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DISTRIBUTIVE GROUPOIDS AND PRERADICALS I.

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Abstract: A theory of preradicals and their compositions for the class of distributive groupoids is developed.

Key words: Groupoid, preradical.

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The main open problem in the theory of distributive groupoids is whether free distributive idempotent groupoids are cancellative. In solving this problem, it appears useful to have at hand an auxiliary theory dealing with congruences which are more or less preserved by homomorphisms. In this way, we come to the notion of preradical known from the theory of modules (see e.g. [1]) and the purpose of the present note is to study preradicals and some of their generalizations for various classes of groupoids but mainly for the class of distributive groupoids. As for details concerning basic definitions, terminology and notation as well as for further references, the reader is referred to [2].

1. Basic notions. Let A be a non-empty abstract class of groupoids (i.e., A is closed under isomorphic images). A semipreradical r (for A) assigns to each $G \in A$ a congruence r(G) of G in such a way that f(r(G)) = r(H) whenever

$G,H \in A$ and f is an isomorphism of G onto H. We shall say that r satisfies the condition

- (A) if $H \in A$ whenever $G \in A$, H is a subgroupoid of G and a block of $r(G)$;

- (B) if $G/r(G) \in A$ for every $G \in A$;

- (C) (resp. (D)) if $f(r(G)) \subseteq r(H)$ whenever $G,H \in A$ and f is an injective (resp. projective) homomorphism of G into H;

- (E) if $f(r(G)) \subseteq r(H)$ whenever $G,H \in A$ and f is a homomorphism of G into H;

- (F) if $r(H) = H \times H$ whenever $G,H \in A$, H is a subgroupoid of G and a block of $r(G)$;

- (G) if $r(H) = id_H$ for every $G \in A$ such that $H = G/r(G)$ belongs to A;

- (H) (resp. (I)) if $r(H) \cap (f(G) \times f(G)) \subseteq f(r(G))$ whenever $G,H \in A$ and f is an injective (resp. projective) homomorphism of G into H;

- (K) if $r(H) \cap (f(G) \times f(G)) \subseteq f(r(G))$ whenever $G,H \in A$ and f is a homomorphism of G into H.

A semipreradical satisfying (E) is said to be a preradical. Every preradical satisfies both (C) and (D) and the converse is true provided A is closed under factorgroupoids (resp. subgroupoids). A semipreradical satisfying (A) and (F) (resp. (B) and (G)) is said to be idempotent (resp. a semiradical). A semipreradical satisfying (A), (C) and (H) (resp. (B), (D) and (I)) is said to be hereditary (resp. cohereditary).

Let r be a semipreradical. A groupoid $G \in A$ is said to be r-torsion (resp. r-torsionfree) if $r(G) = G \times G$ (resp. $r(G) = id_G$).

Let r, s be semipreradicals. For $G \in A$ put $(r \cap s)(G) =$

$= r(G) \cap s(G)$ and denote by $(r+s)(G)$ the congruence generated by $r(G) \cup s(G)$. We obtain thus two semipreradicals $r \cap s$ and $r+s$. Further, we shall write $r \circ s = s \circ r$ if $r(G) \circ s(G) = s(G) \circ r(G)$ for every $G \in A$. In that case, $r \circ s = s \circ r = r+s$. The following results are clear.

1.1. Proposition. Let r and s be semipreradicals.
(i) If both r and s satisfy (C) (resp. (D),(E),(H)) then $r \cap s$ satisfies the condition.
(ii) If both r and s satisfy (C) (resp. (D),(E)) then $r+s$ satisfies the condition.

1.2. Lemma. Let r be a semipreradical, $G,H \in A$ and let f be a homomorphism of G into H such that $\ker(f) \circ r(G) = r(G) \circ \ker(f)$. If $a,b,c \in G$ are such that $f(a) = f(b)$ and $(b,c) \in r(G)$ then $(a,d) \in r(G)$ for some $d \in G$ with $f(c) = f(d)$.

1.3. Proposition. Let r and s be semipreradicals such that $\ker(f) \circ r(G) = r(G) \circ \ker(f)$ and $\ker(f) \circ s(G) = s(G) \circ \ker(f)$ whenever $G,H \in A$ and f is a projective homomorphism of G onto H. If both r and s satisfy (I) then $r+s$ satisfies (I).

