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MONOMORPHISMS AND EPIMORPHISMS OF INVERSE SYSTEMS

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Abstract: Monomorphisms and epimorphisms in a category Pro-C are studied. Characterizations of such morphisms are obtained in case C = SET or C is a topological category over SET.

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0. INTRODUCTION. Given a category C, Pro-C denotes the category of inverse systems in C and their morphisms, following Grothendieck's definition [6]. The notion of inverse systems and the pro-categories have been widely used in Algebraic Topology and, after the work of Mardešić and Segal [10,11], they are a fundamental tool in the study of Shape Theory, in all its aspects. Nevertheless, there exist, up to author's knowledge, no characterizations of monomorphisms and epimorphisms in Pro-C yet.

In this note we give some necessary and sufficient conditions in order to recognize special morphisms in that category. We shall be mainly concerned with those (Pro-C)-morphisms having as domain or codomain a rudimentary system, i.e. a system formed by a single object of C. Such morphisms are interesting since they play a central role in Shape Theory and in recent investigations in Categorical Topology, concerning the connections between (epi-) reflective

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and (epi-) pro-reflective subcategories [3,4,5,12,13].

Most of the results of the paper are contained in section 2 where we characterize monomorphisms and epimorphisms of Pro-SET having rudimentary domain or codomain; then we extend those results to any topological category over SET [7,8]. This is possible since the following holds: if $U: C \to SET$ is a topological functor, so is its extension Pro-U: Pro-$C \to$ Pro-SET which, therefore, preserves and reflects monomorphisms and epimorphisms.

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1. NOTATIONS AND PRELIMINARY RESULTS. The main reference for this note is Ch.I of [11]. The categorical terminology comes from [9].

DEFINITIONS. Let C be any category.

1.1. An inverse system $\underline{K} = (K_i, p_{ij}, I)$ in C is a collection $\{K_i\}_{i\epsilon I}$ of C-objects indexed over a directed set $(I, \leq)$, endowed with bonding morphisms p_{ij}: $K_j \to K_i$, whenever $i \leq j$, in such a way that p_{ii} = identity and $p_{ij} \cdot p_{jk} = p_{ik}$, for $i \leq j \leq k$.

1.2. A morphism $\underline{p}$: $X \to \underline{K}$ from a C-object X (= rudimentary system) to a system $\underline{K}$, is given by a family $\{ p_i : X \to K_i \mid i\epsilon I \}$ of C-morphisms, such that $p_{ij} \cdot p_j = p_i$, for all $i \leq j$; then $\underline{p}$ is a natural source in C ([9],p.133).

1.3. A morphism $\underline{q}$: $\underline{H} \to Y$ from a system $\underline{H} = (H_a, q_{ab}, A)$ to a C-object Y (= rudimentary system), is an equivalence class of C-morphisms from some H_a to Y. $q_a : H_a \to Y$ and $q_b : H_b \to Y$ are two representatives of $\underline{q}$ iff there is a $c\epsilon A$, $c \geq a,b$, such that $q_a \cdot q_{ac} = q_b \cdot q_{bc}$.

Let us call a morphism with rudimentary codomain $\underline{q}$: $\underline{H} \to Y$ full iff it admits a representative $q_a : H_a \to Y$, for all $a\epsilon A$.

1.4. A morphism of systems $\underline{f}$: $\underline{H} \to \underline{K}$ is a family $\{ \underline{f}_i : \underline{H} \to K_i \mid i\epsilon I \}$ of morphisms of type 1.3., such that $\underline{f}_i = p_{ij} \cdot \underline{f}_j$, whenever $i \leq j$. The composition is defined in the obvious way.

Inverse systems in $\mathcal{C}$ and their morphisms form the category Pro-$\mathcal{C}$.

