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A NOTE ON H-HIGH SUBGROUPS OF ABELIAN GROUPS

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Abstract: The purpose of this note is to determine all subgroups H of an abelian group G such that each H-high subgroup of G is an intersection of Γ-isotype subgroups of G.

Key words: H-high subgroups; Γ-isotype, isotype and pure subgroups.

Classification: 20K99

All groups in this paper are abelian, we shall follow the notation and terminology of [4]. Let $\mathbb{P}$ be the set of all primes and $\Gamma = (\alpha_p)_{p \in \mathbb{P}}$ a sequence, where each α_p is either an ordinal or the symbol ∞ which is considered to be larger than any ordinal. A subgroup A of a group G is said to be Γ-isotype in G if $p^\beta A = A \cap p^\beta G$ for every prime p and for every ordinal $\beta \leq \alpha_p$. About Γ-isotype subgroups see [3] (references). Since $(p^\alpha G)_p = p^\alpha(G_p)$ for each ordinal α and each prime p, we shall write only $p^\alpha G_p$. It is natural to use the symbol $p^\alpha G[p]$ for $(p^\alpha G)[p]$.

The concept of an H-high subgroup was introduced into the structure theory of abelian groups by J.M. Irwin and E.A. Walker (see [5],[6]). If H is a subgroup of a group G then each H-high subgroup of G is neat in G though not necessarily pure in G. The subgroup H is said to be a center of purity in G (J.D. Reid [10]) if each H-high subgroup of G is pure in G.

The question of determining all centers of purity (J.M. Irwin, E.A. Walker [5],[6]) was settled by R.S. Pierce [9] (see also [10]). The class of all groups in which every subgroup is a center of purity (i.e. in which each neat subgroup is pure) was described by C. Megibben [8] (see also [10],[11]). The results of R.S. Pierce and C. Megibben were generalized by V.S. Rochlina [11], W. J. Keane [7] and J. Bečvář [1] in three different directions. In the paper [1] there are determined all centers of Γ-isotypness, i.e. such subgroups H of G for that each H-high subgroup of G is Γ-isotype in G.

This note is a supplement to my paper [1], its purpose is to determine all subgroups H of an arbitrary group such that all H-high subgroups are intersections of Γ-isotype subgroups. The proof of the main theorem essentially utilizes the result from [2].

A description of such subgroups H of a group G for that each H-high subgroup of G is an intersection of Γ-isotype subgroups of G is contained already in the following lemma (compare with Proposition 2.1 [10], Lemma [9], Lemma 2 [11], Lemma [1] and Lemma 2.5 [7]).

Lemma: Let G be a group, H a subgroup of G and $\Gamma = (\alpha_p)_{p \in \mathbb{P}}$. Then there is an H-high subgroup of G that is not an intersection of Γ-isotype subgroups of G if and only if there are a prime p, an ordinal $\beta < \alpha_p$ and elements $0 \neq h \in H[p]$, $g \in p^\beta G$ such that $\langle g-h, p^\beta G[p]\rangle \cap H = 0$.

Proof: Let M be an H-high subgroup of G that is not an intersection of Γ-isotype subgroups of G. By Theorem 1 [2], there are a prime p, an ordinal $\beta < \alpha_p$ and an element $g \in p^\beta G \setminus M$ such that $pg \in M$ and $p^\beta G[p] \subseteq M$. Since $pg \in M \cap pG = pM$, there is an element $m_1 \in M$ such that $pg = pm_1$. Hence $g-m_1 \in G[p] = M[p] \oplus H[p]$, i.e. $g-m_1 = m_2+h$, where $m_2 \in M[p]$ and $0 \neq h \in H[p]$. Now $\langle g-h, p^{\beta}G[p]\rangle \cap H \subseteq M \cap H = 0$.

Conversely suppose that there are a prime p, an ordinal $\beta < \alpha_p$ and elements $0 \neq h \in H[p]$, $g \in p^{\beta}G$ such that $\langle g-h, p^{\beta}G[p]\rangle \cap H = 0$. Let M be an H-high subgroup of G containing $\langle g-h, p^{\beta}G[p]\rangle$. Since $g \in p^{\beta}G \setminus M$, $pg = p(g-h) \in M$ and $p^{\beta}G[p] \subseteq M$, we have that M is not an intersection of Γ-isotype subgroups of G by Theorem 1 [2].

