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ON CONGRUENCES IN DIRECT SUMS OF ALGEBRAS

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Abstract: Some results concerning congruences on direct products of finitely many universal algebras not extending to products of infinitely many factors are generalized to direct sums of algebras.

Key words: Universal algebra, variety, direct product, direct sum, congruence relation.

Classification: Primary 08A30, 08B25

When Fraser and Horn [1] and Hu [4] established a Mal'cev type characterization of varieties V of universal algebras in which every congruence on the direct product $A \times B \in V$ is of form $\alpha \times \beta$ for some $\alpha \in$ Con A, $\beta \in$ Con B solving a problem from Grätzer [3], they noticed that their result does not generalize to arbitrary direct products (i.e. of infinitely many factors), because of the congruences induced by nonprincipal filters on the index sets which are always skew. The attempt by Nelson [5] to include the filtered products of congruences led primarily to negative results, as well.

We will show that in order to generalize the above mentioned result to infinitely many factors the notion of the direct sum (called also weak direct product) is much more fitting than that of the direct product. The generalization runs quite smoothly in the expected way and "everything preserved by finite direct products is preserved by direct sums".

1. Preliminaries. For unexplained symbols and notions the reader should consult Grätzer [3].

$N = \{0, 1, 2, \dots\}$ is the set of all natural numbers.

For an algebra A Con A denotes its congruence lattice with the least element $0_A = 0$ and the largest element $1_A = 1$. If A is a subalgebra of an algebra B and $\beta \in$ Con B then $A \wedge \beta = A^2 \cap \beta \in$ Con A denotes the restriction of β to A.

Given a subalgebra A of the direct product $\prod(B_j : j \in J)$ and $X \subseteq J$

$$A \upharpoonright X = \{a \upharpoonright X : a \in A\}$$

is the subalgebra of the direct product $\prod(B_j : j \in X)$ formed by the restrictions $a \upharpoonright X$ of the functions from A to X. The kernel of the natural projection $A \longrightarrow A \upharpoonright X$ is a congruence on A denoted by π_X. If $\alpha \in$ Con A is a congruence on A, its image under this projection generates a congruence on $A \upharpoonright X$ denoted by $\alpha \upharpoonright X$. For $X = \{j\}$ we write $\pi_X = \pi_j$ and $\alpha \upharpoonright X = \alpha(j)$. If $\beta_j \in$ Con B_j $(j \in J)$ then $\prod(\beta_j : j \in J)$ denotes the congruence on $\prod(B_j : j \in J)$ defined, as usual, componentwise.

For $a, b \in \prod(B_j : j \in J)$ we put

$$[\![a = b]\!] = \{j \in J : a(j) = b(j)\}$$

and $[\![a \neq b]\!] = J - [\![a = b]\!]$. A subalgebra $A \in \prod(B_j : j \in J)$ is called a direct sum of the algebras B_j $(j \in J)$ provided for each $a \in A$ and each $b \in \prod(B_j : j \in J)$ holds $b \in A$ iff $[\![a \neq b]\!]$ is finite. Obviously, every direct sum is a subdirect product of the system of algebras in mind, and for finite systems the notions of direct sum and direct product coincide. Let us remark that the direct sum of an arbitrary system of algebras needs neither to exist (meaning to be nonempty) nor to be uniquely determined as a subalgebra of the direct product.

We state without proof the following two easy lemmas:

LEMMA 1. Let A be a subalgebra of the direct product $\prod(B_j: j \in J)$ and $\alpha, \beta \in \mathrm{Con}\, A$. Then $\alpha \subseteq \beta$ iff for each $\langle a,b\rangle \in \alpha$ holds

$$\alpha \wedge \pi_{[a=b]} \subseteq \beta \quad .$$

LEMMA 2. Let A be a direct sum of a system of algebras $(B_j: j \in J)$ and let $X \subseteq J$ be finite. Then there is a natural isomorphism

$$A \cong \prod(B_j: j \in X) \times A \wedge J - X \ .$$

2. Results. Let V be a variety of algebras and $n \in N$. A function F with domain V such that for each $A \in V$ $F_A = F$ is a n-ary operation on Con A is called a (n-ary) congruence operation on V. A congruence operation F is preserved by (finite) direct products iff for any (finite) system $(B_j: j \in J)$ of algebras from V and all $\alpha_j^k \in \mathrm{Con}\, B_j$ $(j \in J,\ 1 \leq k \leq n)$ holds

$$F(\prod(\alpha_j^1: j \in J), \dots, \prod(\alpha_j^n: j \in J)) = \prod(F(\alpha_j^1, \dots, \alpha_j^n): j \in J) \ .$$

F is preserved by direct sums iff for any direct sum A of the algebras B_j and congruences $\alpha_j^k \in \mathrm{Con}\, B_j$ as above holds

$$F(A \wedge \prod(\alpha_j^1: j \in J), \dots, A \wedge \prod(\alpha_j^n: j \in J)) = A \wedge \prod(F(\alpha_j^1, \dots, \alpha_j^n): j \in J) \ .$$

THEOREM 1. Let F be an n-ary congruence operation on the variety V. The following conditions are equivalent:

(i) F is preserved by direct products of two factors;

(ii) F is preserved by finite direct products;

(iii) F is preserved by direct sums.

