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REPLACEABLE NETS AND IMPROPER COLLINEATIONS

V. HAVEL

Abstract: In this Note there is answered the question what is the mutual connection between the following properties of a given net: (i) to be replaceable (in the sense of T.G. Ostrom) and (ii) to admit an improper collineation onto itself (in the sense of V.D. Belousov).

Key words: Net, affine plane, proper and improper collineation, replaceable net, Ostrom net.

Classification: Primary 51A10, 51A99
Secondary 20N99

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§ 1. Fundamental notions. Under an incidence structure (more precisely, regular incidence structure) we understand a couple $(\mathcal{P},\mathcal{L})$, where $\mathcal{P}$ is a set and $\mathcal{L}$ is a non-void set of some at least two-element subsets of the set $\mathcal{P}$, satisfying (i) for all $a, b \in \mathcal{P}$, $a \neq b$, there exists at most one $c \in \mathcal{L}$ such that $a, b \in c$. As a consequence of (i), the incidence structure $(\mathcal{P},\mathcal{L})$ satisfies also (ii) for all $a, b \in \mathcal{L}$, $a \neq b$, there exists at most one $c \in \mathcal{P}$ such that $c \in a, b$.

Some denotations: Let $(\mathcal{P},\mathcal{L})$ be an incidence structure. Elements of $\mathcal{P}$, respectively of $\mathcal{L}$ will be called points, respectively lines. Non-disjoint distinct lines a, b have just one point in common; this point will be called point of intersection and will be denoted by $a \sqcap b$.

Two distinct points a, b lying on the same line are said to be joined and the line containing a and b will be denoted by $a \sqcup b$.

We say an incidence structure $\mathcal{I} = (\mathcal{P},\mathcal{L})$ is embedded into an incidence structure $\mathcal{I}' = (\mathcal{P}',\mathcal{L}')$ if $\mathcal{P}$ is a subset of $\mathcal{P}'$ and every line of $\mathcal{I}$ is a subset of a line of $\mathcal{I}'$; sometimes we shall use only a shorter formulation " $\mathcal{P}$ is embedded into $\mathcal{I}'$ ".

Now we will formulate some further conditions which may be satisfied in an incidence structure $\mathcal{I} = (\mathcal{P},\mathcal{L})$; (iii) any two distinct points of $\mathcal{I}$ are joined.

(Join condition)

(iv'), resp. (iv). For every line a of $\mathcal{I}\{x \in \mathcal{L} \mid x = a \vee x \cap a = \emptyset\}$ is a partition of a subset (depending on a) of $\mathcal{P}$, respectively a partition of $\mathcal{P}$.

(Weak, respectively strong parallelity condition)

If an incidence structure $\mathcal{I} = (\mathcal{P},\mathcal{L})$ satisfies the weak parallelity condition then $/\!/ = \{(a,b) \in \mathcal{L} \times \mathcal{L} \mid a = b \vee a \cap b = \emptyset\}$ is an equivalence relation on $\mathcal{L}$ called parallelity relation or briefly: parallelity.

If an incidence structure $\mathcal{I} = (\mathcal{P},\mathcal{L})$ satisfies the weak parallelity condition then

$$\#(\mathcal{L}/\!/) = \#\{\{x \in \mathcal{L} \mid x = a \vee x \cap a = \emptyset\} \mid a \in \mathcal{L}\}$$

will be called the degree of $\mathcal{I}$.

An incidence structure $\mathcal{I} = (\mathcal{P},\mathcal{L})$ satisfying the strong parallelity condition and having degree ≥ 3 is called a net.

In a net $\mathcal{I} = (\mathcal{P},\mathcal{L})$ it holds $\#a = \#b$ for all $a, b \in \mathcal{L}$ and this cardinal number is called the order of the net.

A net satisfying the join condition is called an <u>affine plane</u>. Let $\mathcal{I} = (\mathcal{P},\mathcal{L})$, $\mathcal{I}' = (\mathcal{P}',\mathcal{L}')$ be incidence structures and σ a mapping of $\mathcal{P}$ into $\mathcal{P}'$, we shall denote this mapping $\sigma: \mathcal{P} \longrightarrow \mathcal{P}'$ also by $\sigma: \mathcal{I} \longrightarrow \mathcal{I}'$. We say σ is <u>join preserving</u> if for any two distinct joined points a, b of $\mathcal{I}$ with distinct images $\sigma(a) \neq \sigma(b)$ it follows that $\sigma(a)$, $\sigma(b)$ are joined in $\mathcal{I}'$.

