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Abstract: Locally finite cylindric algebras of any dimension are shown to be equivalent as a category to a variety of heterogeneous algebras. The notion of relative homomorphism of cylindric algebras generalizing the notions of homomorphism and relativization map is introduced. Based on some metalogical considerations uniform relativizations and uniform relative homomorphisms of locally finite cylindric algebras are studied.

Key words: Cylindric algebra, locally finite, neat reduct, heterogeneous algebra, relativization, homomorphism.

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Cylindric algebras provide an algebraization of the predicate calculus. But from the metalogical point of view, the locally finite (dimensional) ones corresponding to finitary first order theories are of main interest. For any infinite ordinal α the class Lf_α of locally finite cylindric algebras of dimension α is a proper subclass of the variety CA_α of all α-dimensional cylindric algebras, closed under subalgebras, homomorphic images and finite direct products. However, for α infinite Lf_α suffers from a defect - it is not a variety, being not closed under infinite direct products; it is not even an axiomatic class - the closeness under ultraproducts fails. One aim of this note is to show that Lf_α considered as a category with ordinary homomorphisms is equivalent to a variety of heterogeneous algebras, providing so with a supplement the results of Andréka, Németi [2] describing the first order theory of Lf_{α} .

The second part starts from the definition of the notion of relative homomorphism generalising both the notions of homomorphism and relativization map. As communicated to the author by Németi [10], this question was raised by Henkin. As expected, cylindric algebras with relative homomorphisms form again a category.

After it, the results will be applied to obtain the definitions of uniform relativization and uniform relative homomorphism of locally finite cylindric algebras which are more sound with the concepts of relativization of formulas and relative interpretation between first order theories known from the first order logic, contributing to the solution of Henkin's question and providing the paper of Zlatoš [14] with some necessary background.

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1. The equivalence of Lf_{α} and HCA_{α}

For fundamentals on heterogeneous algebras and category theory the reader is referred to Birkhoff, Lipson [3] and to Mac Lane [8], respectively. Concerning cylindric algebras we accept the terminology and notation of Henkin, Monk, Tarski [6], with some minor modifications to be just introduced.

For typographical reasons algebras are denoted by underlined capital Latin letters instead of (nonunderlined) German ones. Their universes are denoted by the same letters without underlining.

In the definition of α-dimensional cylindric algebras, α is allowed to be an arbitrary set, not just an ordinal. For $\underline{A} \in CA_\alpha$ and $p \subseteq \alpha$, $\underline{Rd}_p\underline{A}$ denotes the reduct of $\underline{A}$ obtained by deleting the cylindrifications c_i for $i \in \alpha \sim p$ and diagonal elements d_{ij} for $\langle i,j\rangle \in {}^2\alpha \sim {}^2p$. Clearly, $\underline{Rd}_p\underline{A} \in CA_p$. Let us recall that for a CA_α $\underline{A}$ and $x \in A$ the dimension set of x is

$$\Delta x = \{ i \in \alpha : c_i x \neq x \} ,$$

and $\underline{A}$ is locally finite ($\underline{A} \in Lf_\alpha$) iff Δx is finite for each $x \in A$. Since $Lf_\alpha = CA_\alpha$ for α finite, we assume that α is a fixed infinite set containing a distinquished element θ, from now. Nevertheless, all the results bellow trivially remain true for finite α, as well. $Fin\,\alpha$ denotes the set of all finite subsets of α.

A heterogeneous cylindric algebra $\underline{G}$ of dimension α consists of the following data:

- a nonempty set G_p for each $p \in Fin\,\alpha$, together with operations $+, \cdot, -, 0, I, c_i, d_{ij}$ $(i,j \in p)$ of usual arities converting G_p into a cylindric algebra $\underline{G}_p \in CA_p$;
- unary operations $\overrightarrow{pq}: G_p \longrightarrow G_q$, $\overleftarrow{pq}: G_q \longrightarrow G_p$ for all $p \subseteq q \in Fin\,\alpha$, subject to conditions

(1) $\overrightarrow{pp} = Id \restriction G_p$, (2) $\overrightarrow{qr} \circ \overrightarrow{pq} = \overrightarrow{pr}$ $(p \subseteq q \subseteq r)$,

(3) $\vec{pq}: \underline{G}_p \longrightarrow \underline{\mathrm{Rd}}_p \underline{G}_q$ is a homomorphism of CA_p's $(p \subseteq q)$,

(4) $c_i \vec{pq} x = \vec{pq} x \quad (p \subseteq q,\ i \in q \sim p,\ x \in G_p)$,

(5) $\overleftarrow{pq} \circ \vec{pq} = \mathrm{Id} \restriction G_p \quad (p \subseteq q)$,

(6) $\vec{pq}(\overleftarrow{pq} x) = c_{(q \sim p)} x \quad (p \subseteq q,\ x \in G_q)$,

(where $c_{(r)} x = \begin{cases} x & \text{if } r = \emptyset, \\ c_{i_1} \ldots c_{i_k} x & \text{for } r = \{i_1, \ldots, i_k\} \in \mathrm{Fin}\,\alpha \end{cases}$);

All the conditions above can be easily reformulated into identities, hence, heterogeneous algebras of dimension α form a variety HCA_α .

