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THE APPROXIMATION AND EXTENSION OF UNIFORMLY CONTINUOUS BANACH SPACE VALUED MAPPINGS

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Abstract: The aim of this paper is to present several results concerning the extension of uniformly continuous mappings and the approximation of uniformly continuous mappings by Hölder mappings in the setting of infinite-dimensional Banach spaces. In particular, we show that every uniformly continuous mapping between Hilbert spaces can be uniformly approximated by Lipschitz mapping.

Key words and Phrases: Uniformly continuous, equi-uniformly continuous, Lipschitz, Hölder, Lipschitz for large distances, modulus of continuity, $c_0(I)$, $\ell_\infty(I)$, L^p.

Classification: Primary 26A16, 41A30, 54C20
Secondary 54E15

The principal results of the paper are the following. Theorem 1 establishes the equivalence of extending a uniformly continuous mapping with values in an ell-infinity space and uniformly approximating the mapping by Lipschitz mappings, while Theorem 3 establishes a similar result for uniformly continuous mappings between subsets of Hilbert spaces. Corollary 2 establishes that every uniformly continuous mapping between Hilbert spaces can be uniformly approximated by Lipschitz mappings. Finally, Corollary 3 characterizes the subsets of Hilbert space for which every uniformly continuous Hilbert space valued mapping can be extended; these are precisely the U-embedded sets studied in $[LR]_{1-3}$. This result generalizes the extension theorem found in [GZ].

Notation and definitions. Assume that (M,d) and (N,e) are metric spaces. The family of uniformly continuous mappings from M to N will be denoted by $U(M,N)$. If $N = R$ is the set of real numbers with the standard absolute value metric, the notation $U(M)$ will be used.

Given a mapping $f:M \longrightarrow N$, the modulus of continuity of f is defined by $\omega_f(t) = \sup\{e(f(x),f(y)):d(x,y) \leq t\}$ for $t>0$. The general term modulus of continuity will denote any non-decreasing mapping $\omega:[0,+\infty) \longrightarrow [0,+\infty)$ which satisfies $\lim_{t\to 0} \omega(t) = 0$. A modulus of continuity is subadditive if

$$\omega(s + t) \leq \omega(s) + \omega(t) \text{ for } s,t>0.$$

It is well known that every concave modulus of continuity is subadditive.

A mapping $f:M \longrightarrow N$ is Hölder of class α, $0<\alpha\leq 1$, if

$$\|f\|_\alpha = \sup\{e(f(x),f(y))/d(x,y)^\alpha : x,y\in M,\ x\neq y\}<+\infty.$$

If $\alpha = 1$ and $\|f\|_1<+\infty$, then f is called a Lipschitz mapping. The family of all Hölder mappings of class α is denoted by $\Lambda_\alpha(M,N)$ and the family of all Lipschitz mappings is denoted by $\mathrm{Lip}(M,N)$.

The pair (M,N) has the contraction-extension property with respect to Λ_α if each Hölder mapping $f:S \longrightarrow N$ of class α defined on a subset S of M can be extended to a Hölder mapping $F:M \longrightarrow N$ of class α such that $\|F\|_\alpha = \|f\|_\alpha$. If $\alpha = 1$, we will simply use the phrase "contraction-extension property".

$\overline{\Lambda_\alpha}(M,N)$ (resp. $\overline{\mathrm{Lip}}(M,N)$) will denote the family of mappings $f:M \longrightarrow N$ that can be uniformly approximated by members of $\Lambda_\alpha(M,N)$ (resp. $\mathrm{Lip}(M,N)$); that is, for each $\epsilon > 0$ there exists a mapping g in the respective family such that

$$e(g(x),f(x)) < \varepsilon \quad \text{for every } x \text{ in } M.$$

A mapping $f:M \longrightarrow N$ is <u>Hölder of class</u> α for large distances if for each $\varepsilon > 0$, there exists a constant K_ε such that $d(x,y) > \varepsilon$ implies $e(f(x),f(y)) < K_\varepsilon d(x,y)^\alpha$. If $\alpha = 1$, we say that the mapping is <u>Lipschitz for large distances</u>.

