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POSITIVE SOLUTIONS OF SOME QUASI-LINEAR ELLIPTIC PROBLEMS

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Abstract: In this paper we prove the existence of positive solution $u \in C^{2,\alpha}(\overline{\Omega})$ of the quasi-linear elliptic problem

$$\begin{cases} -\sum D_i(a_{i,j}(u(x))D_j u(x)) + a_o(u(x))u(x) = g(x,u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $g: \overline{\Omega} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a sublinear function.

Key words: Quasi-linear elliptic equations, positive solutions, Schauder fixed point theorem.

Classification: 35J65

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1. Introduction. In this note we prove the existence of positive solution $u \in C^{2,\alpha}(\overline{\Omega})$ of the quasi-linear elliptic problem

$$(1) \quad \begin{cases} -\sum D_i(a_{ij}(u(x))D_j u(x)) + a_o(u(x))u(x) = g(x,u(x)), x \in \Omega, \\ u(x) = 0, x \in \partial\Omega, \end{cases}$$

where $g: \overline{\Omega} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a C^1-function satisfying sublinear condition (see Section 4).

The purpose of this paper is to obtain analogous results as for semilinear elliptic problems with sublinear nonlinearity (see e.g. Amann [2]).

The main idea is to use some results from the linear theory of elliptic problems combined with the Schauder fixed point theorem, the continuity of Němyckij's operator in Hölder spaces and the result of Kramer [9]. Boccardo [3] proved the existence of a positive eigenfunction for a class of quasi-linear operators using a similar method but he was working in Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ and satisfying condition

(S) there exists $M > 0$ such that for every pair of points $x, y \in \Omega$ there exist points $x = z_0, z_1, z_2, \dots, z_n = y$ such that the segments with endpoints z_i, z_{i+1} $(i = 0, 1, 2, \dots, n-1)$ are subsets of Ω and

$$\sum_{i=1}^{n-1} |z_i - z_{i+1}| \le M|x-y| .$$

<u>Remark 1.</u> For details about domains satisfying condition (S) see Kufner, John, Fučík [7]. We need this condition to be true imbedding $C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$ (see [7, Thm. 1.2.14]).

We suppose that real functions $a_{i,j}, a_0 : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the following assumptions:

$$(2) \quad \begin{cases} a_{ij}(s) = a_{ji}(s) & \forall s \in \mathbb{R}, \\ \alpha|\xi|^2 \le \sum a_{ij}(s)\, \xi_i\, \xi_j \le \beta|\xi|^2 & \forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R}, \\ 0 \le a_0(s) \le \gamma & \forall s \in \mathbb{R}, \end{cases}$$

where α, β, γ are some positive constants.

Moreover let

$$(3) \qquad a_{ij} \in C^2(\mathbb{R}),\ a_0 \in C^1(\mathbb{R}).$$

Assume that $g : \overline{\Omega} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a C^1-function. We put $X = \{u \in C^{2,\alpha}(\overline{\Omega});\ u = 0 \text{ on } \partial\Omega\}$ with the norm of $C^{2,\alpha}(\overline{\Omega})$, $Y = C^{1,\alpha}(\overline{\Omega})$, $Z = C^{0,\alpha}(\overline{\Omega})$ (see [7] for usual Hölder space notation).

2. Some auxiliary assertions. The purpose of this section is to prove some auxiliary results which we shall need in the following sections.

Let $w \in Y$ be fixed. We shall denote

$$L(w)v = -\sum D_i(a_{ij}(w(x))D_j v) + a_o(w(x))v.$$

Put $a^w_{ij}(x) = a_{ij}(w(x))$, $a^w_o(x) = a_o(w(x))$, $x \in \bar{\Omega}$. From (2) it follows

$$a^w_{ij}(x) = a^w_{ji}(x) \qquad \forall x \in \Omega,$$

$$(2') \qquad \alpha|\xi|^2 \le \sum a^w_{ij}(x)\,\xi_i\,\xi_j \le \beta|\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \quad \forall x \in \bar{\Omega},$$

$$0 \le a^w_o(x) \le \gamma \qquad \forall x \in \bar{\Omega},$$

where the positive constants α, β, γ are independent of $w \in Y$.