Any $\underline{G} \in HCA_\alpha$ raises to a direct system of finite dimensional cylindric algebras $\langle \underline{G}_p, \vec{pq};\ p \subseteq q \in \mathrm{Fin}\,\alpha \rangle$ over the directed poset $\langle \mathrm{Fin}\,\alpha, \subseteq \rangle$. We put

$$\underline{\mathrm{Dl}}\ \underline{G} = \varinjlim \langle \underline{G}_p, \vec{pq} \rangle .$$

The last formula needs an explanation since the algebras $\underline{G}_p$ are not of the same type. Nevertheless, the standard direct limit (colimit) construction, as described e.g. in Grätzer [4], still works since the types of the $\underline{G}_p$'s form a direct system, too, and all the necessary preservation properties are guaranteed by the identities (1) - (6) of HCA_α . In this way $\underline{\mathrm{Dl}}\ \underline{G}$ naturally becomes an algebra from Lf_α . Details are left to the reader.

From the metalogical point of view the transition from $\underline{G}$ to $\underline{\mathrm{Dl}}\ \underline{G}$ presents the effect of reconstructing the cylindric algebra of a theory in variables v_i $(i \in \alpha)$ from its parts formed by formulas depending only on finitely many variables

v_i $(i \in p)$ where p runs over $\mathrm{Fin}\,\alpha$. Similarly, if V is a set and $\underline{G}_p$ is the cylindric algebra of all subsets of the cartesian space ${}^pV$, $\underline{\mathrm{Dl}}\ \underline{G}$ can be identified with the cylindric algebra of all subsets of ${}^\alpha V$ depending only on finitely many coordinates (i.e. a relation $r \subseteq {}^\alpha V$ belongs to $\mathrm{Dl}\ \underline{G}$ iff there is a finite subset $p \subseteq \alpha$ such that for all $a, b \in {}^\alpha V$ holds: if $a \upharpoonright p = b \upharpoonright p$ then $a \in r$ iff $b \in r$ - consult [6], [7]).

Given a homomorphism $h: \underline{G} \longrightarrow \underline{H}$ in HCA_α the direct limit construction yields a homomorphism

$$\underline{\mathrm{Dl}}\ h = \underrightarrow{\lim}\ h_p : \underline{\mathrm{Dl}}\ \underline{G} \longrightarrow \underline{\mathrm{Dl}}\ \underline{H}\ .$$

We have obtained a functor $\underline{\mathrm{Dl}}: \mathrm{HCA}_\alpha \longrightarrow \mathrm{Lf}_\alpha$.

On the contrary, to each $\underline{A} \in \mathrm{Lf}_\alpha$ one can assigne a heterogeneous cylindric algebra $\underline{\mathrm{Nr}}\ \underline{A}$ - the neat reduct complex of $\underline{A}$ - in the following way: For $p \in \mathrm{Fin}\,\alpha$ $(\underline{\mathrm{Nr}}\ \underline{A})_p = \underline{\mathrm{Nr}}_p\underline{A}$ is the p-th neat reduct of $\underline{A}$ ($\mathrm{Nr}_p\underline{A} = \{x \in A: \Delta x \subseteq p\}$ and $\underline{\mathrm{Nr}}_p\underline{A} \in \mathrm{CA}_p$ is a subalgebra of $\underline{\mathrm{Rd}}_p\underline{A}$ - see [6]), for $p \subseteq q$ $\overrightarrow{pq}: \mathrm{Nr}_p\underline{A} \longrightarrow \mathrm{Nr}_q\underline{A}$ is simply the inclusion map and $\overleftarrow{pq}: \mathrm{Nr}_q\underline{A} \longrightarrow \mathrm{Nr}_p\underline{A}$ is the generalized cylindrification $c_{(q \sim p)}$. Obviously, an algebra from HCA_α was obtained. The intuitive metalogical or geometrical meaning of this construction is selfexplanating.

If $f: \underline{A} \longrightarrow \underline{B}$ is a homomorphism in Lf_α then the system of restrictions $f_p = f \upharpoonright \mathrm{Nr}_p\underline{A}: \underline{\mathrm{Nr}}_p\underline{A} \longrightarrow \underline{\mathrm{Nr}}_p\underline{B}$ defines a homomorphism $\underline{\mathrm{Nr}}\ f = \langle f_p : p \in \mathrm{Fin}\,\alpha \rangle : \underline{\mathrm{Nr}}\ \underline{A} \longrightarrow \underline{\mathrm{Nr}}\ \underline{B}$ in HCA_α. In this way a functor $\underline{\mathrm{Nr}}: \mathrm{Lf}_\alpha \longrightarrow \mathrm{HCA}_\alpha$ was obtained.