<u>Section One</u>. Our first goal is to establish an approximation-extension theorem for mappings with values in Banach spaces of the form $l_\infty(I)$ or $c_0(I)$. This will require two preliminary results.

<u>Lemma 1</u>: Assume that $f:M \longrightarrow (B,\| \|)$ is a bounded uniformly continuous mapping on the metric space M, where B denotes either the Banach space $l_\infty(I)$ or $c_0(I)$ with the usual supremum norm $\| \|$. Then for every $\varepsilon > 0$, there exists a Lipschitz mapping $l:M \longrightarrow B$ such that

$$\|l(x) - f(x)\| < \varepsilon \quad \text{for every } x \text{ in } M.$$

The proof of Lemma 1 for $B = l_\infty(I)$ is found in $[LR]_3$. The result for $c_0(I)$ follows from the result for $l_\infty(I)$ since by [Li] $c_0(I)$ is a Lipschitz retract of $l_\infty(I)$.

The next result shows that in certain cases a Hölder mapping f of class α for large distances can be approximated by a Hölder mapping which is only a "finite distance" from f.

<u>Lemma 2</u>: Assume the pair (M,B) has the contraction-extension property with respect to Λ_α, for some $0 < \alpha \leq 1$, where (M,d) is a metric space and $(B,\| \|)$ is a Banach space. Assume that $f:M \longrightarrow B$ is a uniformly continuous mapping. Then f is Hölder of class α for large distances if and only if there exists a Hölder mapping $l:M \longrightarrow B$ of class α such that

$(*)\quad \sup\{\|f(x)-l(x)\| : x\in M\} < +\infty .$

Proof. First, assume that a Hölder mapping l of class α exists such that $(*)$ is satisfied. A straightforward computation shows that

$$\|f(x)-f(y)\| < 2s + \|l(x)-l(y)\| < 2s + \|l\|_1[d(x,y)]^{\alpha}$$

for each pair of points x and y in M. If $d(x,y) > \varepsilon$, it follows that

$$\|f(x)-f(y)\| < [C + \|l\|_1]\,[d(x,y)]^{\alpha}\ , \text{ where } C = 2s/\varepsilon^{\alpha},$$

so f is Hölder of class α for large distances.

Conversely, assume that f is Hölder of class α for large distances. Choose $\delta > 0$ such that $\omega_f(\delta) < 1$ and choose K such that $\|f(x)-f(y)\| \leq K[d(x,y)]^{\alpha}$ whenever $d(x,y) > \varepsilon$. Let D be a maximal δ-discrete subset of M (i.e. $d(x,y) \geq \delta$ for each distinct pair of points x and y in D and if $p \notin D$, there exists x in D such that $d(x,p) < \delta$). One easily verifies that $f|D: D \longrightarrow B$ is a Hölder mapping of class α with $\|f|D\|_{\alpha} \leq K$. Hence by assumption, this mapping can be extended to a Hölder mapping $l: M \longrightarrow B$ such that $\|l\|_{\alpha} \leq K$.

Now given x in M, choose p in D such that $d(x,p) < \delta$. Then

$\|f(x)-l(x)\| \leq \|f(x)-f(p)\| + \|l(p)-l(x)\|$ (since $f(p) = l(p)$)

$$\leq \omega_f(\delta) + \|l\|_1\,[d(p,x)]^{\alpha}$$
$$< 1 + K\delta^{\alpha},$$

so $(*)$ is satisfied.

<u>Theorem 1</u>: Let B denote either the Banach space $l_\infty(I)$ or $c_0(I)$ with the usual supremum norm $\|\ \|$. The following statements are equivalent for a uniformly continuous mapping $f: S \longrightarrow B$

defined on a subset S of the normed linear space M.

(i) f can be extended to a uniformly continuous mapping $F:M \longrightarrow B$.

(ii) f is Lipschitz for large distances.

(iii) For each $\varepsilon > 0$, there exists a Lipschitz mapping $l:S \longrightarrow B$ such that $\|f(x) - l(x)\| < \varepsilon$ for every x in S.

Proof. We first assume that $B = l_{\infty}(I)$.

(i) $\longrightarrow$ (ii) follows from [CK] or [Li], where it is noted that every member of $U(M,B)$ is Lipschitz for large distances.

