquad \lambda_{1,opt} = \frac{\sqrt{1+\lambda}+1}{\sqrt{1+\lambda}-1}, \quad \lambda = \frac{(n-2)^2}{n-1},$$

which corresponds to the known regularity results [4 - 6] for systems with $\lambda_1 < \lambda_{1,opt}$.

2. Before stating the general theorem about singular solution, we shall consider its special case in R^3; so from now on $n = 3$. Let us consider the elliptic system (0.6) with coefficients (like in [2])

$$(2.1)\quad a_{ij}^{kh} = \delta_{ij}\delta^{kh} + B_i^k B_j^h, \quad B_i^k = \delta_{ki} + 2n_k n_i.$$

We shall use test functions of the form

$$(2.2)\qquad \Phi^k = f(r)n_k, \quad f \in C_0^\infty(0,1),$$

where $n_k = x_k r^{-1}$, $r = |x|$. We have

$$(2.3)\qquad \Phi^h_{x_j} = r^{-1} f \delta_{hj} + (f' - r^{-1}f)n_h n_j$$

and after a calculation (using 1.1)), we obtain

$$(2.4)\qquad (a_{ij}^{kh}\Phi^h_{x_j})_{x_i} = h(r)n_k,$$

where the function h(r) is a second-order differential expression in f(r) given by

$$(2.5)\qquad h = 10(f'' + 2r^{-1}f').$$

Integrating equation (0.6) by parts we obtain

(2.6) $$\int_{B_1} u^k n_k h \, dx = 0$$

Now we can introduce the spherical mean value of u

(2.7) $$w(r) = \int_{\partial B_r} v \, d\Omega = r^{-2} \int_{\partial B_r} v \, dS, \quad v = u^k n_k.$$

Equation (2.6) then reads

(2.8) $$\int_{B_1} v \, h \, dx = \int_0^1 r^2 \, h(r) \, dr \int_{\partial B_r} v(r,\Omega) \, d\Omega = 0$$

so that, using (2.3), we arrive at the ordinary differential e-quation for w(r) (using 2.5))

(2.9) $$\int_0^1 r^2 \, w(f'' + 2r^{-1} f') \, dr = 0, \qquad \forall f \in C_0^\infty .$$

This equation is equivalent to

(2.10) $$2w' + r \, w'' = 0$$

and the basis of its solutions is

(2.11) $$w_1 = \text{const.}, \; w_2 = \text{const. } r^{-1}.$$

The second solution w_2 does not correspond to the H^1-function (this is a general fact, that there is only one H^1-solution, otherwise the uniqueness in the Dirichlet problem would be violated).

Let us consider the Dirichlet boundary value problem with the data

(2.12) $$u^k = \varphi^k \text{ at } \partial B_1$$

and let u^k be the solution. If the data are such that

(2.13) $$\omega = \int_{\partial B_1} \varphi^k n_k \, dS \neq 0,$$

then $w(1) = \omega$ and thus $w(t) = \omega$, $\forall t \in (0,1)$. It implies that the solution u^k cannot be continuous at the origin, as we shall show. If u^k is continuous at the origin then we can write $u^k =$ $= u^k(0) + \bar{u}^k$, $|\bar{u}^k| < \varepsilon$ in B_σ. We obtain

(2.14) $$w(\sigma) \le \int_{B_\sigma} \frac{\partial}{\partial x_k} u^k(0)\, dx + 3\,\varepsilon \int_{\partial B_\sigma} d\Omega \le c\,\varepsilon$$

and this contradicts $w(\sigma) = \omega \neq 0$.

Equations (2.7) and (2.11) mean that the mean flux of the field u^k through ∂B_r is constant and thus u^k cannot be regular at the origin provided this flux is nonzero. The solution can be regular only if $\omega = 0$. There are regular solutions, e.g.

(2.15) $$u^k = \varepsilon^{klm} a_l x_m, \; a_l = \text{constants},$$

where ε^{klm} is the Levi-Civita tensor. This raises an interesting question: are all solutions regular, if $\omega = 0$?

Let us now consider the more general problem in R^n, $n \ge 3$ for equation (0.6) with coefficients given by (2.1), where

(2.16) $$B_i^k = F\delta_{ki} + Gn_k n_i, \; F, G = \text{constants}, \; F > 0.$$

We shall assume that this equation has a radial solution of the form

(2.17) $$u^k_{rad} = g(r) n_k,$$

which is discontinuous at the origin.

Under these circumstances we can state

Theorem 2. If u^k is a solution to the system (0.6), (2.1) and (2.16) such that $\omega = \int_{\partial B_1} u^k n_k \, ds \neq 0$, then u^k is discontinuous at the origin.

Proof. It is analogous to the one in 3-dimensional case discussed above. We write the equation (0.6) with the test function of the form (2.2). Then we shall obtain equation (2.4), where $h(r)$ is a function depending on f, f' and f'', perhaps more complicated than (2.5). Equation (2.6) remains true and in the equations (2.7) and (2.8) r^2 has to be replaced by r^{n-1}. In this way we get a second-order differential equation for w, analogous to (2.9) and (2.10). But $g(r)$ from (2.17) must be a solution of this equation, because the spherical mean w_{rad} of u^k_{rad} is equal to g times the area of the unit sphere. It can be shown that this equation is of the Euler type, so that g must be proportional to a power $r^{-\alpha}$, $\alpha \geq 0$. We have already noted that there may be only one H^1-solution; thus w is proportional to g. Then the discontinuity of u^k follows by the same argument as above.

Remark. If we parametrize solutions by their boundary data, then the space of discontinuous solutions contains the open dense subset of the set of all solutions (the same is true in the H^1-norm on B_1). Thus the discontinuity of a solution is a generic property.

Let R be a subspace of regular solutions to the De Giorgi system (0.6), (2.1) and (2.16) with $\lambda_1 \geq \lambda_{1,opt}$. One could generally expect that R is dense in the space of all solutions; in particular, that the singular solution may be approximated by regular ones (with H^1-norms converging to $+\infty$). This is excluded by Theorem 2.

Conjecture:

(i) R is closed in H^1-norm.

(ii) R has a finite codimension in the space of all solutions.

(iii) This codimension is one if λ_1 is sufficiently small.

(iv) The standard Hölder-continuity estimate is true on R.

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