Theorem 1. The functors $\underline{Dl}$ and $\underline{Nr}$ are pairwise inverse equivalences between the categories HCA_α and Lf_α .

Proof. One has to show that there are natural isomorphisms Ψ: $\underline{Id}\restriction HCA_\alpha \xrightarrow{\sim} \underline{Nr} \circ \underline{Dl}$ and Θ: $\underline{Dl} \circ \underline{Nr} \xrightarrow{\sim} \underline{Id}\restriction Lf$, i.e. that for all $h: \underline{G} \longrightarrow \underline{H}$ in HCA_α and $f: \underline{A} \longrightarrow \underline{B}$ in Lf_α the following two diagrams commute:

$$
\begin{array}{ccc}
\underline{G} & \xrightarrow{\quad h \quad} & \underline{H} \\
\Psi_{\underline{G}} \downarrow & & \downarrow \Psi_{\underline{H}} \\
\underline{Nr}\ \underline{Dl}\ \underline{G} & \xrightarrow[\underline{Nr}\ \underline{Dl}\ h]{} & \underline{Nr}\ \underline{Dl}\ \underline{H}
\end{array}
\qquad
\begin{array}{ccc}
\underline{Dl}\ \underline{Nr}\ \underline{A} & \xrightarrow{\underline{Dl}\ \underline{Nr}\ f} & \underline{Dl}\ \underline{Nr}\ \underline{B} \\
\Theta_{\underline{A}} \downarrow & & \downarrow \Theta_{\underline{B}} \\
\underline{A} & \xrightarrow[f]{} & \underline{B}
\end{array}
$$

The verification based on (1) - (6) is straightforward.

The functor $\underline{Dl}$ can be regarded via the composition with the embedding $\underline{J}: Lf_\alpha \longrightarrow CA_\alpha$ as a functor $\underline{Dl}' = \underline{J} \circ \underline{Dl}: HCA_\alpha \longrightarrow CA_\alpha$. On the other hand, the neat reduct complex construction applies to arbitrary cylindric algebras not just to locally finite ones, giving an extension $\underline{Nr}': CA_\alpha \longrightarrow HCA_\alpha$ of $\underline{Nr}$.

Corollary 1. The functors $\underline{Dl}'$ and $\underline{Nr}'$ are adjoint, (of course, $\underline{Dl}'$ is the left and $\underline{Nr}'$ the right adjoint).

Proof. Both possible ways, either to constitute the natural isomorphism of hom-sets $CA_\alpha(\underline{Dl}'\underline{G}, \underline{A}) \cong HCA_\alpha(\underline{G}, \underline{Nr}'\underline{A})$ or via the unit Ψ': $\underline{Id}\restriction HCA_\alpha \longrightarrow \underline{Nr}' \circ \underline{Dl}'$ and counit Θ': $\underline{Dl}' \circ \underline{Nr}' \longrightarrow \underline{Id}\restriction CA_\alpha$ extending Ψ and Θ , respectively, are straightforward. (Note that Ψ' is still an isomorphism.)

Corollary 2. Lf_α is a full coreflective subcategory of CA_α .

Proof. $\underline{Dl} \circ \underline{Nr}'$ is the right adjoint to the inclusion functor $\underline{J}$. The natural embedding $\underline{Dl}\ \underline{Nr}'\underline{A} \longrightarrow \underline{A}$ is the co-reflection of the object $\underline{A} \in CA_\alpha$ in Lf_α. (Of course, this result can be established also directly. Namely, in every $\underline{A} \in CA_\alpha$ the finite dimensional elements form a subalgebra of $\underline{A}$ which is isomorphic to $\underline{Dl}\ \underline{Nr}'\underline{A}$, and the inclusion map has the desired universal property.)

From Theorem 1 follows the fact proved by Andréka, Gergely, Németi, Sain [1] that the category Lf_α is both complete and cocomplete. The proof of existence of products in Lf_α and their description is due already to Preller [11]. We can give an alternative description of Lf_α-products: To compute ${}_1\mathrm{P}(\underline{A}_t: t \in T)$ in Lf_α one has first to compute $\mathrm{P}(\underline{Nr}\ \underline{A}_t: t \in T)$ componentwise, meaning

$$(\mathrm{P}(\underline{Nr}\ \underline{A}_t: t \in T))_p = \mathrm{P}(\underline{Nr}_p\underline{A}_t: t \in T)$$

the right side product being already direct, and pass to the direct limit

$${}_1\mathrm{P}(\underline{A}_t: t \in T) \cong \underline{Dl}\ \mathrm{P}(\underline{Nr}\ \underline{A}_t: t \in T).$$