If σ is bijective and both σ, σ^{-1} are join preserving then σ will be called a <u>collineation</u>. A collineation σ will be said to be <u>proper</u> if for every $a \in \mathcal{L}$ it follows $\{\sigma(x) \mid x \in a\} \in \mathcal{L}'$; otherwise σ is called <u>improper</u>. A collineation $\sigma: \mathcal{I} \longrightarrow \mathcal{I}$ is called an <u>autocollineation</u> of $\mathcal{I}$. Another denotation for a proper collineation, respectively for a proper autocollineation is <u>isomorphism</u>, respectively <u>automorphism</u>. If there exists an isomorphism $\sigma: \mathcal{I} \longrightarrow \mathcal{I}'$ then $\mathcal{I}$ and $\mathcal{I}'$ are said to be <u>isomorphic</u>.

A net $\mathcal{N} = (\mathcal{P},\mathcal{L})$ is said to be <u>replaceable</u> if $id_{\mathcal{P}}$ (the identity mapping of $\mathcal{P}$) is an improper collineation of $\mathcal{N}$ onto some net $\mathcal{N}^* = (\mathcal{P},\mathcal{L}^*)$ with $\mathcal{L}^* \neq \mathcal{L}$; $\mathcal{N}^*$ is then a <u>replacing net</u> of $\mathcal{N}$.

§ 2. <u>Replaceable nets versus nets admitting improper autocollineations</u>

Proposition 1. a) If there exists an improper collineation of a net $\mathcal{N} = (\mathcal{P},\mathcal{L})$ then $\mathcal{N}$ is replaceable.

b) If there exists an improper collineation of an incidence structure $\mathcal{I} = (\mathcal{P},\mathcal{L})$ onto an incidence structure $\mathcal{I}' = (\mathcal{P}', \mathcal{L}')$ then $id_{\mathcal{P}}$ is an improper collineation of $\mathcal{I}$ on a convenient incidence structure $(\mathcal{P}, \mathcal{L}^*)$ with $\mathcal{L}^* \neq \mathcal{L}$.

Proof. b) Let there exist an improper collineation $\varkappa :$ $: \mathcal{I} \longrightarrow \mathcal{I}'$. We put $\mathcal{L}^* = \{\{\varkappa^{-1}(x) \mid x \in a\} \mid a \in \mathcal{L}'\}$ and get $\mathcal{I}^* =$ $= (\mathcal{P}, \mathcal{L}^*)$, an incidence structure which is isomorphic to $\mathcal{I}'$. As it is seen, $\mathcal{L}^* \neq \mathcal{L}$ and $id_{\mathcal{P}} : \mathcal{I} \longrightarrow \mathcal{I}^*$ is an improper collineation.

a) Let there exist an improper collineation $\varkappa : \mathcal{N} \longrightarrow \mathcal{N}'$. We put again $\mathcal{L}^* = \{\{\varkappa^{-1}(x) \mid x \in a\} \mid a \in \mathcal{L}'\}$ and obtain a net $\mathcal{N}^* =$ $= (\mathcal{P}, \mathcal{L}^*)$ isomorphic with $\mathcal{N}'$. Here $\mathcal{L}^* \neq \mathcal{L}$ and $id_{\mathcal{P}} : \mathcal{N} \longrightarrow \mathcal{N}^*$ is an improper collineation so that $\mathcal{N}$ is replaceable. $\square$

Proposition 2. a) A net admits an improper autocollineation if and only if it admits an isomorphic replacing net.

b) An incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ admits an improper autocollineation if and only if $id_{\mathcal{P}}$ is an improper collineation of $\mathcal{I}$ onto some incidence structure $\mathcal{I}^* = (\mathcal{P}, \mathcal{L}^*)$ isomorphic to $\mathcal{I}$.

