Valeriu Popa

Theorems on multifunctions satisfying a rational inequality

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 24 (1983), No. 4, 673--680

Persistent URL: http://dml.cz/dmlcz/106265

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
24,4 (1983)

# THEOREMS ON MULTIFUNCTIONS SATISFYING A RATIONAL INEQUALITY

V. POPA

Abstract: We prove a fixed point theorem for a sequence of multifunctions satisfying a rational inequality which generalizes theorem 3 from [1].

Key words: Multifunction, fixed point.

Classification: 54H25

---

In [1] B. Fisher gave the following theorem

Theorem 1. Let S and T be mappings of the complete metric space into itself such that for all x, y in X either

$$(1) \qquad d(Sx,Ty) \le \frac{c\ d(x,Sx).d(y,Ty)+b\ d(x,Ty).d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

if $d(x,Sx)+d(y,Ty) \neq 0$ where $b \ge 0$ and $1 < c < 2$, or

$$d(Sx,Ty) = 0$$

otherwise. Then each of S and T has a fixed point and these points coincide.

We now prove a similar common fixed point theorem for two multifunctions $T_1$ and $T_2$ and for a sequence of multifunctions which generalize theorem 1.

The method used is a combination of methods used in [1]-[3].

Let (X,d) be a metric space. We denote by CB(X) the set of all nonempty closed bounded subsets of $(X,d)$ and by $H$ the Hausdorff-Pompeiu metric on $CB(X)$

$$H(A,B) = \max\{\sup_{x\in A} d(x,B);\ \sup_{y\in B} d(y,A)\}$$

where $A,B\in CB(X)$ and

$$d(x,A) = \inf_{y\in A} d(x,y).$$

Let $A,B\in CB(X)$ and $k>1$. In what follows, the following well-known fact will be used: For each $a\in A$, there is a $b\in B$ such that

$$d(a,b)\leq k\, H(A,B).$$

Let $(X,d)$ be a metric space, we denote

$$\delta(A,B) = \sup\{d(a,b);\ a\in A \text{ and } b\in B\}$$

where $A,B\in CB(X)$. If $A$ consists of a single point "a" we write $\delta(A,B) = \delta(a,B)$. If $\delta(A,B)=0$ then $A=B=\{a\}$ (Lemma 1 [4]).

Let $T:X\longrightarrow X$ be a multifunction. Denote

$$F(T) = \{x\in X: x\in Tx\}.$$

<u>Lemma</u>. Let $(X,d)$ be a metric space and $T_1,T_2:(X,d)\longrightarrow$ $\longrightarrow CB(X)$ be two multifunctions. If

$$(2)\qquad H^p(T_1x,T_2y)\leq\frac{c.d(x,T_1x).d^p(y,T_2y)+bd(x,T_2y).d^p(y,T_1x)}{\delta(x,T_1x)+\delta(y,T_2y)}$$

holds for all $x,y\in X$ for which $\delta(x,T_1x)+\delta(y,T_2y)\neq 0$ where $p\geq 1$, $b\geq 0$ and $1<c<2$ and $F(T_1)\neq\emptyset$, then $F(T_2)\neq\emptyset$ and $F(T_1) =$ $= F(T_2)$.

<u>Proof</u>. Let $u\in F(T_1)$, then $u\in T_1u$ and if $d(u,T_2u)\neq 0$ then by (2) we have

$$d^p(u,T_2u)\leq H^p(T_1u,T_2u)\leq$$

$$\leq\frac{c.d(u,T_1u).d^p(u,T_2u)+b\, d(u,T_2u).d^p(u,T_1u)}{\delta(u,T_1u)+\delta(u,T_2u)}\leq$$

$$\leq \frac{c \cdot d(u,T_1u) \cdot d^p(u,T_2u) + b\, d(u,T_2u) \cdot d^p(u,T_1u)}{d(u,T_1u) + d(u,T_2u)}$$

which implies $d(u,T_2u) = 0$. Since $T_2u$ is closed, this shows that $u \in T_2u$, which implies $F(T_1) \subset F(T_2)$. Analogously, $F(T_2) \subset F(T_1)$.