Remark. In the special case $\alpha = \omega$ (the set of all natural numbers 0, 1, ...) the type of algebras in HCA_α can be done less cumbersome. Any algebra $\underline{G} \in HCA_\omega$ is completely determined by its components $\underline{G}_n$ and unary operations $j_n = \overrightarrow{n\ n{+}1}$: $G_n \longrightarrow G_{n+1}$, $e_n = \overleftarrow{n\ n{+}1}: G_{n+1} \longrightarrow G_n$ $(n \in \omega)$. (Let us recall that $n = \{0, 1, \ldots, n-1\} \in \mathrm{Fin}\,\omega$ for $n \in \omega$, so that $\omega \subseteq \mathrm{Fin}\,\omega$.) Finally, we give the description of the variety (equivalent to) HCA_ω in this presentation: An algebra $\underline{G}$ in

HCA_ω consists of a system $\langle \underline{G}_n : n \in \omega \rangle$ of algebras $\underline{G}_n \in CA_n$, homomorphisms $j_n : \underline{G}_n \longrightarrow \underline{Rd}_n \underline{G}_{n+1}$ and mappings $e_n : G_{n+1} \longrightarrow G_n$ satisfying

$$c_n j_n x = j_n x ,$$

$$e_n j_n x = x ,$$

$$j_n e_n y = c_n y$$

for all $n \in \omega$, $x \in G_n$, $y \in G_{n+1}$.

The reader can compare the heterogeneous cylindric algebras of dimension ω with the heterogeneous clones of Taylor [13]. The analogy is transparent. The former provide an algebraization of the first order logic while the latter do the same for the equational logic.

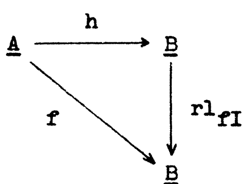
2. Uniform relativizations and relative homomorphisms

Let $\underline{A} \in CA_\alpha$, $a \in A$. The relativization of $\underline{A}$ with respect to a is an algebra of the same type type as $\underline{A}$ denoted by $\underline{Rl_a A}$ with carrier $Rl_a \underline{A} = \{ x \in A : x \leq a \}$. The Boolean operations and diagonal elements in $\underline{Rl_a A}$ are defined in such a (unique possible) way that they are preserved by the canonical map $rl_a = \langle a \cdot x : x \in A \rangle : A \longrightarrow Rl_a \underline{A}$. The cylindrifications c_i^a on $\underline{Rl_a A}$ are defined by $c_i^a x = a \cdot c_i x$ for $a \geq x \in A$ (see [6]).

In general $\underline{Rl_a A}$ needs not to be a CA_α, and even if this is the case, the relativization map $rl_a : \underline{A} \longrightarrow \underline{Rl_a A}$ needs not to be a homomorphism unless $\Delta a = \emptyset$ - otherwise it does not preserve the cylindrifications (see Németi [9]). So it seems meaningful to try to define a minimal class of maps between

CA_α's closed under identity and composition containing both homomorphisms and relativization maps. Let us regard the relativization maps in form $rl_a = \langle a \cdot x : x \in A \rangle : A \longrightarrow A$ for this purpose.

Let $\underline{A}, \underline{B} \in CA_\alpha$. A map $f: A \longrightarrow B$ is called a relative homomorphisms from $\underline{A}$ to $\underline{B}$ if there is a homomorphism $h: \underline{A} \longrightarrow \underline{B}$ such that for each $x \in A$ holds $fx = fI \cdot hx$, i.e. iff f decomposes into a homomorphism and a relativization map



Obviously, every relative homomorphism $f: \underline{A} \longrightarrow \underline{B}$ preserves joins and meets (regarding f as a map $f: \underline{A} \longrightarrow \underline{Rl}_{fI}\underline{B}$, it preserves all the Boolean opérations and diagonal elements, as well).

We subsume all the basic facts about relative homomorphisms in the theorem bellow.

Theorem 2. (i) Every homomorphisms of cylindric algebras is a relative homomorphism. A relative homomorphism f is a homomorphism iff $fI = I$.

(ii) For every $\underline{A} \in CA_\alpha$, $a \in A$ the relativization map rl_a is a relative endomorphism of $\underline{A}$.

(iii) If $f: \underline{A} \longrightarrow \underline{B}$, $g: \underline{B} \longrightarrow \underline{C}$ are relative homomorphisms of α-dimensional cylindric algebras then $g \circ f: \underline{A} \longrightarrow \underline{C}$ is a relative homomorphism, as well.

(iv) Every class of mappings between CA_α's containing all the homomorphisms and relativization maps closed under composition has to contain all relative homomomorphisms between them.

