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UNIFORM WEIGHT OF UNIFORM QUOTIENTS

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Abstract: Uniform weight of uniform quotients is estimated and it is shown that the estimation cannot be improved. In particular, examples of nonmetrizable uniform quotients of metric spaces are given.

Key-words: uniform space, quotient, uniform weight, metric space

Classification: 54E15 , 54C10

The uniform weight of a uniform space is the smallest cardinality of a base for uniform covers or of a base for uniform vicinities of diagonal. We shall look how the uniform weight behaves by uniform quotients. This question reduces to investigation of quotients of metric spaces. Some cases when a uniform quotient of a metric space is metrizable as a uniform space are treated e.g. in [C],[Hi],[M]. An example that a uniform quotient of a metric space is not pseudometrizable is given in [M], however, we were not able to check all the details. The similar examples presented in this paper are simple and the spaces used have additional nice properties: in the first example, the local character of the quotient space is uncountable and the domain space is discrete, in the second one the quotient map is at most 2 to 1 and the uniform quotient is also a topological quotient, hence, it is metrizable as a topological space.

Ordinal number is understood here as the set of smaller ordinals, initial ordinals are cardinals (thus $n+1$ is the set $\{0,1,\dots,n\}$, but because of better understanding we shall denote that set by $\bar{n}$). By ${}^{A}B$ we denote the set of all mappings on A into B.

Thus ${}^{\omega}\omega$ is the set of all mappings on ω into ω and we shall endowed it with the pointwise order: $f<g$ if $fn\leq gn$ for all $n\in\omega$. The $\mathrm{cof}({}^{\omega}\omega)$ is the smallest cardinality of a cofinal set in ${}^{\omega}\omega$ and it is consistent with ZFC that $\mathrm{cof}({}^{\omega}\omega)$ equals to any cardinal which is not gre ter than 2^{ω} and has uncountable cofinality,[He].

Uniform spaces are given by means of the set of uniform covers,and if u is a uniformity then $E(u)$ denotes the corresponding set of uniform vicinities of the diagonal in $X\times X$.A pseudometric d on X is called uniformly continuous on (X,u) if the uniformity induced by d is smaller than u (i.e.,d is a uniformly continuous function on $(X,u)\times(X,u)$).

In the sequel,$q:(X,u)\rightarrow(Y,v)$ is a uniform quotient mapping between uniform spaces,i.e.,v is the biggest uniformity on Y making $q:(X,u)\rightarrow(Y,v)$ uniformly continuous.The uniformity v may be described by means of uniformly continuous pseudometrics d on (Y,v) ($d\circ(q\times q)$ is uniformly continuous on (X,u)),or as the set of covers of Y,initiating a normal sequence in the image $q(u)$.We shall describe the quotient in a way more convenient for our purposes,using the technique described e.g. in [DR].

For $r>0$ and a uniformly continuous pseudometric d on (X,u) we denote $M_d(r)=\{(qa,qb)\mid a,b\in X,d(a,b)<1/r\}$; if f is an increasing mapping $\omega\rightarrow\omega-(0)$,then $M_d(f)=\cup\{M_d(f(p0))\circ M_d(f(p1))\circ\ldots\circ M_d(f(pn))\mid n\in\omega,p$ is a permutation on $\bar{n}\}$ (sometimes,the index d will be omitted).

THEOREM..*The collection* $\{M_d(f)\mid f\in{}^{\omega}(\omega-(0))$ *is increasing,*d *is a uniformly continuous pseudometric on* $(X,u)\}$ *is a base of* (Y,v).

Proof. Let $V\in E(v)$ and take a sequence $\{V_n\}\subset E(v)$ such that $V\supset V_{p0}\circ V_{p1}\circ\ldots\circ V_{pn}$ for each $n\in\omega$ and each permutation p on $\bar{n}$ (e.g. take a uniformly continuous pseudometric e on (Y,v) such that $(a,b)\in V$ provided $e(a,b)<1$ and define $V_n=\{(a,b)\mid e(a,b)<2^{-n-1}\}$). Thus $V\supset M_d(f)$,where d is a uniformly continuous pseudometric on (X,u) such that $(qa,qb)\in V$ provided $d(a,b)<1$,and f is an increasing map on ω into $\omega-(0)$ such that $M_d(fn)\subset V_n$.It remains to show that the collection $\{M_d(f)\}$ is a base for a uniformity;the only non-trivial part is to show that for each d,f there are e,g such that $M_d(f)\supset M_e(g)\circ M_e(g)$.To do that,it suffices to put $e=d,gn=f(2n+2)$: $M_d(g(p0))\circ\ldots M_d(g(pn))\circ M_d(g(p0))\circ\ldots\circ M_d(g(pn))\subset M_d(f(2(p0)+2)\circ$

$M_d(f(2(pn)+2))\cdot M_d(f(2(p0)+1))\cdot\ldots\cdot M_d(f(2(pn)+1))$.