Remark 2. Using assumption (3) and the author's result [4, Thm 1], we obtain that $a^w_{ij} \in Y$, $a^w_o \in Z$ for all $w \in Y$. Hence we are able to apply the Schauder's theory and the L^p-theory for the Dirichlet problem

$$(4) \qquad \begin{cases} L(w)u = f \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

$f \in Z$, for each fixed $w \in Y$. Namely, the Dirichlet problem (4) is uniquely solvable and satisfies the a priori estimates:

$$(5) \qquad \|u\|_X \le c\|f\|_Z,$$

$$(6) \qquad \|u\|_{W^{2,p}(\Omega)} \le c\|f\|_{L^p(\Omega)},$$

where the constant $c > 0$ is independent of $f \in Z$ and $w \in Y$ (see Agmon, Douglis, Nirenberg [1, Thm 7.3, 15.2]).

Remark 3. Let $w \in Y$ be fixed. We shall write L instead of $L(w)$ in this remark. Let us denote by $\mu_j(m)$, resp. $\sigma_j(m)$, the positive eigenvalues of the eigenvalue problem with an indefinite weight:

$$(7)\qquad \begin{cases} Lu = \mu\, m(x)u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

resp.

$$(8)\qquad \begin{cases} -\Delta u = \sigma m(x)u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where m is a C^1-function in $\overline{\Omega}$, $m \not\equiv 0$. If $m(x) > 0$ in $\Omega_1 \subset \Omega$, meas $\Omega_1 > 0$, it is known (see e.g. de Figueiredo [5, Prop.1,10]) that (7), resp. (8), has a sequence of such eigenvalues, with a variational characterization. Moreover $\mu_1(m)$, resp. $\sigma_1(m)$, is simple and the corresponding eigenfunctions are of the same sign in Ω. Lastly $m < \hat{m}$ in Ω implies $\mu_j(\hat{m}) < \mu_j(m)$, resp. $\sigma_j(\hat{m}) < \sigma_j(m)$, and $\mu_j(m)$, resp. $\sigma_j(m)$, is a continuous function of m in the norm of $L^{N/2}(\Omega)$ (see [5, Prop. 1.12A and 1.12B]).

<u>Lemma 1.</u> <u>For each</u> $w \in Y$ <u>it is</u>

$$\mu_1(m) \in [\alpha\sigma_1(m), (\beta + \gamma/\sigma_1(1))\, \sigma_1(m)].$$

<u>Proof.</u> Let us denote by u_1, resp. v_1, the first positive eigenfunction of (7), resp. (8). From the variational characterization of $\mu_1(m)$, $\sigma_1(m)$ and integration by parts we obtain

$$\mu_1(m) \int_\Omega m(x)|u_1(x)|^2 dx = \int_\Omega Lu_1(x)u_1(x)\,dx \geq \alpha \int_\Omega |\nabla u_1(x)|^2 dx \geq$$
$$\geq \alpha\sigma_1(m) \int_\Omega m(x)|u_1(x)|^2 dx.$$

On the other hand we obtain

$$\mu_1(m) \int_\Omega m(x)|v_1(x)|^2 dx \leq \int_\Omega Lv_1(x)v_1(x)\,dx \leq \beta \int_\Omega |\nabla v_1(x)|^2 dx +$$
$$+ \gamma \int_\Omega |v_1(x)|^2 dx \leq (\beta + \gamma/\sigma_1(1))\, \sigma_1(m) \int_\Omega m(x)|v_1(x)|^2 dx$$

and the lemma is proved. Q.E.D.

Let $0 \leq \mu < \alpha \tilde{\sigma}_1(m)$. We are interested in a priori estimates of the solution $u(w) \in X$ of

(9) $\quad L(w)u(w)(x) = \mu m(x)u(w)(x) + f(x), \; x \in \Omega,$

where $f \in Z$ is given.

Lemma 2. There exists a constant $c > 0$ independent of $w \in Y$ and $f \in Z$ such that

(10) $$\|u(x)\|_X \leq c\|f\|_Z.$$

Proof. Using Riesz-Fréchet representation theorem it is possible to write the equation (9) in the operator form

(11) $$u - \mu Tu = \tilde{f},$$

where $T: W_0^{1,2}(\Omega) \longrightarrow W_0^{1,2}(\Omega)$ is linear symmetric compact operator and μ has a positive distance from the spectrum of T (see Lemma 1). It follows from Taylor [8, Thm 6.4C] that

$$\|u\|_{W_0^{1,2}(\Omega)} \leq \text{const.} \, \|\tilde{f}\|_{W_0^{1,2}(\Omega)}$$

with a constant independent of $w \in Y$ and $\tilde{f} \in W_0^{1,2}(\Omega)$. Since $\tilde{f}$ is a representant of f, we obtain