<u>Theorem</u>: Let G be a group, H a subgroup of G and $\Gamma = (\alpha_p)_{p \in \mathbb{P}}$. The following are equivalent:

(i) Each H-high subgroup of G is an intersection of Γ-isotype subgroups of G.

(ii) For each prime p, each ordinal $\beta < \alpha_p$ and each elements $0 \neq h \in H[p]$, $g \in p^{\beta}G$, it is $\langle g-h, p^{\beta}G[p]\rangle \cap H \neq 0$.

(iii) For each prime p, one of the following two conditions holds:

(a) $H_p = 0$;

(b) for each ordinal $\beta < \alpha_p$ either $p^{\beta}G_p$ is elementary and $p^{\beta}G/H \cap p^{\beta}G$ is torsion or $H \cap p^{\beta}G_p \neq 0$.

<u>Proof</u>: The assertions (i) and (ii) are equivalent by the previous lemma.

(ii) $\rightarrow$ (iii). Suppose $H_p \neq 0$ for some prime p and let $\beta < \alpha_p$ be an ordinal such that $H \cap p^{\beta}G_p = 0$. If $0 \neq h \in H[p]$ and $g \in p^{\beta}G$ then $\langle g-h, p^{\beta}G[p]\rangle \cap H \neq 0$ by (ii). Hence $n(g-h) + x = \bar{h} \neq 0$, where n is an integer, $x \in p^{\beta}G[p]$ and $\bar{h} \in H$. Consequently $png = p\bar{h} \in H$ and $p^{\beta}G/H \cap p^{\beta}G$ is a torsion group. If $g \in p^{\beta}G_p$ then $png \in H \cap p^{\beta}G_p = 0$; if $p|n$ then $ng + x = \bar{h} \in H \cap p^{\beta}G_p = 0$ - a contradiction, hence $(p,n) = 1$ and $o(g) = p$.

(iii) $\rightarrow$ (ii). Let p be a prime, $\beta < \alpha_p$ an ordinal, $0 \neq h \in H[p]$ and $g \in p^\beta G$. With respect to (iii) we can suppose that $p^\beta G_p$ is elementary and $p^\beta G / H \cap p^\beta G$ is torsion (otherwise we are through). If g is of infinite order then there is an integer n such that $ng \in H$ and hence $0 \neq pn(g-h) \in \langle g-h, p^\beta G[p] \rangle \cap$ $\cap H$. If g is of finite order then write $g = g_1 + g_2$, where $pg_1 = 0$, $o(g_2) = m$ and $(m,p) = 1$. Now, $0 \neq (m(g_1+g_2-h) - mg_1) \in$ $\in \langle g-h, p^\beta G[p] \rangle \cap H$.

Corollary: Let G be a group and H a subgroup of G. Each H-high subgroup of G is an intersection of isotype subgroups of G if and only if for each prime p one of the following conditions holds:

(i) $H_p = 0$,

(ii) for each ordinal β either $p^\beta G_p$ is elementary and $p^\beta G / H \cap p^\beta G$ is torsion or $H \cap p^\beta G_p \neq 0$.

Corollary: Let H be a subgroup of a group G. Each H-high subgroup of G is an intersection of pure subgroups of G if and only if one of the following two conditions holds:

(i) G/H is torsion and for each prime p, either $H_p = 0$ or $H \cap p^n G_p = 0$ implies $p^{n+1} G_p = 0$ for any natural number n;

(ii) for each prime p, either $H_p = 0$ or $H \cap p^n G_p \neq 0$ for any natural number n.

Remark: The class of all groups G in which each H-high subgroup is an intersection of Γ-isotype subgroups of G for each subgroup H of G obviously coincides with the class of all groups in which each neat subgroup is an intersection of Γ-isotype subgroups of G. This class has been described in [3], where it is also shown that this class coincides also with the class of all groups in which each neat subgroup is Γ-isotype (see also Proposition [1]).

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