Proof. (i) and (ii) are trivial, (iv) was already proved (see the last diagram). As for (iii), let $h: \underline{A} \longrightarrow \underline{B}$, $k: \underline{B} \longrightarrow \underline{C}$ be the homomorphisms inducing f, g, respectively. Then $g \circ f$ is induced by $k \circ h$. Let us compute for $x \in A$:

$$(g \circ f)x = g(fI \cdot hx) = gfI \cdot gI \cdot khx = (g \circ f)I \cdot (k \circ h)x \ .$$

A deepper study of the category CA_α with relative homomorphisms is postponed, as usual, into some future works. Let us turn our attention to another but related problem concerning locally finite cylindric algebras.

The following type of relativization can be found quite often (if not exclusively) in the first order logic: Given a unary predicate (or a formula with a single free variable) P such that $\exists x\, P(x)$ holds, the value of any variable occuring in a formula φ is bounded by P (e.g.

$$\varphi^P(z) \equiv (P(z) \Rightarrow \exists x(P(x) \wedge \forall y(P(y) \Rightarrow x \neq z \wedge R(x,y,z))))$$

corresponds in this way to $\varphi(z) \equiv \exists x \forall y(x \neq y \wedge R(x,y,z)))$. In the cylindric set algebra of all subsets of the Cartesian space ${}^\alpha V$ this presents the effect of relativization with respect to an element ${}^\alpha U$ where U is a nonempty subset of V. However, for $\underline{A} \in Lf_\alpha$, $a \in A$, $\Delta a \subseteq \{0\}$, $c_0 a = I$, the element $\prod_{i \in \alpha} s_i^0 a$ corresponding to ${}^\alpha U$ needs not to exist, so that the procedure of relativization by meeting with a single element as described in [6] does not apply. Nevertheless, work ing locally one can still obtain a reasonable construction following the relativization from the predicate calculus. The previous results provide us with the necessary tools.

Let us recall from [6] that s^i_j $(i,j \in \alpha)$ denotes the substitution operator of the j-th coordinate in place of the i-th one, i.e. for $i \neq j$ $s^i_j x = c_i(d_{ij} \cdot x)$ and $s^i_i x = x$. The reader should keep in mind that every s^i_j is an endomorphism of the Boolean part of a CA_α $\underline{A}$ and that the substitutions are preserved by homomorphisms of cylindric algebras (see [6]).

Let $\underline{A} \in Lf_\alpha$, $a \in A$, and $\Delta a \subseteq \{0\}$. For each $p \in Fin\,\alpha$ we put

$$^p a = \prod_{i \in p} s^0_i a \quad ,$$

in particular, $^{\emptyset}a = I$. Obviously, $^p a \in Nr_p\underline{A}$ for aech finite p. Let $\underline{A}{\restriction}^p a$ denote the relativization $\underline{Rl}_{^p a}\underline{Nr_p A}$ of the neat reduct $\underline{Nr_p A}$ with respect to $^p a$. According to [6, Theorem 2.2.13] $\underline{A}{\restriction}^p a \in CA_p$ for each $p \in Fin\,\alpha$. We would like to convert the system $\langle \underline{A}{\restriction}^p a \; ; \; p \in Fin\,\alpha \rangle$ of finite dimensional cylindric algebras into an α-dimensional heterogeneous cylindric algebra. For $p \subseteq q \in Fin\,\alpha$ we put

$$\overrightarrow{pq}x = {}^q a \cdot x \qquad (x \in A{\restriction}^p a),$$

$$\overleftarrow{pq}y = {}^p a \cdot c_{(q \sim p)}y \qquad (y \in A{\restriction}^q a).$$

(In the sequel the signs of cylindric algebra operations always denote the operations from the original algebra $\underline{A}$ so that the operations in the $\underline{A}{\restriction}^p a$'s have to be expressed according to their definition.)

The just described construction gives the following result:

Theorem 3. Let $\underline{A} \in Lf_\alpha$, $a \in A$, $\Delta a \subseteq \{0\}$. Then $\langle \underline{A}{\restriction}^p a, \overrightarrow{pq}, \overleftarrow{pq} : p \subseteq q \in Fin\,\alpha \rangle$ is a heterogeneous algebra of the same type as the neat reduct complex $\underline{Nr}\ \underline{A}$ satisfying conditions (1), (2), (3), (4) and (6). It satisfies condition (5), i.e. it is a HCA_α iff $c_0 a = I$.