1.5. Given two (Pro-$\mathcal{C}$)-morphisms $\underline{f}: \underline{H} \to \underline{K}$ and $\underline{g}: \underline{K} \to \underline{R} = (R_c, r_{cd}, C)$, we can define their composition $\underline{g}\cdot\underline{f}$ to be given by the natural source in Pro-$\mathcal{C}$ $\{\underline{g}_c \cdot \underline{f} : \underline{H} \to \underline{K} \to R_c \mid c \in C\}$. Also, we can think of $\underline{g}\cdot\underline{f}$ as the natural source in Pro-$\mathcal{C}$ $\{ g_c \cdot \underline{f}_{\psi(c)} : \underline{H} \to K_{\psi(c)} \to R_c \mid c \in C \}$, where g_c is a representative of $\underline{g}_c$ and where $\psi: C \to I$ is a suitable function.

1.6. A pre-order $(I, \leq)$ is cofinite provided for all $j \in I$ the set $\{i \in I \mid i \leq j\}$, of its predecessors, is finite.

An inverse system with cofinite index set will be called a cofinite system.

PROPOSITION 1.7. Let $\underline{q}': \underline{H}' \to Y$. There exist an isomorphism $\underline{h}: \underline{H} \to \underline{H}'$ in Pro-$\mathcal{C}$ and a full morphism $\underline{q}: \underline{H} \to Y$ such that $\underline{q} = \underline{q}' \cdot \underline{h}$ and $\underline{H}$ is cofinite.

Proof. Let $\underline{H}' = (H'_a, q'_{ab}, A')$ and let $\tilde{A}$ denote the subset of A' of all those indexes $a \in A'$ for which there exists a representative $q'_a: H'_a \to Y$ of $\underline{q}'$. Since $\tilde{A}$ is a directed cofinal subset of A', if we let $\underline{\tilde{H}}$ denote the subsystem of $\underline{H}'$ indexed over $\tilde{A}$, then the restriction morphism ([11], p.8) $\underline{h}': \underline{\tilde{H}} \to \underline{H}'$ is an isomorphism in Pro-$\mathcal{C}$. To conclude apply Theor.2, p.10 of [11] to obtain an isomorphism $\underline{\tilde{h}}: \underline{H} \to \underline{\tilde{H}}$, with $\underline{H}$ cofinite, and put $\underline{q} = \underline{q}' \cdot \underline{h}$, where $\underline{h} = \underline{h}' \cdot \underline{\tilde{h}}$.

1.8. By the preceding result, every time we are given a (Pro-$\mathcal{C}$)-morphism with rudimentary codomain $\underline{q}: \underline{H} \to Y$, we may suppose, without loss of generality, that $\underline{H}$ is cofinite and $\underline{q}$ is full.

As a consequence, for every such $\underline{q}$ we can choose a natural sink $q^* = \{q_a : H_a \to Y \mid a \in A\}$ of representatives of $\underline{q}$. If $a \in A$, take a unique representative q_{a_i} of $\underline{q}$ for each predecessor a_i of a and let q_a be the common value of the compositions $q_{a_i} \cdot q_{a_i a}$, for all i.

We say that q^* is a sink representing the (Pro-$\mathcal{C}$)-morphism $\underline{q}$. Obviously $\underline{q}$ does not determine q^* uniquely. If $\tilde{q}^* = (\tilde{q}_a)_A$ is another sink representing $\underline{q}$, then for all $a \in A$ there is $b \geq a$ such that $q_b = \tilde{q}_b$. We express this fact by saying that the sinks q^* and $\tilde{q}^*$ are cofinally equal or, simply, cofinal.

One easily verifies that if q^*, $\tilde{q}^*$ are sinks representing (Pro-$\mathcal{C}$)-morphisms $\underline{q}$, $\underline{\tilde{q}}: \underline{H} \to Y$, respectively, then q^* and $\tilde{q}^*$ cofinal implies $\underline{q} = \underline{\tilde{q}}$.

1.9. Recall from [7,8] that a functor $U: \mathcal{C} \to \mathcal{D}$ is topological if it admits all initial (and final) liftings. In particular, every topological functor is cotopological and preserves and reflects monomorphisms and epimorphisms [7,8]. A concrete category $\mathcal{C} = (C, U: C \to SET)$ is a <u>topological category</u> over SET when the forgetful functor U is topological.