COROLLARY 1. *If the uniform weight of (X,u) is κ, then the uniform weight of its quotient (Y,v) is less or equal to $\kappa\cdot\mathrm{cof}({}^{\omega}\omega)$.*

Proof. Clearly, if $f<g$ then $M_d(f)\supset M_d(g)$. If e is a uniformly continuous pseudometric on a pseudometric space (X,d), then $M_e(f)\supset M_d(g)$ for some convenient g (since for each n there exists m such that $d(a,b)<1/m$ implies $e(a,b)<1/n$).

COROLLARY 2. *Assume that X is a uniform space with uniform weight not smaller than $\mathrm{cof}({}^{\omega}\omega)$. If X has a monotone base, then any uniform quotient of X has a monotone base, too.*

Proof. A uniform space is said to admit cardinal κ if κ-many uniform covers have a common uniform refinement. A uniform space X has a monotone base iff X admits any cardinal smaller than its uniform weight. Since every quotient of X admits the cardinals admitted by X, our assertion follows from Corollary 1.

In fact, we have proved more, namely that if X is the space from Corollary 2, then its uniform quotient is either uniformly discrete or has the same uniform weight as X has.

We shall show now that the estimation given in Corollary 1 of the uniform weight of (Y,v) cannot be improved, i.e. that a uniform quotient of a metric space has uniform weight equal to $\mathrm{cof}({}^{\omega}\omega)$.

EXAMPLE 1. *There is a complete countable metric space X which is topologically discrete and has a uniform quotient Y such that every point of Y has local character equal to $\mathrm{cof}({}^{\omega}\omega)$.*

Denote $Y=(\emptyset)\cup\cup\{{}^{\overline{n}}(\omega-(0))\mid n\in\omega\}$, $X=\omega\times Y$, q the projection X onto Y. The metric d on X is defined as follows (by $z=y^\frown n$ for $y\in{}^{\overline{k}}(\omega-(0))$ we describe the situation when $z\in{}^{\overline{k+1}}(\omega-(0))$, z extends y and $z(k+1)=n$, by $z=\emptyset^\frown n$ we mean $z\in{}^{\overline{0}}(\omega-(0))$, $z(0)=n$)

$$d((x_1,y_1),(x_2,y_2))=d((x_2,y_2),(x_1,y_1))=\begin{cases}1/n & \text{if } y_2=y_1{}^\frown n,\ x_2=0,\ x_1=?\\ 1 & \text{otherwise.}\end{cases}$$

The metric d is complete and induces the discrete topology on X. On Y, we take the quotient uniformity along $q:(X,d)\to Y$. For $y\in Y$ and $n\in\omega-(0)$, $M(n)(y)=\{z\in Y\mid \text{either } z=y^\frown k \text{ and } k>n \text{ or } y=z^\frown k \text{ and } k>n\}$ hence $M(f)(\emptyset)\supset\{z\in Y\mid z\in{}^{\overline{k}}(\omega-(0)),\ z(n)>f(n) \text{ for each } n\le k,\ k\in\ldots$

Suppose now that $\{M(f)(\emptyset)|f\in F\}$ is a local base at $\emptyset$ in Y and $|F|<\mathrm{cof}({}^{\omega}\omega)$. Then there is $g\in{}^{\omega}(\omega-(0))$ which is not bounded from above by any $f\in F$. We can find $f\in F$ such that $M(g)(\emptyset)\supset M(f)(\emptyset)$ and and $n\in\omega$ with $gn>fn$. Take now such a $s\in{}^{\bar{n}}(\omega-(0))$ that $s(k)=fk+1$ for all $k\leq n$; then $s\in M(f)(\emptyset)-M(g)(\emptyset)$, which is a contradiction. Indeed, if $s\in M(g)(\emptyset)$, then there is $\{u_i\}_0^k\in Y$ such that $u_0=\emptyset, u_k=s$ and $(u_i,u_{i+1})\in M(g(pi))$ for $i<k$. For each $j\leq n$ there exists $i\leq k$ such that $u_{i+1}=u_i{}^\frown s(j)$, hence there exists an injection $\phi:\bar{n}\to\bar{k}$ with $s(j)>g(\phi j)$. Since both f,g are increasing and $gn\geq sn$, we have $\phi n<n$, consequently $\phi i\geq n$ for some $i<n$, but then $si=fi+1>g(\phi i)>fn$, hence $fi>fn$, which is not possible. The same procedure works for other points $y\in Y$.

The map q from Example 1 cannot be expected to be finite-to-one and the space Y cannot be the topological quotient of X. We shall now construct another example, where the map q is at most 2 to 1 and is also the topological quotient, but the cardinality of X is uncountable. It follows from one result of Arhangelskij in [A] that the quotient space Y is metrizable as a topological space.