(ii) $\longrightarrow$ (iii). Let $\varepsilon > 0$. From [AP], the pair (M,B) has the contraction-extension property; hence by assumption (ii) and Lemma 2, there exists a Lipschitz mapping $l_1:S \longrightarrow B$ such that

$$\sup \{\|l_1(x) - f(x)\| : x \in S\} < +\infty.$$

By Lemma 1, there exists a Lipschitz mapping $l_2:S \longrightarrow B$ such that

$$\|l_2(x) - (f(x) - l_1(x))\| < \varepsilon \quad \text{for every } x \text{ in } S,$$

so $l = l_1 + l_2$ is the Lipschitz mapping required in (iii).

(iii) $\longrightarrow$ (i). The contraction-extension property guarantees that every Lipschitz mapping $l:S \longrightarrow B$ can be extended to a Lipschitz mapping $L:M \longrightarrow B$. Since f can be uniformly approximated by members of $Lip(S,B)$, it follows from either $[LR]_2$ or [Pt] that f can be extended to a member of $U(M,B)$.

The equivalence of statements (i)-(iii) for $B = c_0(I)$ follows from their equivalence for $l_{\infty}(I)$ and the fact established in [Li] that $c_0(I)$ is a Lipschitz retract of $l_{\infty}(I)$.

It is noted in [CK] and [Li] that condition (ii) in Theorem 1 is satisfied for every member of $U(S,B)$ whenever S is a convex subset of a normed linear space. Therefore, we have established the following result.

Corollary 1: Let B denote either the Banach space $l_\infty(I)$ or $c_0(I)$. If S is a convex subset of a normed linear space, every member of U(S,B) can be uniformly approximated by members of Lip(S,B).

Remark: The $l_\infty(I)$ version of Corollary 1 was probably known to Gehér, but it was not explicitly stated in [G].

Section Two. Following the pattern used in section one, we will first prove an approximation theorem for bounded mappings and then use a general lemma to establish an approximation theorem for unbounded mappings.

Given $p > 1$, define

$$\alpha(p) = \begin{cases} 1/2 & 1 < p < 2 \\ 1/p & 2 \leq p. \end{cases}$$

For $p \geq 1$, L^p will denote any ell-p space based on a sigma-finite measure space.

Lemma 3: Let $f: M \longrightarrow L^p$ $(p > 1)$ be a bounded uniformly continuous mapping defined on a metric space (M,d). For each $\varepsilon > 0$, there exists a Hölder mapping $l: M \longrightarrow L^p$ of class $\alpha(p)$ such that

$\| f(x) - l(x) \|_p < \varepsilon$ for every x in M.

Proof. Let $\alpha = \alpha(p)$. Without loss of generality, we may assume that f[M] is contained in the unit ball of L^p -- $f[M] \subseteq \{ v : \| v \|_p \leq 1 \}$.

Furthermore, we may assume that L^p is isometrically embedded as a linear subspace of $B = l_\infty(I)$ for some set I.

Define $Y = \{z \in B : \inf\{\|z - w\| : w \in L^p\} < 4\}$. By ([Li], Theorem 9(a)), there exists a retraction

$$r : Y \longrightarrow L^p$$

such that $\omega_r(t) \leq Ct^\alpha$ for $0 < t \leq 4$ and a fixed constant C.

Given $0 < \varepsilon < 1$, by Lemma 1 there exists a Lipschitz mapping $l : M \longrightarrow B$ such that

$\|l(x) - f(x)\|_\infty < [\varepsilon/(C + 1)]^{1/\alpha}$ for every x in M.

Since $\varepsilon < 1$, it follows that $l[M]$ is a subset of Y, so the mapping

$$h = r \circ l : M \longrightarrow L^p$$

is well defined. We claim that h is a Hölder mapping of class α.