<u>Theorem 2</u>. Let $(X,d)$ be a complete metric space and $T_1$, $T_2 : X \longrightarrow CB(X)$ two multifunctions such that for all $x,y \in X$ the inequality (2) holds if $\delta(x,T_1x) + \delta(y,T_2y) \neq 0$ where $p \geq 1$, $b \geq 0$ and $1 < c < 2$. Then $T_1$ and $T_2$ have common fixed points and $F(T_1) = F(T_2)$.

<u>Proof</u>. Choose a real number k with

$$(3) \qquad 1 < k < \left(\frac{2}{c}\right)^{1/p} .$$

Let $x_0 \in X$ and $x_1 \in T_1x_0$. Then there is an $x_2 \in T_2x_1$ so that $d(x_1,x_2) \leq k\, H(T_1x_0,T_2x_1)$. Suppose $x_3,x_4,\dots,x_{2n-1},x_{2n},\dots$ could be chosen so that $x_{2n-1} \in T_1x_{2n-2}, x_{2n} \in T_2x_{2n-1}$ and

$$d(x_{2n-1},x_{2n}) \leq k\, H(T_1x_{2n-2},T_2x_{2n-1})$$

$$d(x_{2n-2},x_{2n-1}) \leq k\, H(T_1x_{2n-2},T_2x_{2n-3}).$$

Suppose first of all that

$$\delta(x_{2n-2},T_1x_{2n-2}) + \delta(x_{2n-1},T_2x_{2n-1}) = 0$$

for some n. Then $x_{2n-2} = \{T_1x_{2n-2}\} = x_{2n-1} = \{T_2x_{2n-1}\}$ and $x_{2n-2} = x_{2n-1}$ is a common fixed point for $T_1$ and $T_2$.

Similarly $\delta(x_{2n-1},T_2x_{2n-1}) + \delta(x_{2n},T_1x_{2n}) = 0$ for some n implies that $x_{2n-1} = x_{2n}$ is a common fixed point for $T_1$ and $T_2$.

Now suppose that $\delta(x_{2n-2},T_1x_{2n-2}) + \delta(x_{2n-1},T_2x_{2n-1}) \neq 0$ for $n=1,2,\dots$ . Then by (2) we have successively

$$d^p(x_{2n-1},x_{2n}) \leq k^pH^p(T_1x_{2n-2},T_2x_{2n-1}) \leq$$

$$\leq k^p.\ \frac{cd(x_{2n-2},T_1x_{2n-2}).d^p(x_{2n-1},T_2x_{2n-1})+bd(x_{2n-2},T_2x_{2n-1}).}{(x_{2n-2},T_1x_{2n-2}) + (x_{2n-1},T_2x_{2n-1})}$$

$$\frac{.d^p(x_{2n-1},T_1x_{2n-2})}{\dots} \leq \frac{k^p.c.d(x_{2n-2},x_{2n-1}).d^p(x_{2n-1},x_{2n})}{d(x_{2n-2},x_{2n-1})+d(x_{2n-1},x_{2n})}.$$

If $d(x_{2n-1},x_{2n}) = 0$, then $x_{2n-1} = x_{2n}$ is a common fixed point for $T_1$ and $T_2$. If $d(x_{2n-1},x_{2n}) \neq 0$ then

$d(x_{2n-1},x_{2n}) \leq (ck^p-1).d(x_{2n-2},x_{2n-1})$ for $n=0,1,2,\dots$ . Similarly we have

$d(x_{2n},x_{2n+1}) \leq (ck^p-1).d(x_{2n-1},x_{2n})$ for $n=0,1,2,\dots$ .

Repeating the above argument, we obtained

$d(x_n,x_{n+1}) \leq (ck^p-1)^n\ d(x_o,x_1)$ for $n=0,1,2,\dots$ .

Since $0<(ck^p-1)<1$ by (3), then by routine calculation one can show that $\{x_n\}$ is a Cauchy sequence and since X is complete, we have $\lim x_n = u$ for some $u \in X$.