EXAMPLE 2. *There is at most 2 to 1 mapping q defined on a Baire space D^{ω} such that uniform weight of the quotient along q is $\mathrm{cof}({}^{\omega}\omega)$. The quotient space is topologically metrizable.*

Let D be a cofinal set in ${}^{\omega}(\omega-(0))$ endowed with the uniformly discrete uniformity and $X=D^{\omega-(0)}$ be endowed with the Baire metric $d(\{x_i\},\{y_i\})=1/n$, where n is the first coordinate with $x_n\neq y_n$. Choose a countable subset $\{c_n\}$ in D and for every $f\in D, n\in\omega$, define $a_f^n, b_f^n\in X$:

$$a_f^n(i)=\begin{cases} f & \text{if } i=1 \\ c_{2n} & \text{if } i>1 \end{cases} \qquad b_f^n(i)=\begin{cases} f & \text{if } i=1 \\ c_{2n} & \text{if } 1<i\leq fn \\ c_{2n+1} & \text{if } i>fn \end{cases}.$$

The quotient map $q:X\to Y$ is defined by means of the equivalence: $qa=qb$ if either $a=b$ or there is $f\in D, n\in\omega$ such that either $a=a_f^{n+1}$, $b=b_f^n$ or $a=b_f^n, b=a_f^{n+1}$. Since $M(n)=\{(a,b)\in Y\times Y|$ there are $x,y\in X$ with $qx=a, qy=b, x_i=y_i$ for all $i\leq n\}$, the pair (qa_f^0, qa_f^n) always belongs to $M(f)$.
Suppose that the uniform quotient Y of X has a base $\{M(f)|f\in F\}$ of cardinality less than $\mathrm{cof}({}^{\omega}\omega)$ and take $g\in{}^{\omega}\omega$ such that $g<f$ for

no $f\in F$. We may suppose that $g0$ and all $f0$ for $f\in F$ are bigger than 1. There is some $f\in F$ such that $M(g)\supset M(f)$ and $n\in\omega-(0)$ such that $gn>fn$. We shall show that $(qa_f, qa_f^{n+1})\notin M(g)$, which contradicts the previous facts. If $(qa_f, qa_f^{n+1})\in M(g)$, then there are points $u_i\in X$ for $i\leq k$ and a permutation p on $\bar{k}$ such that $u_0=a_f^0, u_k=a_f^{n+1}$, $(qu_i, qu_{i+1})\in M(g(pi))$ for all $i<k$. Since $g0>1$, for every $i\leq n+1$ there is $\phi i\leq k$ such that $qu_{\phi i}=qa_f^i$ and the mapping $\phi:\overline{n+1}\to\bar{k}$ preserves ordering; moreover, for every $i\leq n$ there must be a ψi such that $\phi i\leq\psi i<\phi(i+1)$ and $gp\psi i\leq fi$ (since $(qu_{\phi i}, qu_{\phi(i+1)})=(qa_f^i, qa_f^{i+1})\in$ $\in M(g(p\phi i))\bullet M(g(p(1+\phi i)))\bullet\ldots\bullet M(g(p(\phi(i+1)-1))))$ but that is impossible because there is at most $n-1$ points in $\bar{k}$ in which g has value less or equal to fn.

At the end we would like to add a remark concerning the behaviour of uniform pseudoweight by quotients. Similarly as pseudocharacter in topological spaces, uniform pseudoweight of a uniform space (X,u) is the least cardinality κ for which there exists $v\subset u$ with $|v|=\kappa$ and such that the meet of v coincides with that of u - for separated spaces it means that $\cap E(v)$ is the diagonal. We shall now provide an example showing that there is no simple connection between uniform pseudoweights of a space and its quotient.

EXAMPLE 3. *For each cardinal κ there is a uniform quotient $q:X\to Y$ such that X has countable uniform pseudocharacter and uniform pseudocharacter of Y is not smaller than κ.*

Let κ be an infinite regular cardinal and Y be the uniform space with the underlying set $\kappa\times 2$ and with the base of uniform covers

$$\{(y)\,|\,y\in Y\}\cup\{((\alpha,0),(\alpha,1))\,|\,\alpha>\beta\}\quad\text{for } \beta\in\kappa.$$

Uniform pseudoweight of Y is κ. We shall show that Y is a uniform quotient of a space having countable pseudoweight.

For each cofinal set S in κ we may find a monotone sequence $\{S_n\}$ such that each S_n is cofinal in S, $\cap S_n=\emptyset$. Let X_S be the uniform space with the same underlying set as Y has and with the base of uniform covers

$$\{(y)\,|\,y\in Y\}\cup\{((\alpha,0),(\alpha,1))\,|\,\alpha\in S_n, \alpha>\beta\}\quad\text{for } \beta\in\kappa, n\in\omega.$$

Uniform pseudoweight of X_S is ω, and the uniformity of Y is the biggest uniformity contained in the uniformities of the above spaces X_S. Thus Y is a uniform quotient of the sum of spaces X_S.

R E F E R E N C E S

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