Let A denote the class of groupoids. We define two semipreradicals id and $t\ell$ by $id(G) = id_G$ and $t\ell(G) = G \times G$ for every groupoid G. Obviously, both id and $t\ell$ satisfy all the ten conditions (A),...,(K).

2. Composition of semipreradicals. Let A be a non-empty abstract class of groupoids. Consider two semipreradicals r and s and suppose that s satisfies (B). We define a semipreradical $r:s$ by $(r:s)(G) = f^{-1}(r(H))$, f being the natural

projection of G onto $H = G/s(G)$. The following assertions can be verified easily.

2.1. Proposition. (i) $s \subseteq r{:}s$.

(ii) If r satisfies (B) then $r{:}s$ satisfies (B).

(iii) If r satisfies (D) then $r, r+s \subseteq r{:}s$.

(iv) If r satisfies (C) and s satisfies (C) and (H) then $r{:}s$ satisfies (C).

(v) If both r and s satisfy (D) (resp. (E)) then $r{:}s$ satisfies (D) (resp. (E)).

(vi) If A is closed under factorgroupoids, r satisfies (D) and (F) and s satisfies (H) then $r{:}s$ satisfies (F).

(vii) If r satisfies (F) and s satisfies (C) and (H) then $r{:}s$ satisfies (F).

(viii) If r satisfies (G) and s satisfies (G) and (I) then $r{:}s$ satisfies (G).

(ix) If s satisfies (G) then $s{:}s = s$.

(x) If r satisfies (H) and s satisfies (C) and (H) then $r{:}s$ satisfies (H).

(xi) If A is closed under factorgroupoids (resp. subgroupoids), r satisfies (D) and (H) and s satisfies (H) then $r{:}s$ satisfies (H).

(xii) If r satisfies (I) then $r{:}s \subseteq r+s$.

2.2. Lemma. Let $G, H \in A$ and let f be a homomorphism of G into H such that $\ker(f) \circ s(G) = s(G) \circ \ker(f)$.
Suppose that $f(s(G)) = s(H) \cap (f(H) \times f(H))$ and $r(H/s(H)) \cap \cap (g(G/s(G)) \times g(G/s(G))) \subseteq g(r(G/s(G)))$, g being the induced homomorphism of $G/s(G)$ into $H/s(H)$. Then $(r{:}s)(H) \cap (f(G) \times \times f(G)) \subseteq f((r{:}s)(G))$.

2.3. <u>Proposition</u>. Suppose that $ker(f) \circ s(G) = s(G) \circ ker(f)$ whenever $G, H \in A$ and f is a (projective) homomorphism of G into H. If r satisfies (K) (resp. (I)) and s satisfies (D) and (K) (resp. (I)) then $r:s$ satisfies (K) (resp. (I)).

2.4. <u>Proposition</u>. Suppose that every groupoid from A is idempotent. Let s satisfy (A), (C) and (F) and let r satisfy (D) and (F). Further, let either r satisfy (A) or let A be closed under factorgroupoids. Then $r:s$ satisfies (F).

2.5. <u>Proposition</u>. Let $r+s$ satisfy (B) and (G) and let both r and s satisfy (D). Then $r+s = r:s$.

2.6. <u>Lemma</u>. Suppose that r satisfies (B) and let q be a semipreradical. Then $(q:r):s = q:(r:s)$.

2.7. <u>Lemma</u>. Let r_i, $i \in I$, be a non-empty family of semipreradicals. Then $(\sum r_i):s = \sum r_i:s$.

2.8. <u>Lemma</u>. Suppose that r satisfies (D). Let s_i, $i \in I$, be a non-empty family of semipreradicals satisfying (B) such that the semipreradical $\sum s_i$ satisfies (B). Then $\sum r:s_i \subseteq r: \sum s_i$.