In the sequel $\underline{H}$, $\underline{K}$ and $\underline{R}$ will always denote inverse systems $\underline{H} = (H_a, q_{ab}, A)$, $\underline{K} = (K_i, p_{ij}, I)$ and $\underline{R} = (R_c, r_{cd}, C)$, unless otherwise specified.

2. MONOMORPHISMS AND EPIMORPHISMS WITH RUDIMENTARY DOMAIN OR CODOMAIN.

<u>DEFINITION 2.1</u>. Let $A = (A, \leq)$ be a directed set. A sink $\{q_a: H_a \to Y \mid a \in A\}$ is said to be an <u>epicofinal sink</u> iff the following holds:
given $f, g: Y \to Z$ with the property that for all $a \in A$ there is an index $b \geq a$ such that $f \cdot q_b = g \cdot q_b$, then $f = g$.
Every epicofinal sink is an episink ([9], p.127).
If $\underline{q}: \underline{H} \to Y$ has a representing epicofinal sink, then every sink representing $\underline{q}$ is epicofinal.

<u>THEOREM 2.2</u>. $\underline{q}: \underline{H} \to Y$ is an epimorphism in Pro-$\mathcal{C}$ iff $\underline{q}$ has a representing epicofinal sink.

Proof. Let $\underline{f}, \underline{g}: Y \to \underline{K}$ be such that $\underline{f} \cdot \underline{q} = \underline{g} \cdot \underline{q}$. This means that $f_i \cdot \underline{q} = g_i \cdot \underline{q}$ for all $i \in I$. Let $q^* = (q_a)_A$ be an epicofinal sink representing $\underline{q}$; the sinks $f_i \cdot q^*$ and $g_i \cdot q^*$ are cofinal since they represent the same (Pro-$\mathcal{C}$)-morphism, hence for all $a \in A$ there is an index $b \geq a$ such that $f_i \cdot q_b = g_i \cdot q_b$, so that $f_i = g_i$, by the hypothesis on q^*, and this is true for all $i \in I$. It follows $\underline{f} = \underline{g}$ and $\underline{q}$ is an epimorphism.
Let now $\underline{q}$ be an epimorphism in Pro-$\mathcal{C}$ and let q^* be a sink which represents $\underline{q}$. Let $f, g: Y \to Z$ be $\mathcal{C}$-morphisms such that for all $a \in A$ there is a $b \geq a$ with $f \cdot q_b = g \cdot q_b$. Since $f \cdot q_b$ and $g \cdot q_b$ represent, respectively, $f \cdot \underline{q}$ and $g \cdot \underline{q}$, it follows that $f \cdot \underline{q} = g \cdot \underline{q}$, hence $f = g$, since $\underline{q}$ is an epimorphism. Then q^* is epicofinal.

<u>PROPOSITION 2.3</u>. In SET one has:

i) $\{p_i: X \to K_i \mid i \in I\}$ is a monosource iff it separates points of X.

ii) $\{q_a: H_a \to Y \mid a \in A\}$ is an episink iff it covers points of Y, i.e. for all

$y \in Y$ there are $a \in A$ and $h \in H_a$ with $y = q_a(h)$.

As we have already seen, an epicofinal sink is a particular episink, hence, in SET it covers points of the codomain. The next theorem shows that a natural sink q^* in SET is epicofinal iff it covers points in a special way.

THEOREM 2.4. A natural sink $q^* = (q_a)_A$ in SET is epicofinal iff the following property holds:

(α) for every $y \in Y$ there exists $a \in A$ such that $q_b^{-1}(y) \neq \emptyset$, for all $b \geq a$.