Proof. The verification of conditions (1) - (4) is easy. As for the condition (6) we assume that the set $q \sim p$ contains exactly two elements i and j. The general case follows then by induction. Let $y \in A{\restriction}^q a$. Then

$$\overrightarrow{pq}(\overleftarrow{pq}y) = {}^q a.{}^p a\cdot c_i c_j y = {}^q a\cdot c_i c_j(s_i^0 a\cdot y)$$

$$= {}^q a.{}^{q\sim\{i\}}a\cdot c_i(s_i^0 a\cdot c_j y) = {}^q a\cdot c_i({}^q a\cdot c_j y)$$

which is the generalized cylindrification $c_{(q\sim p)}y$ in $\underline{A}{\restriction}^q a$. If $c_0 a = I$, observe that for any $r \in Fin\,\alpha$ $c_{(r)}{}^r a = I$. Let us compute for $x \in A{\restriction}^p a$, $p \subseteq q \in Fin\,\alpha$

$$\overleftarrow{pq}(\overrightarrow{pq}x) = {}^p a\cdot c_{(q\sim p)}({}^q a\cdot x) = {}^p a\cdot x\cdot c_{(q\sim p)}{}^{q\sim p}a = x$$

which proves (5). To show the necessity, it suffices to realize that

$$\overleftarrow{\emptyset\{0\}}\,(\overrightarrow{\emptyset\{0\}}\,I) = c_0 a .$$

Passing to the direct limit

$$\underline{Dl}\,\langle \underline{A}{\restriction}^p a, \overrightarrow{pq}, \overleftarrow{pq} \rangle = \underrightarrow{\lim}\,\langle \underline{A}{\restriction}^p a, \overrightarrow{pq} \rangle$$

an algebra of the same type as $\underline{A}$ called the uniform relativization of $\underline{A}$ with respect to a and denoted by $\underline{A}{\restriction}^\alpha a$ is obtained. Theorems 1 and 3 yield immediately

Corollary. $A{\restriction}^\alpha a \in Lf_\alpha$ or, which is the same, $\underline{A}{\restriction}^\alpha a \in CA_\alpha$ iff $c_0 a = I$.

A slight modification of the direct limit construction gives the following direct description of $\underline{A}\restriction^{\alpha} a$ (cf. [41]): We start with the set $\{x \in A: x \leq {}^{p}a$ for some $p \in Fin\,\alpha$, $\Delta x \subseteq p\} = \{x \in A: x \leq {}^{\Delta x}a\}$ and introduce the following equivalence on it:

$$x \equiv_a y \quad \text{iff} \quad {}^{p}a.x = {}^{p}a.y \quad \text{for some} \quad p \in Fin\,\alpha ,$$

$$\Delta x \subseteq p \;, \; \Delta y \subseteq p$$

$$\text{iff} \quad {}^{\Delta x \cup \Delta y}a.x = {}^{\Delta x \cup \Delta y}a.y \;.$$

Let $[x]_a = [x]$ denote the block of equivalence of the element x in $\equiv_a$. We introduce the cylindric algebra operations on the set $\{x \in A: x \leq {}^{\Delta x}a\}/\equiv_a$ as follows:

$$[x] + [y] = [{}^{\Delta(x+y)}a.(x+y)] \;, \quad [x]\cdot[y] = [x\cdot y] \;,$$

$$-[x] = [{}^{\Delta x}a - x] \;, \quad 0 = [0] \;, \quad I = [I] \;,$$

$$c_i[x] = [{}^{\Delta x}a.c_i x] \;, \quad d_{ij} = [{}^{\{i,j\}}a.d_{ij}] \;.$$

Now, we would like to have a result on transitivity of uniform relativizations. Let us start with a lemma.

<u>Lemma 1</u>. Let $\underline{A} \in Lf_\alpha$, $a,b \in A$, $a \geq b$, $\Delta a \subseteq \{0\}$, $\Delta b \subseteq \{0\}$. Then in $\underline{A}\restriction^{\alpha} a$ holds

$$\Delta[b] \subseteq \{0\} \qquad \text{and} \qquad {}^{P}[b] = [{}^{P}b] \;.$$

Moreover, if $c_0 b = I$ (hence, also $c_0 a = I$), then $c_0[b] = I$ in $\underline{A}\restriction^{\alpha} a$, as well.

Proof. The first and the last conditions can be verified immediately. To prove the second one it is enough to show that for $i \in \alpha \sim \{0\}$ holds $s_i^0[b] = [s_i^0 b]$. A simple computation gives

$$s_i^0[b] = c_0[{}^{\{0,i\}}a \cdot d_{0i} \cdot b] = [{}^{\{0,i\}}a \cdot c_0(s_i^0 a \cdot d_{0i} \cdot b)]$$

$$= [{}^{\{0,i\}}a \cdot s_i^0 b] = [s_i^0 b] .$$

Theorem 4. Let $\underline{A}$, a, b be as in Lemma 1. Then there is a natural isomorphism

$$(\underline{A}\restriction^{\alpha} a)\restriction^{\alpha}[b] \cong \underline{A}\restriction^{\alpha} b .$$

Moreover, $\underline{A}\restriction^{\alpha} b \in Lf_{\alpha}$ iff $c_0 b = I$ iff $c_0 a = I$ and $a \leq c_0 b$.