If we now suppose that $d(u,T_1u) \neq 0$ then

$$d^p(x_{2n},T_1u) \leq H^p(T_2x_{2n-1},T_1u) \leq$$

$$\leq \frac{cd(u,T_1u).d^p(x_{2n-1},T_2x_{2n-1})+ bd(x_{2n-1},T_1u).d^p(u,T_2x_{2n-1})}{\sigma(u,T_1u) + \sigma(x_{2n-1},T_2x_{2n-1})} \leq$$

$$\leq \frac{cd(u,T_1u).d^p(x_{2n-1},x_{2n}) + bd(x_{2n-1},T_1u).d^p(u,x_{2n})}{d(u,T_1u) + d(x_{2n-1},x_{2n})}$$

and on letting n tend to infinity we have $d(u,T_1u) \leq 0$. It follows that $d(u,T_1u) = 0$. Since $T_1u$ is closed, this shows that $u \in T_1u$. By lemma $u \in T_2u$ and $F(T_1) = F(T_2)$.

If $T_1 = T_2$ we have the following theorem:

<u>Theorem 3</u>. Let $(X,d)$ be a complete metric space and let $T:(X,d) \longrightarrow CB(X)$ be a multifunction such that

$$H^p(Tx,Ty) \leq \frac{cd(x,Tx).d(y,Ty)^p + bd(x,Ty).d(y,Tx)^p}{\delta(x,Tx) + \delta(y,Ty)}$$

holds for all $x,y \in X$ for which $\delta(x,Tx) + \delta(y,Ty) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$, then $T$ has fixed points.

If $T_1$ and $T_2$ are single valued mappings we have the following theorem:

Theorem 4. Let $T_1$ and $T_2$ be mappings of a complete metric space $(X,d)$ into itself such that for all $x$, $y$ in $X$ either

$$d^p(T_1x,T_2y) \leq \frac{cd(x,T_1x).d^p(y,T_2y) + b\, d(x,T_2y).d^p(y,T_1x)}{d(x,T_1x) + d(y,T_2y)}$$

if $d(x,T_1x) + d(y,T_2y) \neq 0$ where $p \geq 1$, $b \geq 0$ and $1 < c < 2$ or $d(T_1x,T_2y) = 0$ otherwise, then $T_1$ and $T_2$ have a unique common fixed point $u$.

Proof. The existence follows from the theorem 2. Now suppose that $T_1$ and $T_2$ have a second fixed point $u'$. Then $d(u,T_1u) + d(u',T_2u') = 0$ implies $d(T_1u,T_2u') = 0$ and so $u = T_1u$, $u' = T_2u'$ and $T_1u = T_2u'$ and so the common fixed point of $T_1$ and $T_2$ is in this case unique.
We note that without the extracondition "$d(x.T_1x)+d(y,T_2y) = 0$ implies $d(T_1x,T_2y) = 0$" the common fixed point is not necessarily unique. (Ex., pp. 40, [1].)

Remark. If $p=1$ then theorem 1 is obtained.

Theorem 5. Let $(X,d)$ be a complete metric space and $\{T_n\}_{n \in N}$ a sequence of multifunctions of $X$ into $CB(X)$ such that

$$(4) \quad H^p(T_1x,T_ny) \leq \frac{cd(x,T_1x).d^p(y,T_ny) + bd(x,T_ny).d^p(y,T_1x)}{\delta(x,T_1x) + \delta(y,T_ny)}$$

holds for $x$, $y$ in $X$ for which $\delta(x,T_1x) + \delta(y,T_ny) \neq 0$, where $n \geq 2$, $p \geq 1$, $b \geq 0$, $1 < c < 2$, then $\{T_n\}_{n \in N}$ has a common fixed point and $F(T_1) = F(T_n)$.

The proof follows by theorem 2 and lemma.

Let X be a nonempty set and e and d two metrics on X and $f:X \to X$ a single valued mapping. For such mappings Maia [5] proved a fixed point theorem which was generalized in many directions by Iséki [6], I.A. Rus [7],[8], K.L. Singh [9] and others. I gave in [10] and [11] some generalizations of Maia's theorem for multifunctions.