Proof. a) By proposition 1 b), if the structure $\mathcal{I} =$ $= (\mathcal{P}, \mathcal{L})$ admits an improper autocollineation $\varkappa$ then $id_{\mathcal{P}}$ is an improper collineation of $\mathcal{I}$ onto a convenient incidence structure $\mathcal{I}^* = (\mathcal{P}, \mathcal{L}^*)$ where $\mathcal{L}^*$ is different from $\mathcal{L}$ and $\varkappa : \mathcal{I} \longrightarrow \mathcal{I}^*$ is an isomorphism. Conversely, if $id_{\mathcal{P}}$ is an improper collineation of a given structure $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ onto some structure $\mathcal{I}^* = (\mathcal{P}, \mathcal{L}^*)$ isomorphic to $\mathcal{I}$ then there is an isomorphism $\varkappa : \mathcal{I} \longrightarrow \mathcal{I}^*$ and this $\varkappa$ is at the same time an improper autocollineation of $\mathcal{I}$.

b) The argumentation from point a) can be carried over onto point b) analogously as it was made in the proof of proposition 1. $\square$

Example. Investigate the net $\mathfrak{N} = \{\{1,2,3,4,5,6,7,8,9\}, \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}\}$ from Fig. 1. The mapping

$1 \mapsto 4 \mapsto 8 \mapsto 2 \mapsto 6 \mapsto 9 \mapsto 1, \quad 3 \mapsto 3, \quad 5 \mapsto 5, \quad 7 \mapsto 7$

is an improper autocollineation of $\mathfrak{N}$. Further, the net $\mathfrak{N}^* = \{\{1,2,3,4,5,6,7,8,9\}, a'_1, a'_2, a'_3, b'_1, b'_2, b'_3, c'_1, c'_2, c'_3\}\}$ from Fig. 2 is a replacing net of $\mathfrak{N}$ and is isomorphic to $\mathfrak{N}$.

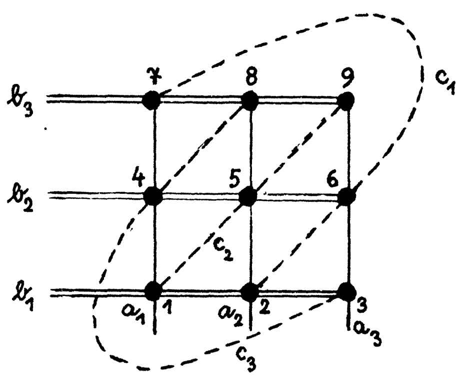

Fig. 1

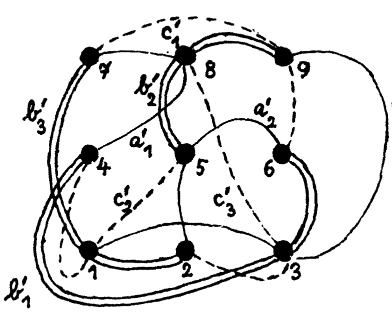

Fig. 2

A net will be called Ostrom net of degree k and of dimension 2 if it is isomorphic with the net $F_{(k-1)} = (F^4, \{\{(x_1, x_2, ux_1+v_1, ux_1+v_1, x_2, ux_2+v_2) \mid x_1, x_2 \in F\} \mid u, v_1, v_2 \in F\} \cup \{\{(a_1 y_1, a_2, y_2) \mid y_1, y_2 \in F\} \mid a_1, a_2 \in F\}$

Theorem 1 (J. Klouda). A net of degree k and of order $(k-1)^2$ admits an improper autocollineation if and only if it is Ostrom net of degree k and of dimension 2. An isomorphic replacing net for $F_{(k-1)}$ $(F = GF(k-1))$ is the net
$F_{[k-1]} = (F^4, \{\{(x_1, ux_1+v_1, x_2, ux_2+v_2) \mid x_1, x_2 \in F\} \mid u, v_1, v_2 \in F\} \cup$
$\cup \{\{(a_1, y_1, a_2, y_2) \mid y_1, y_2 \in F\} \mid a_1, a_2 \in F\}$
with isomorphism $\varkappa : F_{(k+1)} \to F_{[k-1]}, (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4)$.
The set of all improper autocollineations of $F_{(k-1)}$ is $\{\lambda \circ \varkappa \mid \lambda \in Aut\, F_{(k-1)}\}$ where $Aut\, F_{(k-1)}$ is the set of all automorphisms of $F_{(k-1)}$ (determined f.e. in [1], Theorem 4 and in [6]).