Proof. Let q^* be a sink which satisfies condition (α); let $f, g\colon Y \to Z$ be maps such that for all $a \in A$ there is $b \geq a$ with $f \cdot q_b = g \cdot q_b$. It is easy to verify that in this situation one has $f \cdot q_c = g \cdot q_c$ for all $c \geq b$, too. Let us prove that $f = g$. If y is any point in Y, then there exists $a \in A$ such that $q_b^{-1}(y) \neq \emptyset$ for all $b \geq a$. It is possible to choose b in order to have $f \cdot q_b = g \cdot q_b$, at the same time. Then $f(y) = (f \cdot q_b)(h) = (g \cdot q_b)(h) = g(y)$, for some $h \in q_b^{-1}(y)$.
Suppose now that q^* is an epicofinal sink and that (α) does not hold. Then there exists $y_0 \in Y$ such that for every $a \in A$ there is $b(a) \geq a$ with $q_{b(a)}^{-1}(y_0) = \emptyset$. Define two maps $f, g\colon Y \to Y$ as follows. $f = 1_Y$ and, if $Y' = \bigcup_{a \in A} \operatorname{Im} q_{b(a)}$, let $g_{|Y'}$ = identity, $g_{|Y-Y'}$ = constant map of value $\bar{y} \in Y'$. Then f and g are two maps which differ, at least, in y_0 and with the property that for every $a \in A$ there is $b(a) \geq a$ such that $f \cdot q_{b(a)} = g \cdot q_{b(a)}$. But this last equality, by epicofinality of q^*, implies $f = g$, which is a contradiction.

COROLLARY 2.5. $\underline{q}\colon \underline{H} \to Y$ is an epimorphism in Pro-SET iff $\underline{q}$ admits a representing sink q^* which satisfies condition (α) above.

Also (Pro-SET)-epimorphisms with rudimentary domain have a nice characterization.

THEOREM 2.6. $\underline{p}\colon X \to \underline{K}$ is an epimorphism in Pro-SET iff for all $i \in I$ such that $p_i\colon X \to K_i$ is not onto, there exists an index $j \geq i$ with $\operatorname{Im} p_{ij} \subset \operatorname{Im} p_i$

Proof. If p_i is onto for all $i \in I$, then $\underline{p}$ is an epimorphism in Pro-SET. Suppose that $\underline{p}$ is an epimorphism in Pro-SET and let $p_i\colon X \to K_i$ be not surjective. Let us consider maps $f_i, g_i\colon K_i \to K_i$, given by $f_i = 1_{K_i}$, $g_{i|\operatorname{Im} p_i}$ = identity,

$g_i\big|_{K_i - \mathrm{Im}\, p_i}$ = constant map of value $\bar{x} \in \mathrm{Im}\, p_i$.

Since f_i and g_i represent (Pro-SET)-morphisms $\underline{f}$, $\underline{g}$: $\underline{K} \to K_i$ and since $f_i \cdot p_i = g_i \cdot p_i$, it follows that $\underline{f} \cdot \underline{p} = \underline{g} \cdot \underline{p}$, so that $\underline{f} = \underline{g}$, $\underline{p}$ being an epimorphism. This equality means that there is a $j \geq i$ such that $f_i \cdot p_{ij} = g_i \cdot p_{ij}$, which implies $\mathrm{Im}\, p_{ij} \subset \mathrm{Im}\, p_i$, since, by definition $\mathrm{Im}\, g_i = \mathrm{Im}\, p_i$.

Conversely, let $\underline{p}$: $X \to \underline{K}$ be a (Pro-SET)-morphism with the property that for all i in I such that p_i is not onto, there is a $j \geq i$ with $\mathrm{Im}\, p_{ij} \subset \mathrm{Im}\, p_i$. Let us prove that $\underline{p}$ is an epimorphism in Pro-SET. Let $\underline{f}$, $\underline{g}$: $\underline{K} \to \underline{H}$ be such that $\underline{f} \cdot \underline{p} = \underline{g} \cdot \underline{p}$. We may suppose, without any restriction, that $\underline{K}$ and $\underline{H}$ are indexed over the same directed set I and that $\underline{f}$, $\underline{g}$ admit as representatives the <u>level maps</u> $(f_i, 1_I)$, $(g_i, 1_I)$, respectively ([11], Th.3.3). Then, from $\underline{f} \cdot \underline{p} = \underline{g} \cdot \underline{p}$ one obtains that $f_i \cdot p_i = g_i \cdot p_i$, for all $i \in I$. If p_i is not onto, let $j \geq i$ be as in the hypothesis: $p_{ij} \cdot p_j = p_i$; hence $f_i \cdot p_{ij} \cdot p_j = g_i \cdot p_{ij} \cdot p_j$. Now, let $z \in K_j$, then there exists an $x \in X$ such that $p_{ij}(z) = p_i(x)$; it follows $(f_i \cdot p_{ij})(z) = f_i(p_i(x)) = g_i(p_i(x)) = (g_i \cdot p_{ij})(z)$, hence $f_i \cdot p_{ij} = g_i \cdot p_{ij}$, that is $\underline{f} = \underline{g}$ in Pro-SET. Hence $\underline{p}$ is an epimorphism.