PROOF. (i) $\Rightarrow$ (ii) follows directly by induction, (iii) $\Rightarrow$ (i) is trivial. It is enough to prove (i) & (ii) $\Rightarrow$ (iii). For notational convenience we assume that $F(\alpha, \beta) = \alpha \cdot \beta = \alpha\beta$ is a binary congruence operation on V. The general case can be treated similarly. Let $\alpha_j, \beta_j \in \mathrm{Con}\, B_j$ $(j \in J)$ and A is a direct sum of the algebras B_j from V. According to Lemma 1 we will be over if we show that for each finite $X \subseteq J$ holds

$$(*) \qquad A \upharpoonright \prod(\alpha_j : j \in J) \cdot A \upharpoonright \prod(\beta_j : j \in J) \wedge \pi_{J-X}$$

$$= A \upharpoonright \prod(\alpha_j \beta_j : j \in J) \wedge \pi_{J-X}$$

Identifying the isomorphic algebras from Lemma 2 we obtain

$$A \upharpoonright \prod(\alpha_j : j \in J) \cdot A \upharpoonright \prod(\beta_j : j \in J) \wedge \pi_{J-X} =$$

$$(\prod(\alpha_j : j \in X) \times (A \wedge J - X) \upharpoonright \prod(\alpha_j : j \in J - X)) \cdot$$

$$(\prod(\beta_j : j \in X) \times (A \wedge J - X) \upharpoonright \prod(\beta_j : j \in J - X)) \wedge \pi_{J-X}$$

$$= \prod(\alpha_j : j \in X) \cdot \prod(\beta_j : j \in X) \times$$

$$(A \wedge J - X) \upharpoonright \prod(\alpha_j : j \in J - X) \cdot (A \wedge J - X) \upharpoonright \prod(\beta_j : j \in J - X)$$

$$\wedge \pi_{J-X} = \prod(\alpha_j : j \in X) \cdot \prod(\beta_j : j \in X) \times 0_A \wedge J - X \quad ,$$

and

$$A \upharpoonright \prod(\alpha_j \beta_j : j \in J) \wedge \pi_{J-X} =$$

$$\prod(\alpha_j \beta_j : j \in X) \times (A \wedge J - X) \upharpoonright \prod(\alpha_j \beta_j : j \in J - X) \wedge \pi_{J-X}$$

$= \prod(\alpha_j \beta_j : j \in X) \times 0_A \wedge J - X$.

Now, the desired equality (*) follows from (ii).

Let $(B_j : j \in J)$ be a system of algebras from a variety V, and A is a subalgebra of their direct product. All congruences α on A which are not of the form $A \wedge \prod(\alpha_j : j \in J)$ for any $\alpha_j \in$ Con B_j $(j \in J)$ are called skew (on A).

THEOREM 2. Let V be a variety of algebras. The following conditions are equivalent:

(i) there are no skew congruences on direct products of two algebras from V;

(ii) there are no skew congruences on finite direct products of algebras from V;

(iii) there are no skew congruences on direct sums of algebras from V.

PROOF. Let us concentrate on (ii) $\Rightarrow$ (iii) only. If A is a direct sum of the system $(B_j : j \in J)$ from V and $\alpha \in$ Con A, it is enough to show

$$A \wedge \prod(\alpha(j) : j \in J) \subseteq \alpha ,$$

the reversed inclusion being trivial. Let $X \subseteq J$ be finite. Using Lemma 2 we obtain

$$A \wedge \prod(\alpha(j) : j \in J) \wedge \pi_{J-X} =$$

$$\prod(\alpha(j) : j \in X) \times (A \wedge J - X) \wedge \prod(\alpha(j) : j \in J - X) \wedge \pi_{J-X}$$

$$= \prod(\alpha(j) : j \in X) \times 0_A \wedge J - X$$

$$\subseteq \prod(\alpha(j) : j \in X) \times \alpha \wedge J - X = \alpha$$

according to (ii). Now, the result follows from Lemma 1.

Since finite meets (trivially) and finite joins (see Fraser-Horn [1]) are preserved by finite direct products, from Theorem it follows that for a direct sum A of any system of algebras $(B_j: j \in J)$ there is an injective 0,1-homomorphism of lattices $\prod(\mathrm{Con}\ B_j: j \in J) \longrightarrow \mathrm{Con}\ A$ given by

$$\langle \alpha_j: j \in J \rangle \mapsto A \wedge \prod(\alpha_j: j \in J) .$$

which is an isomorphism in any variety satisfying the conditions of Theorem 2. Moreover, every congruence operation in the variety preserved by finite direct products is preserved by the above map, too. The commutator $[\alpha, \beta]$ (see Freese-McKenzie [2]) as well as the congruence operation $[\alpha \rightarrow \beta] = \bigvee(\gamma : [\gamma, \alpha] \subseteq \beta)$, called mutator by Zlatoš [6], can serve as examples.

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