Now we prove a fixed point theorem for a sequence of multifunctions in a set with two metrics.

Theorem 6. Let X be a metric space with two metrics e and d. If X satisfies the following conditions:

(1) $e(x,y) \leq d(x,y)$; $\forall x,y \in X$,

(2) X is complete with respect to e,

(3) two multifunctions $T_1, T_2 : X \to X$ are punctually closed and punctually bounded with respect to both metrics,

(4) $T_1$ or $T_2$ is u.s.c. with respect to e,

(5) the inequality (2) holds for all x, y in X for which $\delta(x,T_1x) + \delta(y,T_2y) \neq 0$, where $p \geq 1$, $b \geq 0$, $1 < c < 2$, then $T_1$ and $T_2$ have a common fixed point and $F(T_1) = F(T_2)$.

Proof. Analogously as in the proof of the theorem 2, for any $x_0 \in X$ we can construct a sequence $\{x_n\}$ such that $x_{2n+1} \in T_1x_{2n}$, $\{x_n\}$ being a Cauchy sequence with respect to d. Therefore, by $e \leq d$, $\{x_n\}$ is a Cauchy sequence with respect to e and since X is complete with respect to e, $x_n \to x$. As $T_1$ is u.s.c. from the theorem 4 [10], $T_1$ has a closed graph and then from $x_{2n+1} \in T_1x_{2n}$ it results $x \in T_1x$ and from lemma $F(T_1) = F(T_2)$.

**Theorem 7.** Let X be a metric space with two metrics e and d. If X satisfies the following conditions:

(1) The sequence of multifunctions $\{T_n\}_{n\in N}$ is formed by punctually closed and punctually bounded multifunctions with respect to both metrics,

(2) e, d and $T_1$ satisfy conditions (1),(2) and (4) of theorem 6,

(3) the inequality (4) holds for all x, y in X for which $\delta(x,T_1x) + \delta(y,T_ny) \neq 0$, where $n \geq 2$, $p \geq 1$, $b \geq 0$, $1 < c < 2$, then $\{T_n\}_{n\in N}$ has common fixed points and $F(T_1) = F(T_n)$.

The proof follows by theorem 6 and lemma.

## References


[1] B. FISHER: Theorems on mappings satisfying a rational inequality, Comment. Math. Univ. Carolinae 19(1978), 37-44.

[2] J. ACHARI: On a family of maps and fixed points, C.R.Acad. Bulgare des Sciences, 2(1977), 171-174.

[3] TORU KITA: A common fixed point theorem for multivalued mappings in complete metric space, Mathematica Japonicae, 33(1977), 113-116.

[4] I.A. RUS: Fixed point theorems for multivalued mappings in complete metric space, Mathematica Japonicae, 20(1975), 21-24.

[5] D.G. MAIA: Un osservazione sulle contrazione metriche, Rend. Semin. Mat. Univ. Padova, 40(1968),139-143.

[6] K. ISÉKI: A common fixed point theorem, Rend. Semin. Mat. Univ. Padova, 55(1975), 13-14.

[7] I.A. RUS: On a fixed point theorem of Maia, Studia Univ. Babeş-Bolyai, Mathematica, XXII(1977), 40-42.

[8] I.A. RUS: On a fixed point theorem in a set with two metrics, L'Analyse numérique et la théorie de l approximation, 6(1977), 197-202.

[9] K.L. SINGH: A note on common fixed points, Bull. Math. Soc. Sci. Math. R.S.R., 22(70)(1978), 95-98.

[10] V. POPA: Fixed point theorems for multifunctions, Studia Univ. Babeş-Bolyai, Mathematica, XXVII(1982), 21-27.

[11] V. POPA: Fixed point theorems for a sequence of multifunctions, (to appear - Bull. Math. Soc. Sci. Math. R.S.R.)

Higher Education Institute, Bacău, Romania

(Oblatum 17.10. 1983)