<u>PROPOSITION 2.7</u>. $\underline{p}$: $X \to \underline{K}$ is a monomorphism in Pro-SET iff the source $\underline{p} = (p_i)_I$ separates points of X.

Proof. By Proposition 2.3.i).

<u>THEOREM 2.8</u>. $\underline{q}$: $\underline{H} \to Y$ is a monomorphism in Pro-SET iff there exists a sink $q^* = (q_a)_A$ representing $\underline{q}$ which satisfies the following property:

(β) for every $a \in A$ there is $b \geq a$ such that q_b is injective.

Proof. Let us prove that condition (β) is sufficient. Let $\underline{f}$, $\underline{g}$: $\underline{K} \to \underline{H}$ be such that $\underline{q} \cdot \underline{f} = \underline{q} \cdot \underline{g}$. We may suppose, as in the proof of 2.6. , that $\underline{H}$ and $\underline{K}$ are indexed over the same directed set A, $\underline{K} = (K_a, p_{ab}, A)$, and that $\underline{f}$, $\underline{g}$ are represented by level maps $(f_a, 1_A)$, $(g_a, 1_A)$, respectively. In this case the sinks $\{ q_a \cdot f_a : K_a \to Y \mid a \in A \}$ and $\{ q_a \cdot g_a : K_a \to Y \mid a \in A \}$ must be cofinal, since they represent the same (Pro-SET)-morphism with rudimentary codomain; hence for every $a \in A$ there is $c \geq a$ such that $q_c \cdot f_c = q_c \cdot g_c$. Let now $d \in A$, $d \geq a,b,c$ and consider the following diagram

$$\begin{tikzcd}
 & K_b \arrow[r, shift left, "f_b"] \arrow[r, shift right, "g_b"'] \arrow[d, "p_{ab}"] & H_b \arrow[d, "q_{ab}"] \arrow[dr, "q_b"] & \\
K_d \arrow[ur, "p_{bd}"] \arrow[r, "p_{ad}"] \arrow[dr, "p_{cd}"'] & K_a \arrow[r, shift left, "f_a"] \arrow[r, shift right, "g_a"'] & H_a \arrow[r, "q_a"] & Y \\
 & K_c \arrow[u, "p_{ac}"] \arrow[r, shift left, "f_c"] \arrow[r, shift right, "g_c"'] & H_c \arrow[u, "q_{ac}"] \arrow[ur, "q_c"'] &
\end{tikzcd}$$

which gives $q_b \cdot f_b \cdot p_{bd} = q_a \cdot f_a \cdot p_{ad} = q_c \cdot f_c \cdot p_{cd}$ and $q_b \cdot g_b \cdot p_{bd} = q_a \cdot g_a \cdot p_{ad} = q_c \cdot g_c \cdot p_{cd}$.