Given x and y in M, let $v = l(x)$ and $w = l(y)$. Since

$$\begin{aligned} \|v - w\|_\infty &< \|v - f(x)\|_\infty + \|f(x) - f(y)\|_\infty + \|f(y) - w\|_\infty \\ &< \quad \varepsilon \quad + \quad 2 \quad + \quad \varepsilon \\ &< 4, \end{aligned}$$

the estimate $\omega_r(t) \leq Ct^\alpha$ is valid for $t = \|v - w\|_\infty$; hence

$$(*) \quad \|r(v) - r(w)\|_p \leq C\,\|v - w\|_\infty^\alpha .$$

In addition, $\|v - w\|_\infty = \|l(x) - l(y)\|_\infty < \|l\|_1 d(x,y)$, so

$$(**) \quad \|v - w\|_\infty \leq \|l\|_1 d(x,y).$$

Combining $(*)$ and $(**)$, we obtain

$$\|h(x) - h(y)\|_p \leq C\|l\|_1 d(x,y)^\alpha,$$

so h is a Hölder mapping of class α.

Finally, we claim that $\|h(x) - f(x)\|_p < \varepsilon$ for every x in M.

By definition, $\|h(x) - f(x)\|_p = \|r(l(x)) - r(f(x))\|_p \leq$

$\leq \omega_r(\|l(x) - f(x)\|_\infty) \leq C\|l(x) - f(x)\|_\infty^\alpha$

since $\|l(x) - f(x)\|_\infty < 1$; therefore

$\|h(x) - f(x)\|_p \leq C\ [\varepsilon/(C + 1)] < \varepsilon$, which completes the proof.

Lemma 4: Assume that the pair (M,B) has the contraction-extension property with respect to Λ_α , for some $0 < \alpha \leq 1$, where (M,d) is metric space and (B, $\| \ \|$) is a Banach space. Then for every $0 < \beta < \alpha$,

$$\Lambda_\beta(M,B) \subseteq \overline{\Lambda_\alpha(M,B)}.$$

Proof. Assume that f is a member of $\Lambda_\beta(M,B)$ with $\|f\|_\beta = K$. Given $\varepsilon > 0$, define $\delta = (\varepsilon/2K)^{1/\beta}$ and let D be a maximal δ-discrete subset of M.

We claim that $f|D$ is a member of $\Lambda_\alpha(D,B)$. Choose x and y in D with $x \neq y$. Then $d(x,y) \geq \delta$, so $a = d(x,y)/\delta \geq 1$ implies that $a^\alpha \geq a^\beta$; hence $\|f(x) - f(y)\| \leq Kd(x,y)^\beta =$
$= K[d(x,y)/\delta]^\beta \cdot \delta^\beta = Ka^\beta \cdot \delta^\beta \leq Ka^\alpha \delta^\beta = (K\delta^{\beta-\alpha})d(x,y)^\alpha$.

Now extend $f|D$ to a member l of $\Lambda_\alpha(M,B)$ such that $\|l\|_\alpha < K\delta^{\beta-\alpha}$. Given x in M, chcose p in D such that $d(x,p) < \delta$. Then

$$\begin{aligned}\|f(x) - l(x)\| &\leq \|f(x) - f(p)\| + \|l(p) - l(x)\| \\ &\leq Kd(x,p)^\beta + \|l\|_\alpha\ d(x,p)^\alpha \\ &\leq K\delta^\beta + K\delta^\beta = \varepsilon,\end{aligned}$$

which establishes that f is a member of $\overline{\Lambda_\alpha(M,B)}$.

We can now establish the following analogue of Theorem 1.

Theorem 2: Assume that the pair (M,L^p) has the contraction-extension property with respect to Λ_α , for some

$0 < \alpha \leq 1$. Then the following statements are equivalent for a uniformly continuous mapping $f: M \longrightarrow L^p$, $p > 1$.

(i) f is a Hölder mapping of class α for large distances.

(ii) Given $\varepsilon > 0$, there exists a Hölder mapping $l: M \longrightarrow L^p$ of class α such that $\| f(x) - l(x) \|_p < \varepsilon$ for every x in M.

Proof. The proof of (ii) $\longrightarrow$ (i) is essentially the same as the proof of Lemma 2. The proof of (i) $\longrightarrow$ (ii) will be divided into two cases.

<u>Case 1</u>: $0 < \alpha \leq \alpha(p)$.

By Lemma 2, there exists a Hölder mapping

$$l_1: M \longrightarrow L^p$$

of class α such that $\sup \{ \| f(x) - l_1(x) \|_p : x \in M \} < + \infty$.