3. <u>Composition of semipreradicals</u>. Let A be a non-empty abstract class of groupoids. Consider a semipreradical r satisfying (B); in fact, we shall demand a bit more which will be clear from the following. For every ordinal $o \geq 0$, we define a semipreradical ${}^{o}r$ by ${}^{0}r = id$, ${}^{o+1}r = r:{}^{o}r$ and ${}^{o}r = \cup\, {}^{p}r$, $p < o$, if $o > 0$ is limit; here, we assume that $G/{}^{o}r(G) \in A$ for all $G \in A$ and $o \geq 0$. It is clear that

${}^{0}r \subseteq {}^{1}r \subseteq {}^{2}r \subseteq \dots \subseteq {}^{o}r \subseteq {}^{o+1}r \subseteq \dots$ and ${}^{1}r = r$. Moreover, for every groupoid $G \in A$ there exists an ordinal $o = \mathcal{L}(G,r)$ which is the least with ${}^{o}r(G) = {}^{o+1}r(G)$. Setting $\hat{r}(G) = {}^{o}r(G)$ we obtain a semipreradical $\hat{r}$. The following statements are nearly obvious (use 2.1, 2.4 and, occasionally, a transfinite induction).

3.1. Proposition. (i) For every $o \geq 0$, the semipreradicals ${}^{o}r$ satisfy (B).
(ii) $\hat{r}$ is a semiradical satisfying (B).
(iii) If r satisfies (D) then all the semipreradicals ${}^{o}r$ satisfy (D) and $\hat{r}$ satisfies (D).
(iv) If r satisfies (E) then all the semipreradicals ${}^{o}r$ satisfy (E) and $\hat{r}$ is a radical.
(v) If r satisfies (C) and (H) then all the semipreradicals ${}^{o}r$ as well as $\hat{r}$ satisfy (C) and (H).
(vi) If A is closed under factorgroupoids (resp. subgroupoids) and r satisfies (D) and (H) then all the semipreradicals ${}^{o}r$ as well as $\hat{r}$ satisfy (D) and (H).

3.2. Proposition. Suppose that every groupoid from A is idempotent and that A is closed under subgroupoids. Let r satisfy (E) and (F). Then $\hat{r}$ is an idempotent radical.

3.3. Lemma. Let s be a semipreradical such that s satisfies (D) and $r \cap s = id$. Then $\hat{r} \cap s = id$.

3.4. Proposition. Suppose that r satisfies (D). Let s be a semipreradical such that $\hat{s}$ exists, s satisfies (D) and $r \cap s = id$. Then $\hat{r} \cap \hat{s} = id$.

3.5. Lemma. Let s be a semipreradical such that ${}^{k}s$ exists and satisfies (B) for each non-negative integer k. Suppose that $r:s = s:r$. Then $^{n}r:{}^{m}s = {}^{m}s:{}^{n}r$ for all non-negative integers n, m.

Proof. We show that $r:{}^{m}s = {}^{m}s:r$ by induction on m. For $m = 0$, there is nothing to be proved. If $m \geq 1$, then $r:{}^{m}s = r:s:{}^{m-1}s = s:r:{}^{m-1}s = s:{}^{m-1}s:r = {}^{m}s:r$ by 2.6 and the induction hypothesis.

3.6. <u>Lemma</u>. Let s be a semipreradical such that $s:r = s+r$. Then $s:{}^{n}r = s+{}^{n}r$ for each non-negative integer n.

Proof. By induction on n. If $n \geq 1$, then $s+{}^{n}r = s+{}^{n-1}r+{}^{n}r = (s:{}^{n-1}r)+{}^{n}r$ by the induction hypothesis. However, as one may check easily, $(s:{}^{n-1}r)+{}^{n}r = (s+r):{}^{n-1}r = s:r^{n-1}r = s:{}^{n}r$.

3.7. <u>Lemma</u>. Let s be a semipreradical such that ^{k}s exists and satisfies (B) for each non-negative integer k. Suppose that $r:s = s:r = r+s$. Then $^{n}r:{}^{m}s = {}^{m}s:{}^{n}r = {}^{n}r+{}^{m}s$ for all non-negative integers n, m.

Proof. Use 3.5 and 3.6.

3.8. <u>Lemma</u>. Suppose that r satisfies (D). Let s be a semipreradical satisfying (B) such that $r:s \subseteq s:r$. Then $\hat{r}:s \subseteq s:\hat{r}$.