By the assumption that $q_c \cdot f_c = q_c \cdot g_c$ it follows $q_b \cdot f_b \cdot p_{bd} = q_b \cdot g_b \cdot p_{bd}$ and also $f_b \cdot p_{bd} = g_b \cdot p_{bd}$, since q_b is a monomorphism. Finally, one has $f_a \cdot p_{ad} = f_a \cdot p_{ab} \cdot p_{bd} = q_{ab} \cdot f_b \cdot p_{bd} = q_{ab} \cdot g_b \cdot p_{bd} = g_a \cdot p_{ab} \cdot p_{bd} = g_a \cdot p_{ad}$. Hence we have shown that the level maps $(f_a, 1_A)$, $(g_a, 1_A)$ represent the same (Pro-SET)-morphism, that is $\underline{f} = \underline{g}$ and $\underline{q}$ is a monomorphism.

Conversely, let $\underline{q}: \underline{H} \to Y$ be a monomorphism in Pro-SET and let $q^* = (q_a)_A$ be a sink representing $\underline{q}$. Suppose that q^* does not satisfy condition (β), then there exists an index a_0 in A such that for all $b \geq a_0$, q_b is not injective. Hence, for all $b \geq a_0$, there are $h'_b \neq h''_b$ in H_b such that $q_b(h'_b) = q_b(h''_b)$. Define maps $f_a, g_a : H_a \to H_a$, for all $a \in A$, as follows: $f_a = 1_{H_a}$, for all $a \in A$, and if $A_0 = \{b \in A \mid b \geq a_0\}$, let $g_a = 1_{H_a}$, for all $a \in A - A_0$, while, for $b \in A_0$, let $g_b : H_b \to H_b$ be the map which permutes h'_b and h''_b and leaves all other points fixed.

Moreover, for every $a \leq b$ in A let $\tilde{q}_{ab} : H_b \to H_a$ be a map which makes the following diagram commutative

$$\begin{tikzcd}
H_b \arrow[r, "g_b"] \arrow[d, "\tilde{q}_{ab}"] & H_b \arrow[d, "q_{ab}"] \\
H_a \arrow[r, "g_a"] & H_a
\end{tikzcd}$$

Note that $\tilde{q}_{ab}$ will act the same as q_{ab} up to a rearrangement of its values on h'_b, h''_b, $q^{-1}_{ab}(h'_a)$, $q^{-1}_{ab}(h''_a)$, when needed.

From the above it follows that $(H_a, \tilde{q}_{ab}, A)$ is an inverse system in SET, denoted by $\underline{\tilde{H}}$, while $(f_a, 1_A)$, $(g_a, 1_A)$ are level maps which represent (Pro-SET)-morphisms

$\underline{f}$, $\underline{g}$: $\underline{\tilde{H}} \to \underline{H}$. One has $\underline{q}\cdot\underline{f} = \underline{q}\cdot\underline{g}$, in fact $q_a \cdot f_a = q_a \cdot g_a$, for all $a\varepsilon A$, hence $\underline{f} = \underline{g}$, since $\underline{q}$ is a monomorphism. But this last equality means that there exists $b \geq a_0$ such that the diagram

$$\begin{tikzcd}
 & H_{a_0} \arrow[dr, "f_{a_0}"] & \\
H_b \arrow[ur, "\tilde{q}_{a_0 b}"] \arrow[dr, "\tilde{q}_{a_0 b}"'] & & H_{a_0} \\
 & H_{a_0} \arrow[ur, "g_{a_0}"'] &
\end{tikzcd}$$

commutes, that is $g_{a_0} \cdot \tilde{q}_{a_0 b} = \tilde{q}_{a_0 b}$, which is impossible because of the definition of the maps involved. Hence condition (β) must hold when $\underline{q}$ is a monomorphism in Pro-SET.

Let now ($\mathcal{C}$, U:$\mathcal{C} \to$ SET) be a concrete category and let Pro-U: Pro-$\mathcal{C} \to$ Pro-SET, $\underline{K} \mapsto \underline{UK} = (UK_i, Up_{ij}, I)$, be the extension of U to the pro-categories.

<u>THEOREM 2.9</u>. If ($\mathcal{C}$, U:$\mathcal{C} \to$ SET) is a topological category over SET, then Pro-U is a topological functor.