Proof: cf. [5].

§ 3. Some properties of replaceable nets of degree k and of order $(k-1)^2$

Proposition 3. Let $\mathcal{N} = (\mathcal{P}, \mathcal{L})$ be a replaceable net of degree k and of order m having a replacing net $\mathcal{N}^* = (\mathcal{P}^*, \mathcal{L}^*)$ of degree k^* and of order m^*. Then $k^* = k$, $m^* = m$, $m \leq (k-1)^2$ and every line of $\mathcal{N}^*$ is an incidence structure which is embedded into $\mathcal{N}$ and satisfies the join condition and weak parallelity condition. Moreover, every line of $\mathcal{N}^*$ is an affine plane (of order $k-1$) embedded into $\mathcal{N}$ if and only if $m = (k-1)^2$.

Proof. From $\# \mathcal{P} = m^2 = m^{*2}$ it follows $m = m^*$. The number of all joined (non-ordered) couples of distinct points of $\mathcal{N}$ respectively of $\mathcal{N}^*$ is $\frac{1}{2} m^2 k (m-1)$ respectively $\frac{1}{2} m^2 k^* (m-1)$ so that $k = k^*$. As $\mathcal{N}^*$ is a replacing net for $\mathcal{N}$ there exists a line $a^* \in \mathcal{L}^* \setminus \mathcal{L}$. First we shall show that $\#(a^\bullet \cap a) \leq k-1$ for all $a \in \mathcal{L}$: Suppose $\#(a^\bullet \cap a) \geq k$ for some $a \in \mathcal{L}$ and take a point $A \in a^\bullet \setminus a$ (which exists because of $\# a = \# a^\bullet$, $a \neq a^\bullet$). Now we construct the joining lines of A with each of mentioned k points (these joining lines exist since any two distinct points of a line of $\mathcal{N}^\bullet$ must be joined in $\mathcal{N}$). So we get (in $\mathcal{N}$) at least k lines through A intersecting with a , a contradiction. - Now take a point $B \in a^\bullet$.

For every line $b \in \mathcal{L}$ through B we have $\#(a^\bullet \cap b) \leq k-1$ and every point of $a^\bullet \setminus \{B\}$ lies on exactly one such a line. Consequently $m = \# a^\bullet \leq k(k-2)+1 = (k-1)^2$. [1] Every line $\ell^\bullet \in \mathcal{L}^\bullet$ determines the set $\mathcal{L}_{\ell^\bullet} = \{a \cap \ell^\bullet \mid a \in \mathcal{L}, \#(a \cap \ell) \geq 2\}$. This set $\mathcal{L}_{\ell^\bullet}$ is non-void: Actually, $\# \ell^\bullet \geq 2$ and any two distinct points of $\ell^\bullet$ must be joined in $\mathcal{N}$ as $\mathcal{N}^\bullet$ is a replacing net for $\mathcal{N}$. It follows that $(\ell^\bullet, \mathcal{L}_{\ell^\bullet})$ is an incidence structure which is embedded into $\mathcal{N}$ and satisfies the join condition and weak parallelity condition. -

Moreover, a line $a^\bullet \in \mathcal{L}^\bullet \setminus \mathcal{L}$ is an affine plane (of order $k-1$) embedded into $\mathcal{N}$ if and only if for every $A^\bullet \in a^\bullet$ there are precisely k lines from $\mathcal{L}_{a^\bullet}$ going through $A^\bullet$ and any such line intersects $a^\bullet$ in just $k-1$ points. This occurs if and only if $\# a^\bullet = k(k-2)+1 = (k-1)^2$. Further we will show that for $m = (k-1)^2$ every line $\ell^\bullet \in \mathcal{L}^\bullet$ is an affine plane of order $(k-1)^2$ embedded into $\mathcal{N}$: Assume on the contrary that there is a line $c \in \mathcal{L} \cap \mathcal{L}^\bullet$. We take a line $a^\bullet \in \mathcal{L}^\bullet \setminus \mathcal{L}$ (the existence of which was stated above). Then $a^\bullet \cap c = \emptyset$ as in case $a^\bullet \cap c \neq \emptyset$ it would follow $\#(a^\bullet \cap c) = k-1$ and this is impossible while the lines $a^\bullet, c \in \mathcal{L}^\bullet$ are not parallel in