Define $g = f - l_1$. By Lemma 3, there exists a (bounded) Hölder mapping

$$l_2: M \longrightarrow L^p$$

of class $\alpha(p)$ such that $\| g(x) - l_2 \|_p < \varepsilon$ for every x in M. Since l_2 is bounded and $\alpha \leq \alpha(p)$, it is easy to verify that l_2 is also a Hölder mapping of class α.

Hence $l = l_1 + l_2$ is the required approximation of f.

<u>Case 2</u>: $\alpha(p) < \alpha \leq 1$.

We proceed as in Case 1 to choose a Hölder mapping l_1 of class α and a Hölder mapping l_2 of class $\alpha(p)$ such that

$$\| f(x) - l_1(x) - l_2(x) \|_p \quad 2$$

for every x in M. Since $\alpha > \alpha(p)$, we may use Lemma 4 to choose a Hölder mapping h of class α such that

$$\| h(x) - l_2 \|_p < \varepsilon/2$$

for every x in M.

Then $l = h + l_1$ is the required approximation of f.

Section Three. When both the domain and the range of the mapping are subsets of Hilbert spaces, we can use the results in section two to derive not only the Grünbaum-Zarantonello extension theorem found in [GZ], but also a general approximation theorem.

Theorem 3: Let H and K be Hilbert spaces and let S be a subset of H. The following statements are equivalent for a uniformly continuous mapping $f: S \longrightarrow K$.

(i) f can be extended to a uniformly continuous mapping $F: H \longrightarrow K$.

(ii) f is Lipschitz for large distances.

(iii) f can be uniformly approximated by members of Lip(S,K).

Proof. (i) $\longrightarrow$ (ii) is a consequence of the previously mentioned fact that (ii) is valid for every uniformly continuous mapping defined on a convex subset of a normed linear space.

(ii) $\longrightarrow$ (iii). Valentine's theorem in V states that the pair (S,K) has the contraction-extension property; hence by Theorem 2, (ii) implies (iii).

(iii) $\longrightarrow$ (i) follows from Valentine's theorem and the technique presented in $[LR]_2$ or [Pt] (as noted in the proof of (iii) $\longrightarrow$ (i) of Theorem 1).

Since condition (ii) in Theorem 3 is always satisfied for convex subsets of normed linear spaces, we can also state the following result.

Corollary 2: Every uniformly continuous mapping defined on a convex subset of a Hilbert space with range in a Hilbert space can be uniformly approximated by Lipschitz mappings.

Remarks. 1. The original Grünbaum-Zarantonello theorem in [GZ] states the equivalence of conditions (i) and (ii) in Theorem 3, but with condition (ii) expressed in the following terms: there exists a sub-additive modulus of continuity ω such that

$$\omega_f(t) \leq \omega(t) \quad \text{for every } t > 0.$$

In fact, one can establish a result which connects the two formulations. The following statements are equivalent for a uniformly continuous mapping $f: M \to N$ between metric spaces:

(i) f is Lipschitz for large distances.

(ii) $\omega_f(t) = O(t)$ as $t \to +\infty$.

(iii) There exists a subadditive modulus of continuity ω such that

$$\omega_f(t) \leq \omega(t) \quad \text{for every } t > 0.$$

(The equivalence of conditions (i) - (iii) is established in [G].)

2. The authors do not know whether conditions (ii) and (iii) in Theorem 3 are equivalent for other pairs of ell-p spaces. The following example shows that condition (i) is generally not equivalent to either condition (ii) or (iii).

Example: Assume that B is either a separable infinite-dimensional reflexive Banach space or has the form L^1. Assume that B is isometrically embedded as a subspace of $l_\infty(I)$. Then the identity mapping $B \to B$ cannot be extended to a uniformly continuous mapping $l_\infty(I) \to B$.

By ([LT], 2.a.7), No separable infinite-dimensional Banach space is injective, so there does not exist a continuous linear projection $l_\infty(I) \longrightarrow B$. Hence by [Li] there does not exist a uniformly continuous projection $l_\infty(I) \longrightarrow B$.