Proof. It suffices to prove that every (Pro-U)-sink $\{\underline{f}^\alpha : \underline{UK}^\alpha \to \underline{S} \mid \alpha\varepsilon\Lambda\}$, $\underline{S}=(S_a, s_{ab}, A)$ ε Pro-SET, has a unique (Pro-U)-final lifting; that is there exists a unique, up to isomorphisms, $\underline{H}$ ε Pro-$\mathcal{C}$ and a(Pro-U)-final sink $\{\underline{g}^\alpha : \underline{K}^\alpha \to \underline{H} \mid \alpha\varepsilon\Lambda\}$, such that $\underline{S} = \underline{UH}$ and $\underline{g}^\alpha = U\underline{f}^\alpha$, for all $\alpha\varepsilon\Lambda$.
We only sketch the construction of $\underline{H}$, leaving all other easy details to the reader. For every $a\varepsilon A$, let $\{f^\alpha_{\phi(a)} : UK^\alpha_{\phi(a)} \to S_a \mid \alpha\varepsilon\Lambda\}$ be the sink where $f^\alpha_{\phi(a)}$ is a representative of $\underline{f}^\alpha_a : \underline{UK}^\alpha \to S_a$, and let $\{g^\alpha_{\phi(a)} : K^\alpha_{\phi(a)} \to H_a \mid \alpha\varepsilon\Lambda\}$ be its U-final lifting, which there exists by hypothesis. Then, by the properties of the final lifting, it follows that, for all $a \leq b$, $s_{ab} : S_b \to S_a$ comes from a certain $\mathcal{C}$-morphism $q_{ab} : H_b \to H_a$, so that $\underline{H} = (H_a, q_{ab}, A)$ is an inverse system in $\mathcal{C}$.

By virtue of the above theorem, since topological functors (1.9) preserve monomorphisms and epimorphisms, all the results in this section, characterizing monomorphisms and epimorphisms in Pro-SET, can be extended to Pro-$\mathcal{C}$ as well, where $\mathcal{C}$ is any topological category over SET.

<u>REMARKS</u>.

2.10. Let $\underline{K}$ be an inverse system in HComp, the category of compact Hausdorff spaces, and let $\underline{p}: K \to \underline{K}$ be the inverse limit morphism [1] . If $p_i(K)$ is an open set in K_i, for all i, then $\underline{p}$ is an epimorphism in Pro-HComp. This follows from [2] , Th.3.7, p.217, and Th. 2.6. above.

2.11. Let HLcomp be the category of locally compact Hausdorff spaces and let TYCH be the category of completely regular T_1 spaces. Every $X \in$ TYCH admits an HLcomp-expansion [3,11] $\underline{p}: X \to \underline{K}$, where $\underline{K} \in$ Pro-HLcomp is formed by taking all open neighbourhoods of X in βX, its Stone-Čech compactification, directed by reversed inclusion. $\underline{p}$ is an epimorphism in Pro-TYCH, in fact each p_i is an epimorphism in TYCH.

If S is any topological space and $\eta: S \to X$ is its epireflection in TYCH, then $\underline{p} \cdot \eta: S \to X \to \underline{K}$ is an HLcomp-expansion of S which is not an epimorphism in Pro-TOP (TOP being the category of all topological spaces and continuous maps) This may be seen using Th. 2.6.

3. GENERAL MONOMORPHISMS AND EPIMORPHISMS IN Pro-$\mathcal{C}$. Let now $\mathcal{C}$ be an arbitrary category.

LEMMA 3.1. Let $\underline{e}: \underline{H} \to \underline{K}$, $\underline{f}, \underline{g}: \underline{K} \to R$ be given morphisms in Pro-$\mathcal{C}$. Then $\underline{f} \cdot \underline{e} = \underline{g} \cdot \underline{e}$ holds iff there is an index $i \in I$ such that $f_i \cdot \underline{e}_i = g_i \cdot \underline{e}_i$, where f_i, g_i represent $\underline{f}$, $\underline{g}$, respectively.

Proof. Follows from the definitions and from 1.5.