Proof. First, by induction on $o \geq 0$, we show that $^{o}r:s \subseteq s:{}^{o}r$. If o is not limit then we can proceed similarly as in 3.5. Assume that $o > 0$ is limit. We have $^{o}r:s = (\sum {}^{p}r):s = \sum {}^{p}r:s \subseteq \sum s:{}^{p}r \subseteq s:(\sum {}^{p}r) = s:{}^{o}r$ by 2.7, 2.8 and the induction hypothesis. Now, let $G \in A$. There is an ordinal o such that $r(H) = {}^{o}r(H)$ for every factorgroupoid H of G and $(r:s)(G) = ({}^{o}r:s)(G) \subseteq (s:{}^{o}r)(G) = (s:r)(G)$.

4. <u>Composition of semipreradicals</u>. Let A be a non-empty abstract class of distributive idempotent groupoids. Consider two semipreradicals r and s and suppose that r satisfies (E) and s satisfies (A). We define a semipreradical r.s by $(a,b) \in (r.s)(G)$ iff $(a,b) \in s(G)$ and $(a,b) \in r(H)$, H being the block of s(G) containing a (take into account that all the blocks of s(G) are subgroupoids and all the translations of G are endomorphisms). The following observations are clear.

4.1. <u>Proposition</u>. (i) $r.s \subseteq r \cap s$.

(ii) If r satisfies (A) then r.s satisfies (A).

(iii) If s satisfies (C) (resp. (E)) then r.s satisfies (C) (resp. (E)).

(iv) If s satisfies (E) and (F) then s.s = s.

(v) If r satisfies (G) and s satisfies (B), (D) and (G) then r.s satisfies (G).

(vi) If r satisfies (H) then $r.s = r \cap s$.

(vii) If both r and s satisfy (H) then $r.s = r \cap s$ satisfies (H).

4.2. <u>Lemma</u>. Let $G,H \in A$ and let f be a homomorphism of G into H such that $\ker(f) \circ s(G) = s(G) \circ \ker(f)$. Suppose that $s(H) \cap (f(G) \times f(G)) = f(s(H))$ and $r(L) \cap (f(K) \times f(K)) \subseteq f(r(K))$ whenever L is a block of s(H) and K is a block of s(G) such that $f(K) \subseteq L$. Then $(r.s)(H) \cap (f(G) \times f(G)) \subseteq f((r.s)(G))$.

4.3. <u>Proposition</u>. Suppose that $\ker(f) \circ s(G) = s(G) \circ \ker(f)$ whenever $G,H \in A$ and f is a (projective) homomorphism of G into H.

(i) If r satisfies (K) and s satisfies (D) and (K) then r.s satisfies (K).

(ii) If A is closed under subgroupoids and both r and s satisfy (I) then r.s satisfies (I).

4.4. Proposition. Let $r \cap s$ satisfy (A) and (F). Then $r \cap s = r.s$.

4.5. Lemma. Suppose that r satisfies (A) and let q be a preradical. Then $q.(r.s) = (q.r).s$.

4.6. Lemma. Let r_i, $i \in I$, be a non-empty family of preradicals. Then $(\bigcap r_i).s = \bigcap r_i.s$.

4.7. Lemma. Let s_i, $i \in I$, be a non-empty family of semipreradicals satisfying (A) such that the semipreradical $\bigcap s_i$ satisfies (A). Then $r.(\bigcap s_i) \subseteq \bigcap r.s_i$. The equality holds, provided r satisfies (H).

5. Composition of preradicals. Let A be a non-empty abstract class of distributive idempotent groupoids. Consider a preradical r satisfying (A). For every ordinal $o \geq 0$, we define a preradical r^o by $r^0 = t$, $r^{o+1} = r.r^o$ and $r^o = \bigcap r^p$, $p < o$, if $o > 0$ is limit; here, we assume that all the blocks of r^o belong to A. We have $\ldots r^{o+1} \subseteq r^o \subseteq \ldots \subseteq r^2 \subseteq r^1 \subseteq r^0$, $r^1 = r$ and for every groupoid $G \in A$ there exists an ordinal $o = \ell(r,G)$ which is the least with $r^o(G) = r^{o+1}(G)$. Setting $\bar{r}(G) = r^o(G)$, we obtain a preradical $\bar{r}$ and we can formulate the following simple facts.

5.1. Proposition. (i) For every $o \geq 0$, the preradicals r^o satisfy (A).

(ii) $\bar{r}$ is an idempotent preradical satisfying (A).