Proof. The isomorphism is given by

$$[[x]_a]_{[b]} \mapsto [x]_b \quad .$$

The completion of the proof on the base of Lemma 1 and the transitivity result for relativizations of cylindric algebras [6, Theorem 2.2.15] applied to the $\underline{Nr_p A}$'s, ${}^p a$'s and ${}^p b$'s is left to the reader.

Now, we are able to express the notion of relative interpretation between two first order theories (see e.g. Shoenfield [12]) on the language of locally finite cylindric algebras.

Let $\underline{A}$, $\underline{B} \in Lf_{\alpha}$. A pair $\langle f,b \rangle$ where $b \in B$, $\Delta b \subseteq \{0\}$ and $f = \langle f_p : p \in Fin\,\alpha \rangle$ is a system of mappings $f_p : Nr_p\underline{A} \longrightarrow Nr_p\underline{B}$ is called a uniform relative homomorphism from $\underline{A}$ to $\underline{B}$ if there is a system $h = \langle h_p : p \in Fin\,\alpha \rangle$ of homomorphisms of p-dimensional cylindric algebras $h_p : \underline{Nr_p A} \longrightarrow \underline{Nr_p B}$ such that for all $p \subseteq q \in Fin\,\alpha$ and each $x \in Nr_p\underline{A}$ holds

$$f_q x = {}^q b \cdot h_p x .$$

It follows immediately that given any uniform relative homomorphism $\langle f,b\rangle : \underline{A} \longrightarrow \underline{B}$ induced by a system of homomorphisms h, every $f_p : \underline{Nr_p A} \longrightarrow \underline{Nr_p B}$ $(p \in Fin\,\alpha)$ becomes a relative homomorphism induced by the homomorphism h_p and $f_p I = {}^p b$ (particularly, $b = f_{\{0\}} I$) and for all $p \subseteq q \in Fin\,\alpha$ the following diagram commutes:

$$\begin{array}{ccc} \underline{Nr_p A} & \xrightarrow{\overrightarrow{pq}} & \underline{Nr_q A} \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f_q} \\ \underline{B}\restriction{}^p b & \xrightarrow[\overrightarrow{pq}]{} & \underline{B}\restriction{}^q b \end{array}$$

As easily seen, these conditions could be used to obtain an equivalent definition of the notion of uniform relative homomorphism.

Some useful and interesting preservational properties are the following:

Lemma 2. Let $\langle f,b\rangle : \underline{A} \longrightarrow \underline{B}$ be a uniform relative homomorphism of Lf_α's.

(i) If $p,q \in Fin\,\alpha$, $x,y \in A$, $\Delta x \subseteq p$, $\Delta y \subseteq q$ then

$$f_{p\cup q}(x\cdot y) = f_p x \cdot f_q y \ .$$

(ii) If $a \in A$, $\Delta a \subseteq \{0\}$, $i \in \alpha$ and $p \in Fin\,\alpha$ then

(a) $f_{\{i\}} s_i^0 a = s_i^0 f_{\{0\}} a$,

(b) $f_p({}^p a) = {}^p(f_{\{0\}} a)$.

Proof. (i) Let us compute

$$f_{p\cup q}(x\cdot y) = f_{p\cup q}x\cdot f_{p\cup q}y = {}^{p\cup q}b\cdot h_p x\cdot h_q y$$
$$= {}^{p}b\cdot h_p x\cdot {}^{q}b\cdot h_q y = f_p x\cdot f_q y$$

where h is the system of homomorphisms inducing f.

(ii) To prove (a), it is enough to show that both sides of the equality give the same result under the one-one map (it has left inverse) $\overrightarrow{\{i\}\,\{0,i\}} : B\restriction^{\{i\}}b \longrightarrow B\restriction^{\{0,i\}}b$. We suppose that $i \neq 0$, the case $i = 0$ being trivial. Let us compute

$$\begin{aligned} {}^{\{0,i\}}b\cdot f_{\{i\}}\,s_i^0 a = f_{\{0,i\}}s_i^0 a &= {}^{\{0,i\}}b\cdot h_{\{0,i\}}s_i^0 a \\ &= {}^{\{0,i\}}b\cdot s_i^0 h_{\{0,i\}}a \\ &= {}^{\{0,i\}}b\cdot s_i^0({}^{\{0,i\}}b\cdot h_{\{0\}}a) \\ &= {}^{\{0,i\}}b\cdot s_i^0(b\cdot h_{\{0\}}a) = {}^{\{0,i\}}b\cdot s_i^0 f_{\{0\}}a \quad . \end{aligned}$$

(b) is a direct consequence of (i) and (a).

If $\langle f,b\rangle : \underline{A} \longrightarrow \underline{B}$, $\langle g,c\rangle : \underline{B} \longrightarrow \underline{C}$ are two uniform relative homomorphisms of Lf_α's, their composition is defined componentwise

$$\langle g,c\rangle \circ \langle f,b\rangle = \langle\langle g_p \circ f_p \ : \ p \in Fin\,\alpha\rangle,\ g_{\{0\}}b\rangle \quad .$$

Now, we can state an analogue of Theorem 2.