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1) $k, m < \aleph_0$

$\mathcal{N}^*$ and as such they have only one point common. Now choose a line $d \in \mathcal{L}$ which goes through a point $D \in a^*$ and is not parallel to c. Then $c \sqcap d$, D must be contained in some line $d^* \in \mathcal{L}^*$. By the above argumentation used on $c \in \mathcal{L}$, $d^* \in \mathcal{L}^*$ we obtain $\#(c \cap d^*) = k - 1$.

On the other side c and d^* are not parallel lines in $\mathcal{N}^*$ and have a one-point intersection. Thus we have obtained a contradiction. □

Proposition 4. Let $\mathcal{N} = (\mathcal{P}, \mathcal{L})$ be a replaceable net of degree k and of order $(k-1)^2$ with a replacing net $\mathcal{N}^* = (\mathcal{P}, \mathcal{L}^*)$. Then the following conditions are valid: a) For any disjoint $\alpha, \beta \in \mathcal{L}^*$ there exists $c \in \mathcal{L}$ such that $\alpha \cap c, \beta \cap c \neq \emptyset$ (such $c \in \mathcal{L}$ will be called a <u>cross-line</u> of α, β). All cross-lines of given disjoint $\alpha, \beta \in \mathcal{L}^*$ are parallel and every line of $\mathcal{N}$ parallel with such a cross-line is either a cross-line too or is disjoint with both α, β.
b) For any two disjoint $\alpha, \beta \in \mathcal{L}^*$ and for every $c \in \mathcal{L}$ with $c \cap \alpha$, $c \cap \beta \neq \emptyset$ it follows $\alpha \sqcap \beta \in c$.

Proof. a) Let α, β be disjoint lines of $\mathcal{N}^*$. Then all <u>prolongations</u> of lines of the affine plane α (i.e., these lines of $\mathcal{N}$ which contain some line of α) contain altogether

$$k(k-1)(k-1)(k-2) = (k-1)^4 - (k-1)^2$$

points outside α so that these points exhaust $\mathcal{P} \setminus \alpha$. Thus at least one of prolongations must be a cross-line of α, β. If a, b are two non-parallel cross-lines of given disjoint $\alpha, \beta \in \mathcal{L}^*$ then $a \sqcap b \in \alpha \cap \beta$, a contradiction. Thus two distinct cross-lines of given disjoint α, $\beta \in \mathcal{L}^*$ are always parallel.

Let there be given disjoint α, $\beta \in \mathcal{L}^*$ and a point $A \in \alpha$.

Choose an arbitrary $\gamma \in \mathcal{L}^{\circ} \setminus \{\alpha\}$ going through A. Then β, γ are non-parallel lines of $\mathcal{N}^{\circ}$ and the points $A, \beta \sqcap \gamma$ are joined in $\mathcal{N}$ (as they are joined in $\mathcal{N}^{\circ}$). Thus every point $A \in \alpha$ is contained in a cross-line of α, β. Similarly, every point $B \in \beta$ is contained in a cross-line of α, β. We can result: For any given disjoint $\alpha, \beta \in \mathcal{L}^{\circ}$ there exist cross-lines of them. These cross-lines belong to the same parallelity class of $\mathcal{N}$ and such lines of this parallelity class which are not cross-lines of α, β are disjoint with both α, β.

b) Let $\alpha, \beta \in \mathcal{L}^{\circ}$ be not parallel in $\mathcal{N}^{\circ}$ and let $c \in \mathcal{L}$ not contain the point $\alpha \sqcap \beta$. Thus through $\alpha \sqcap \beta$ there go just $k-1$ lines of the affine plane α and just $k-1$ lines of the affine plane β. The prolongations of these lines are not parallel with c. Since $\#(\alpha \cap \beta) = 1$ there are just $2(k-1)$ of such prolongations. As $2(k-1) > k$ ($\Longleftrightarrow k > 2$) we have a contradiction to the assumption that k is the degree of $\mathcal{N}$ (and that consequently through $\alpha \sqcap \beta$ there go just k lines of $\mathcal{N}$).