(12) $$\|u(w)\|_{W_0^{1,2}(\Omega)} \leq \hat{c}\|f\|_{L^2(\Omega)}.$$

Hence using Sobolev imbedding theorems (see [7]) the right hand side of (9) is in $L^p(\Omega)$ for some $p > 2$. Applying the estimate (6) and imbedding theorems we obtain that the right hand side of (9) is in $L^{p_1}(\Omega)$ for $p_1 > p$. Proceeding further we obtain that the right hand side of (9) is in Z. Lastly, applying the estimate (5) and the inequality $\|f\|_{L^2(\Omega)} \leq$ $\leq \text{const.} \, \|f\|_Z$ we obtain

$$\| u(w)\|_X \leq c\|f\|_Z,$$

with a constant independent of $w \in Y$ and $f \in Z$. Q.E.D.

Remark 4. If we denote $L^{-1} = (L(w) - \mu m)^{-1} : Z \to X$ then $L^{-1}f = u(w)$ for f and $u(w)$ from (9). Lemma 2 tells us that $\|L^{-1}\| \leq$ const. with a constant independent of $w \in Y$, where $\|L^{-1}\|$ denotes the usual operator norm.

Lemma 3. *Let*

(13) $L(w_n)u(w_n)(x) = \mu m(x)u(w_n)(x) + f_n(x)$ *in* Ω *and* $w_n \to w$ *in* Y, $f_n \to f$ *in* Z. *Then* $u(w_n) \to u(w)$ *in* X, *for* $n \to \infty$.

Proof. From the assumption (3) and the author's result [4, Thm 2] we obtain

$$a_{ij}(w_n) \to a_{ij}(w) \text{ in } Y, a_o(w_n) \to a_o(w) \text{ in } Z.$$

Hence

$$\sum_j a_{ij}(w_n)D_jv \to \sum_j a_{ij}(w)D_jv \text{ in } Y,$$

$$a_o(w_n)v \to a_o(w)v \text{ in } Z$$

for each $v \in X$. Consequently

(14) $$L(w_n)v \to L(w)v \text{ in } Z$$

for each $v \in X$. Using (14), Remark 4 and denotation $L_n^{-1} = (L(w_n) - \mu m)^{-1}$ we obtain

$$\|u(w_n) - u(w)\|_X = \|L_n^{-1}f_n - L^{-1}f\|_X \leq$$

$$\leq \|L_n^{-1}(L_n - L)L^{-1}f\|_X + \|L_n^{-1}(f_n - f)\|_X \leq$$

$$\leq \text{const.}(\|L_n(L^{-1}f) - L(L^{-1}f)\|_Z + \|f_n - f\|_Z) \to 0.$$

Q.E.D.

Remark 5. There is proved in [4, Thm 2] that a necessary and sufficient condition for the continuity of Němyckij's operator $a_{ij}(\ .\ ):Y \longrightarrow Y$, resp. $a_0(\ .\ ):Z \longrightarrow Z$, is (3). This is the reason why using this method of the proof there is not possible to weaken the condition (3).

Let $m \in C^1(\overline{\Omega})$ be the weight function satisfying the assumptions stated in Remark 3. We are ready, now, to prove the following useful assertion.

<u>Lemma 4</u>. Suppose that $\mu_1(m) > 1$ for all $w \in Y$, $f \in Z$, $f > 0$ in Ω. Then the problem

$$(15) \qquad \begin{cases} L(v)v = m(x)v + f \text{ in } \Omega, \\ \quad v = 0 \text{ on } \partial\Omega \end{cases}$$

has the solution $v \in X$ such that $v > 0$ in Ω and outward normal derivative $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$.

<u>Proof</u>. According to [5, Thm 1.14, 1.17], for each fixed $w \in Y$ there exists the unique solution $v(w) \in X$ of the linear problem

$$(15') \qquad \begin{cases} L(w)v(w) = m(x)v(w) + f \text{ in } \Omega, \\ \quad v(w) = 0 \text{ on } \partial\Omega \end{cases}$$

such that $v(w) > 0$ in Ω and $\frac{\partial v(w)}{\partial \nu} < 0$ on $\partial\Omega$. We shall define the operator $S:Y \longrightarrow X$ by the way $S(w) = v(w)$, where $v(w)$ is the unique solution of (15').