PROPOSITION 3.2. $\underline{e}: \underline{H} \to \underline{K}$ is an epimorphism in Pro-$\mathcal{C}$ iff there is an index $i \in I$ such that $\underline{e}_i: \underline{H} \to K_i$ is an epimorphism, i.e. has a representing epicofinal sink.

Proof. Let $\underline{e}_i$ be an epimorphism. If $\underline{f}, \underline{g}: \underline{K} \to \underline{R}$ are such that $\underline{f} \cdot \underline{e} = \underline{g} \cdot \underline{e}$, then $\underline{f}_c \cdot \underline{e} = \underline{g}_c \cdot \underline{e}$, for all $c \in C$, which means $f_{ic} \cdot \underline{e}_i = g_{ic} \cdot \underline{e}_i$, by the Lemma, hence $f_{ic} = g_{ic}$, for all $c \in C$. It follows $\underline{f} = \underline{g}$, and $\underline{e}$ is an epimorphism.

Conversely, let $\underline{e}$ be an epimorphism and suppose that no $\underline{e}_i$, $i \in I$, is an epimorphism. Then, for all $i \in I$, there are $\underline{f} \neq \underline{g}: K_i \to \underline{R}$ such that $\underline{f} \cdot \underline{e}_i = \underline{g} \cdot \underline{e}_i$. This last implies $f_c \cdot \underline{e}_i = g_c \cdot \underline{e}_i$, for all $c \in C$, then $\underline{f}_c \cdot \underline{e} = \underline{g}_c \cdot \underline{e}$, for all $c \in C$, by the Lemma. By assumption one obtains $\underline{f}_c = \underline{g}_c$, for all $c \in C$, that is $\underline{f} = \underline{g}$, which is a contradiction.

PROPOSITION 3.3. If $\underline{m}$: $\underline{K} \to \underline{R}$ is such that $\underline{m}_c : \underline{K} \to R_c$ is a monomorphism, for some $c \in C$, then also $\underline{m}$ is a monomorphism in Pro-C.

Proof. Suppose $\underline{m}_c$ is a monomorphism in Pro-C and let $\underline{f},\underline{g}$: $\underline{H} \to \underline{K}$ be such that $\underline{m} \cdot \underline{f} = \underline{m} \cdot \underline{g}$. This means $\underline{m}_d \cdot \underline{f} = \underline{m}_d \cdot \underline{g}$, for all $d \in C$, then $\underline{f} = \underline{g}$.

This proposition may be inverted in some cases, such as the following.

PROPOSITION 3.4. Let $\underline{m}$: $\underline{K} \to \underline{R}$ be a monomorphism in Pro-C . If there is an index $c \in C$ with the property that $r_{cd} : R_d \to R_c$ is a monomorphism in C for every $d \geq c$, then $\underline{m}_c : \underline{K} \to R_c$ is a monomorphism in Pro-C.

Proof. Let $\underline{f}$, $\underline{g}$: $\underline{H} \to \underline{K}$ be such that $\underline{m}_c \cdot \underline{f} = \underline{m}_c \cdot \underline{g}$. Then, for all $e \in C$, one has the following commutative diagram, where $d \geq c,e$,

$$\begin{tikzcd}
 & & R_c \\
\underline{H} \arrow[r, shift left, "\underline{f}"] \arrow[r, shift right, "\underline{g}"'] & \underline{K} \arrow[ur, "\underline{m}_c"] \arrow[r, "\underline{m}_d"] \arrow[dr, "\underline{m}_e"'] & R_d \arrow[u, "r_{cd}"'] \arrow[d, "r_{ed}"] \\
 & & R_e
\end{tikzcd}$$

Since r_{cd} is a monomorphism, $\underline{m}_e \cdot \underline{f} = \underline{m}_e \cdot \underline{g}$. Finally $\underline{f} = \underline{g}$ by the assumption and by 1.5.

Note that, in case C = SET or C is a topological category over SET, then one can apply the results of section 2 in order to obtain information about monomorphisms and epimorphisms in Pro-C of the general form $\underline{H} \to \underline{K}$.

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