Theorem 2. A net of degree k and of order $(k-1)^2$ is replaceable if and only if it is Ostrom net of degree k and of dimension 2. Thus a net of degree k and of order $(k-1)^2$ is replaceable if and only if it admits an improper autocollineation.

Proof. If a net is Ostrom net of degree k and of dimension 2 then it is replaceable, by proposition 1 and by theorem 1. -

If a net of degree k and of order $(k-1)^2$ is replaceable then proposition 4 permits to apply the argumentation from the proofs of theorems 2,1 and 2,3 from [5] so that by theorem 2,4 from [5] the given net must be Ostrom net of degree k and of dimension 2. $\square$

Proposition 5 (cf. [2], proposition 7.2 on p. 22 [2]). Let $\mathcal{N} = (\mathcal{P},\mathcal{L})$ be a net of degree k and of order $(k-1)^2$. Then $\mathcal{N}$ is replaceable if and only if the following condition is valid: Any two distinct joined points of $\mathcal{N}$ are contained in just one affine plane of order $k-1$ embedded into $\mathcal{N}$.

Proof. The part b) is obvious because every line of a replacing net $\mathcal{N}^*$ of $\mathcal{N}$ is an affine plane of order $k-1$ embedded into $\mathcal{N}$ and any two different points which are joined (simultaneously in $\mathcal{N}$ and in $\mathcal{N}^*$) determine just one line in $\mathcal{N}$ containing both these points and just one line in $\mathcal{N}^*$ containing both these points.

We go over to part a). Denote by $\mathcal{L}^*$ the set of all affine planes of order $k-1$ embedded into $\mathcal{N}$. We shall show that $(\mathcal{P},\mathcal{L}^*)$ is an incidence structure satisfying the strong parallelity condition: Obviously any two non-joined points cannot be contained in the same $\ell^* \in \mathcal{L}^*$ whereas any two distinct joined points are contained in just one $\ell^* \in \mathcal{L}^*$ by assumption. Thus $(\mathcal{P},\mathcal{L}^*)$ is an incidence structure.

Now investigate a point a and an affine plane $\beta \in \mathcal{L}^*$ not through a. We assert that there exists just one affine plane $\alpha \in \mathcal{L}^*$ going through a and being disjoint to β. Again (as in the proof of proposition 4 a)) we shall show that through any point outside $\beta \in \mathcal{L}^*$ there goes a line of $\mathcal{N}$

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2) The reasoning from [2] (p. 22, i.e. the only reference onto theorem 2 from T.G. Ostrom's "Net with critical deficiency", Pac.J.Math. 14(1964), 1381-1387 and onto theorem 6 from T.G. Ostrom's "Semi-translation planes", Trans.Amer.Math.Soc. 111 (1964), 1-18) seems to be unsatisfactory.

having a non-void intersection with β : Indeed, the total number of points lying on prolongations of lines of the plane β is $k(k-1)(k-1)(k-2)$. As this number is equal to $(k-1)^4 - (k-1)^2$, we obtain in this way all points of $\mathcal{P} \setminus \beta$.

Let us return to a couple formed by a point a and an affine plane $\beta \in \mathcal{L}^{\bullet}$ not containing a and choose a line $b \in \mathcal{L}$ going through a and having a non-void intersection with β. We know that $\#(b \cap \beta) = k-1$. Every couple of distinct points a, x with $x \in b \cap \beta$ is obtained in some $\alpha_x \in \mathcal{L}^{\bullet}$ and the total number of such α_x is $k-1$. Thus through a it goes still the remaining affine plane $\alpha \in \mathcal{L}^{\bullet}$. We assert that $\alpha \cap \beta = \emptyset$: Indeed, if on the contrary, $\alpha \cap \beta \neq \emptyset$, then there exists just one common point c of α, β so that $a = b \sqcap (a \sqcup c)$ and consequently $a \in \beta$, a contradiction. $\square$

R e f e r e n c e s

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