Theorem 5. (i) For every homomorphism $h: \underline{A} \longrightarrow \underline{B}$ of Lf_α's $\langle\langle h\restriction Nr_p\underline{A} \ : \ p \in Fin\,\alpha\rangle,\ I\rangle$ is a uniform relative homomorphism. A uniform relative homomorphism $\langle f,b\rangle : \underline{A} \longrightarrow \underline{B}$ has the above form for some homomorphism h iff $b = I$.

(ii) Let $\underline{A} \in Lf_\alpha$, $a \in A$, $\Delta a \subseteq \{0\}$. Then the pair $\langle rl,\ a\rangle$

$= \langle\langle rl_{p_a} : p \in Fin\,\alpha\rangle, a\rangle$ is a uniform relative endomorphism of $\underline{A}$.

(iii) If $\langle f,b\rangle : \underline{A} \longrightarrow \underline{B}$, $\langle g,c\rangle : \underline{B} \longrightarrow \underline{C}$ are uniform relative homomorphisms of Lf_α's then $\langle g,c\rangle \circ \langle f,b\rangle : \underline{A} \longrightarrow \underline{C}$ is a uniform relative homomorphism, as well.

Proof. (i) The first assertion is trivial. The second one reduces to its following corollary contributing to the results of §1.

Corollary. Let $\underline{G}$, $\underline{H}$ be heterogeneous cylindric algebras of dimension α . A system of mappings $h = \langle h_p : p \in Fin\,\alpha\rangle$ where every $h_p : \underline{G}_p \longrightarrow \underline{H}_p$ is a homomorphism of CA_p's, is a homomorphism of HCA_α's iff for all $p \subseteq q \in Fin\,\alpha$ holds

$$\overrightarrow{pq} \circ h_p = h_q \circ \overrightarrow{pq} \ .$$

Proof. The only if part is trivial. We will show that the preservation of the $\overleftarrow{pq}$'s is a consequence of the preservation of the cylindrifications and the $\overrightarrow{pq}$'s. Let us compute for $p \subseteq q \in Fin\,\alpha$

$$\begin{aligned} h_p \circ \overleftarrow{pq} &= \overleftarrow{pq} \circ \overrightarrow{pq} \circ h_p \circ \overleftarrow{pq} = \overleftarrow{pq} \circ h_q \circ \overrightarrow{pq} \circ \overleftarrow{pq} \\ &= \overleftarrow{pq} \circ h_q \circ c_{(q \sim p)} = \overleftarrow{pq} \circ c_{(q \sim p)} \circ h_q \\ &= \overleftarrow{pq} \circ \overrightarrow{pq} \circ \overleftarrow{pq} \circ h_q = \overleftarrow{pq} \circ h_q \ . \end{aligned}$$

Continuation of the proof of Theorem 5. (ii) is a direct consequence of the definition of uniform relative homomorphism.

(iii) The inclusion $\Delta g_{\{0\}} b \subseteq \{0\}$ is trivial. Let h, k be the systems of homomorphisms inducing f, g, respectively. A simple computation using Lemma 2 gives for every $p \subseteq q \in Fin\,\alpha$,

$x \in Nr_p\underline{A}$

$$(g_q \circ f_q)x = g_q({}^q b \cdot h_p x) = g_q({}^q b) \cdot {}^q c \cdot k_p h_p x$$

$$= {}^q(g_{\{0\}} b) \cdot (k_p \circ h_p)x \ .$$

Every uniform relative homomorphism $\langle f,b\rangle : \underline{A} \longrightarrow \underline{B}$ of Lf_α's raises to a mapping $[f]_b : A \longrightarrow B{\restriction}^\alpha b$ given by

$$[f]_b x = [f_{\Delta x} x]_b = [f_{\Delta x} x] \quad (x \in A) \ .$$

Then $\langle f,b\rangle$ can be quite successfully identified with $[f]_b$. Working with the relative interpretations between first order theories this is really the case. In the particular case of (ii) of Theorem 5 $[rl]_a$ works as follows

$$[rl]_a x = [{}^{\Delta x}a \cdot x]_a = [{}^{\Delta x}a \cdot x] \quad (x \in A) \ .$$

From the metalogical point of view only the uniform relative homomorphisms $\langle f,b\rangle$ with $c_0 b = I$ are of importance. According to Theorems 3 and 5 they seem to be the right subject algebraizing the notion of relative interpretation between first order theories. Unfortunately, this special class of uniform relative homomorphisms is not closed under composition.

The translation of the results of §2 for the case $\alpha = \omega$ into the more elegant language of HCA_ω stated at the end of §1 is left to the reader.

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