(iii) If r satisfies (G) then $\bar{r}$ is an idempotent radical.

5.2. Lemma. Let s be a preradical such that s^k exists and satisfies (A) for each non-negative integer k. Suppose that r.s = s.r. Then $r^n.s^m = s^m.r^n$ for all non-negative integers n, m.

Proof. Similar to that of 3.5.

5.3. Lemma. Let s be a preradical such that s.r. = $s \cap r$. Then $s.r^n = s \cap r^n$ for each non-negative integer n.

Proof. Similar to the of 3.6.

5.4. Lemma. Let s be a preradical such that s^k exists and satisfies (A) for each non-negative integer k. Suppose that r.s. = s.r = $r \cap s$. Then $r^n.s^m = s^m.r^n = r^n \cap s^m$ for all non-negative integers n, m.

Proof. Use 5.2 and 5.3.

5.5. Lemma. Let s be a preradical satisfying (A) such that $s.r \subseteq r.s$. Then $s.\bar{r} \subseteq \bar{r}.s$.

Proof. Similar to that of 3.8.

6. Examples

6.1. Example. Let A be the class of groupoids. For $G \in A$, define t(G) by $(a,b) \in t(G)$ iff $a,b \in G$ and ac = bc, ca = cb for every $c \in G$. It is easy to see that t is a semipreradical satisfying (A), (B), (D), (F) and (H). By 3.1, $\hat{t}$ is an idempotent semiradical satisfying (D) and (H).

6.2. Example. Let A be the class of groupoids and let B be a non-empty abstract class of groupoids closed under subgroupoids and cartesian products. For every $G \in A$, let $m_B(G)$ denote the least congruence of G such that the corresponding factorgroupoid belongs to B. Then m_B is a radical satisfying (A) and (B). Moreover, if G, H are groupoids and f is a projective homomorphism of G onto H such that $\ker(f)\circ m_B(G) = m_B(G)\circ\ker(f)$ then $f(m_B(G)) = m_B(H)$.

6.3. Example. Let A be the class of groupoids. For every $G\in A$, let fr(G) denote the Frattini congruence of G. Then fr is a semiradical satisfying (A), (B) and (D). If G is a non-trivial finitely generated groupoid then $fr(G)\neq G\times G$.

6.4. Example. Let A be the class of regular groupoids. Then t (see 6.1) is a hereditary preradical.

6.5. Example. Let A be the class of quasigroups and B a non-empty abstract class of cancellative groupoids such that B is closed under subgroupoids and cartesian products. Consider the radical m_B from 6.2. Then m_B is a cohereditary radical.

6.6. Example. Let A be the class of quasitrivial groupoids and B that of commutative groupoids. Then m_B is an idempotent radical.

7. Examples. Let A denote the class of distributive idempotent groupoids.

7.1. Example. For every $G\in A$, define p(G) (resp. q(G)) by $(a,b)\in p(G)$ (resp. $(a,b)\in q(G)$) iff $a,b\in G$ and $ac = bc$ (resp. $ca = cb$) for each $c\in G$. Then both p and q are semipreradicals satisfying (A), (B), (D), (F) and (H) and $p\cap q = id$. By 3.1 and 3.4, both $\hat{p}$ and $\hat{q}$ are idempotent semiradicals satisfying (A), (B), (D) and (H) and $\hat{p}\cap\hat{q} = id$. Moreover, $p\circ q = q\circ p$ and $\hat{p}\circ\hat{q} = \hat{q}\circ\hat{p}$ (see [2]).

7.1.1. Proposition. Let M be a generator set of a groupoid $G\in A$ and o the least limit ordinal greater than card(M).

Then $\ell(G,p) \leq o$ and $\ell(G,q) \leq o$.

Proof. Let $(a,b) \in {}^{o+1}p(G)$. Then $(ac,bc) \in {}^{o}p(G)$ for every $c \in G$ and there is an ordinal $u < o$ such that $(ad,bd) \in {}^{u}p(G)$ for every $d \in M$. Now, denote by N the set of all $e \in G$ such that $(ae,be) \in {}^{u}p(G)$. It is clear that N is a subgroupoid of G and $M \subseteq N$. Consequently, $N = G$, $(a,b) \in {}^{u+1}p(G)$ and $(a,b) \in {}^{o}p(G)$.