3. The convexity assumption on S in Corollary 3 is unnecessarily restrictive. Following $[LR]_1$, we say that a subset S of a metric space M is U-embedded if every member of U(S) can be extended to a member of U(M). Even when M is a normed linear space, the U-embedding property is difficult to classify, but it is apparently geometric in nature. Every convex set (or more generally every quasi-convex subset in the sense of [CK]) is U-embedded. For orientation, we only mention here that a hyperbola or elliptic paraboloid is U-embedded in respectively, R^2 or R^3, while a parabola or hyperbolic paraboloid is not U-embedded in respectively, R^2 or R^3.

In $[LR]_3$, the authors proved that a subset S of a normed linear space is U-embedded if and only if every uniformly continuous mapping on S with range in any metric space is Lipschitz for large distances. Therefore, we have also established the following result.

Corollary 3: Let H and K be Hilbert spaces and let S be a subset of H. Every uniformly continuous mapping $f: S \longrightarrow K$ can be extended to a uniformly continuous mapping $F: H \longrightarrow K$ if and only if S is a U-embedded subset of H.

Finally, it should be noted that many of the preceding results can be generalized to results for *families* of mappings. For example, the following results can be established:

If (f_i) is a point-bounded equi-uniformly continuous family of mappings defined on a U-embedded subset S of the Hilbert space H with range in the Hilbert space $(K, \| \|)$, then

(i) for each $\varepsilon > 0$, there exists a family of Lipschitz mappings

$$(l_i): S \to K$$

such that

(a) $\sup\{\|l_i\|_1\} < +\infty$,

(b) $\|f_i(x) - l_i(x)\| < \varepsilon$

for every i and each x in S, and

(ii) there exists a point-bounded equi-uniformly continuous family of mappings $(F_i): H \to K$ such that for every i, F_i is an extension of f_i.

Furthermore, if the U-embedded subset S is also <u>uniformly connected</u> (i.e. S is <u>not</u> the union of two sets A and B such that the distance between A and B is positive), then the results (i) and (ii) stated above are valid without the point-bounded restriction on the family (f_i).

The above generalizations can be established by using the techniques found in the present paper and the results found in $[LR]_3$.

References

[AP] N. ARONSZAJN and P. PANITCHPAKDI: Extensions of uniformly continuous transformations and hyperconvex metric spaces, Pacific J. Math. 6(1956), 405-439.

[CK] H.H. CORSON and V. KLEE: Topological classification of convex bodies, Proc. Symposia Pure Math. AMS (Convexity) 7(1963), 37-51.

[G] L. GEHÉR: Über Fortsetzungs- und Approximationprobleme für stetige Abbildungen von metrischen Räumen, Acta Sci. Math. Szeged 20(1959), 48-66.

[GZ] F. GRÜNBAUM and E. ZARANTONELLO: On the extension of uniformly continuous mappings, Mich. Math. J. 15(1968), 65-74.

[Li] J. LINDENSTRAUSS: On non-linear projections in Banach spaces, Mich. Math. J. 11(1964), 263-287.

[LR]$_1$ R. LEVY and M.D. RICE: The approximation and extension of uniformly continuous mappings, Fifth Prague Topological Symposium, August, 1981, Heldermann-Verlag, Berlin, 1982.

[LR]$_2$ -------------------- : The extension of uniformly continuous mappings from subsets of metric spaces, submitted to Amer. J. Math.

[LR]$_3$ -------------------- : The extension of uniformly continuous Banach space-valued mappings, Special Session on Rings of Continuous Functions, Winter meeting AMS, 1982, to appear in Marcel Dekker Lecture Notes.

[LT] J. LINDENSTRAUSS and L. TZAFRIRI: Classical Banach Spaces I, Sequence Spaces, Springer-Verlag, New York, 1977.

[Pt] P. PTÁK: On extensions of uniformly continuous Banach space-valued mappings, Comment. Math. Univ. Carolinae 20(1979), 101-105.

[V] F.A. VALENTINE: A Lipschitz condition preserving extension for a vector function, Amer. J. Math. 67 (1945), 83-93.

[WW] J.H. WELLS and L.R. WILLIAMS: Embeddings and Extensions in Analysis, Springer-Verlag, New York, 1975.

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