Let us suppose that $w_n \longrightarrow w$ in Y. Applying Lemma 3 we obtain $v(w_n) \longrightarrow v(w)$ in X. This means that S is continuous from Y into X. According to [7, Thm 1.2.14, 1.5.10] we have the compact imbedding $X \hookrightarrow\hookrightarrow Y$ and hence the restriction $\tilde{S} = S|X:X \longrightarrow X$ is completely continuous operator. Applying Lemma 2 we obtain the existence of a sufficiently large ball in X centred at the origin which is mapped by $\tilde{S}$ into itself. Schauder fixed point theorem implies the existence of at least one $v \in X$ such that $S(v) = v$, i.e. v is the solution of (15). Since v is also the solution of (15') with $w = v$ it is $v > 0$ in Ω, $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$. Q.E.D.

The following result is due to Boccardo [3, Thm 1].

<u>Lemma 5</u>. For each positive real number r, we can find a positive eigenvalue μ with the corresponding positive eigenfunction $u \in X$ such that

$$(16) \qquad \begin{cases} L(u)u = \mu u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

and $\|u\|_{L^2(\Omega)} = r$.

<u>Remark 6</u>. More precisely, by a direct application of [3, Thm 1] we obtain a positive eigenfunction $u \in Z$. But under our assumptions on the coefficients of the differential operator L Remark 2 immediately implies that $u \in X$.

The following assertion will be very important in the proof of our main existence theorem.

<u>Lemma 6</u>. There exists a constant $k > 0$ (independent of $u \in X$ and $r > 0$) such that

$$\|u\|_X \leq kr,$$

where $u \in X$, $\|u\|_{L^2(\Omega)} = r$ is the solution of the eigenvalue problem (16).

Proof of this lemma is based on the bootstrap argument used in the proof of Lemma 2 and the uniform estimates (5) and (6) play the key role in proving this assertion.

3. Subsolution, supersolution and the existence of the solution

Definition. A function $\overline{u} \in C^{2,\alpha}(\overline{\Omega})$ is said to be a supersolution of (1) if

$$L(\overline{u})\overline{u} \geq g(x,\overline{u}) \text{ in } \Omega,$$
$$\overline{u} \geq 0 \text{ on } \partial\Omega .$$

A function $\underline{u} \in C^{2,\alpha}(\overline{\Omega})$ is said to be a subsolution of (1) if

$$L(\underline{u})\underline{u} \leq g(x,\underline{u}) \text{ in } \Omega,$$
$$\underline{u} \leq 0 \text{ on } \partial\Omega .$$

Let us formulate, now, the assertion which is proved in more general setting in Kramer [9].

Lemma 7. Suppose $\underline{u} \leq \overline{u}$ (in Ω) are sub- and super-solutions of (1). Then there exists at least one solution $u(x) \in C^{2,\alpha}(\overline{\Omega})$ of (1) satisfying

$$\underline{u}(x) \leq \overline{u}(x) \leq \overline{u}(x) \text{ in } \Omega .$$

Remark 7. The result of Kramer [9] is the generalization of the well known result of Kazdan and Warner for semilinear elliptic problems (see e.g. Fučík [6]).

4. Existence of positive solutions. In this section we shall prove the existence of a positive solution for quasilinear elliptic problem (1) with sublinear nonlinearity $g(x,s)$.

Let the function g satisfy the following conditions:

(17) There are constants $g_0 > 0$, $s_0 > 0$ such that
$g(x,s) \geq g_0 s \qquad \forall x \in \overline{\Omega}, \ \forall 0 < s < s_0 .$

(18) There are continuous functions $g_\infty, c : \overline{\Omega} \longrightarrow R$, with $c(x) \geq 0$ such that

$$g(x,s) \leq g_{\infty}(x)s + c(x) \qquad \forall x \in \overline{\Omega}, \quad \forall s \geq 0.$$

Theorem 1. Suppose that the function g satisfies (17) and (18). Let

$$(19) \qquad \sigma_1(g_0) < \frac{1}{\beta + \gamma/\sigma_1(1)},$$

$$(20) \qquad \sigma_1(g_{\infty}) > \frac{1}{\alpha}.$$

Then the Dirichlet problem (1) has a positive solution.

Remark 8. An analogous theorem for semilinear elliptic problems was firstly proved by Amann [2].

Proof of Theorem 1. Choose the C^1-functions $\hat{g}_{\infty}$, $\hat{c}: \overline{\Omega} \to R$ such that $\hat{c}(x) > 0$,

(21) $g(x,s) \leq \hat{g}_{\infty}(x)s + \hat{c}(x) \qquad \forall x \in \overline{\Omega}, \quad \forall s \geq 0,$

$\hat{g}_{\infty}(x_0) > 0$ for some $x_0 \in \Omega$ and

$$\| g_{\infty} - \hat{g}_{\infty} \|_{L^{N/2}(\Omega)} < \varepsilon$$

for such small $\varepsilon > 0$ that the continuous dependence of $\sigma_1(m)$ on the weight function m (see Remark 3) would imply $\sigma_1(\hat{g}_{\infty}) > \frac{1}{\alpha}$. According to Lemma 1 it is $\mu_1(\hat{g}_{\infty}) > 1$ for all $w \in Y$. Hence using Lemma 4 the problem

$$(22) \qquad \begin{cases} L(u)u = \hat{g}_{\infty}u + \hat{c} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has the solution $\overline{u} \in X$ and $\overline{u} > 0$ in Ω, outward normal derivative $\frac{\partial \overline{u}}{\partial \nu} < 0$ on $\partial\Omega$. Hence the expressions (21) and (22) show that $\overline{u}$ is a supersolution of (1).

The assumption (19) implies that $\mu_1(g_0) < 1$ for all $w \in Y$. Then according to Lemma 5 the eigenvalue problem

$$(23)\quad \begin{cases} L(u)u = \mu g_0 u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a positive eigenfunction $\underline{u} \in X$ corresponding to the eigenvalue $\mu_1 < 1$ and $\|\underline{u}\|_{L^2(\Omega)} = r$. According to Lemma 6 the number $r > 0$ can be chosen such small that $\underline{u} < s_0$ and $\underline{u} < \overline{u}$ in Ω. Then using (17) we obtain

$$L(\underline{u})\underline{u} = \mu_1 g_0 \underline{u} < g(x,\underline{u})$$

which shows that $\underline{u}$ is a subsolution of (15). There are fulfilled all the assumptions of Lemma 7 and there exists a solution $u \in X$ of the problem (1). Note that this solution is such that $u(x) \geq \underline{u}(x) > 0$ for all $x \in \Omega$. Q.E.D.

Remark 9. Consider the eigenvalue problem

$$(24)\quad \begin{cases} -\sum D_i(a_{ij}(u(x))D_j u(x)) + a_0(u(x))u(x) = \lambda f(x,u(x)), \\ x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega \end{cases}$$

where $f: \overline{\Omega} \times R^+ \longrightarrow R$ is a C^1-function, and let us suppose that

$$f_0(x) = \liminf_{s \to 0_+} \frac{f(x,s)}{s}, \quad f_\infty(x) = \limsup_{s \to +\infty} \frac{f(x,s)}{s}$$

are continuous functions. Then if

(i) $f_0(x) \equiv +\infty$ (in particular if $f(x,0) > 0$) and $f_\infty(x) \leq \leq 0$, the problem (24) has a positive solution for all $\lambda > 0$;

(ii) $f_0(x) \equiv +\infty$ and $f_\infty(x_0) > 0$ for some point $x_0 \in \Omega$, the problem (24) has a positive solution for all

$$0 < \lambda < \frac{\alpha \delta_1(1)}{\sup_{x \in \overline{\Omega}} f_\infty(x)};$$

(iii) $0 < \varepsilon \leq f_0(x) < +\infty$ in $\overline{\Omega}$ and $f_\infty(x) \leq 0$, the problem (24) has a positive solution for all

$$\lambda > \frac{\beta \delta_1(1) + \gamma}{\inf_{x \in \overline{\Omega}} f_0(x)};$$

(iv) $0 < \varepsilon \leq f_0(x) < +\infty$ in $\bar{\Omega}$ and $f_\infty(x_0) > 0$ for some $x_0 \in \Omega$, the problem (24) has a solution for all

$$\frac{\beta\delta_1(1) + \gamma}{\inf_{x \in \bar{\Omega}} f_0(x)} < \lambda < \frac{\alpha\delta_1(1)}{\sup_{x \in \bar{\Omega}} f_\infty(x)} .$$

The proof of (i) - (iv) follows immediately from Theorem 1.

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