7.1.2. Corollary. Let $G \in A$ be a finitely generated groupoid. Then $\ell(G,p) \leq o$ and $\ell(G,q) \leq o$, o being the first infinite ordinal.

7.1.3. Proposition. Let M be a generator set of a $\hat{p}$-torsion groupoid $G \in A$ and o the least limit ordinal greater than card(M). Then $\ell(G,p) < o$.

Proof. By 7.1.1, $\ell(G,p) \leq o$, and hence there is an ordinal $u < o$ such that $(a,b) \in {}^{u}p(G)$ for all $a,b \in M$. From this, we see that ${}^{u}p(G) = G \times G$.

7.1.4. Corollary. Let $G \in A$ be a finitely generated $\hat{p}$-torsion groupoid. Then $\ell(G,p)$ is finite.

7.1.5. Proposition. Let $G \in A$ be a medial groupoid and $M \subseteq G$ a subset such that G is generated by M as a left (right) ideal. Denote by o the least limit ordinal greater than card(M). Then $\ell(G,p) \leq o$ ($\ell(G,q) \leq o$).

Proof. Let $(a,b) \in {}^{o+1}p(G)$. Then there is an ordinal $u < o$ such that $(ab,b) \in {}^{u}p(G)$ and $(ac,bc) \in {}^{u}p(G)$ for every $c \in M$. Denote by N the set of all $d \in G$ such that $(ad,bd) \in {}^{u}p(G)$. We have $M \subseteq N$ and $a.ed = ae.ad \; {}^{u}p(G) \; ae.bd = ab.ed \; {}^{u}p(G) \; b.ed$ for all $d \in N$ and $e \in G$. Hence N is a left ideal and $N = G$. Consequently, $(a,b) \in {}^{o}p(G)$.

7.1.6. Corollary. Let $G \in A$ be a left-ideal-free medial groupoid. Then $\ell(G,p) \leq o$, o being the first infinite ordinal.

7.1.7. Lemma. $p:q = q:p = p \circ q = q \circ p = p+q$.

Proof. Suppose that $G \in A$ and $(a,b) \in (p:q)(G)$. Then $d.ac = d.bc$ for all $d,c \in G$. In particular, $da.c = dc.ac = dc.bc = db.c$ and $(a,b) \in (q:p)(G)$. We have proved that $p:q \subseteq q:p$. Similarly the converse inequality, and so $p:q = q:p$. By 2.1(iii), $p+q \subseteq p:q$. Finally, $d.ac = d.bc$, $da.c = db.c$ for all $d,c \in G$, therefore $da = d.ba$, $ba.c = bc$, $(a,ba) \in q(G)$, $(ba,b) \in p(G)$ and $(a,b) \in (q \circ p)(G) = (p \circ q)(G) = (p+q)(G)$.

7.1.8. Proposition. (i) ${}^np:{}^mq = {}^mq:{}^np = {}^np+{}^mq = {}^np \circ {}^mq = {}^mq \circ {}^np$ for all non-negative integers n, m.
(ii) $\hat{p}:q \leq q:\hat{p}$ and $\hat{q}:p \leq p:\hat{q}$.

Proof. Apply 7.1.7, 3.7 and 3.8.

7.2. Example. For every $G \in A$, define two relations $r(G)$ and $s(G)$ by $(a,b) \in r(G)$ iff $a = ab$, $b = ba$ and $(c,d) \in s(G)$ iff $c = dc$, $d = cd$. Denote by $a\ell(G)$ and $ar(G)$ the least transitive relation containing $r(G)$ and $s(G)$, resp. Then both $a\ell$ and ar are idempotent preradicals satisfying (B) (see [2]). By 3.1 and 3.2, $\widehat{a\ell}$ and $\widehat{ar}$ are idempotent radicals satisfying (B). It is easy to see that $p \leq ar$, $q \leq a\ell$, $\hat{p} \leq \widehat{ar}$ and $\hat{q} \leq \widehat{a\ell}$. Further, as proved in [2], $a\ell \cap ar = id$ and we have $\widehat{a\ell} \cap \widehat{ar} = id$ by 3.4.

7.2.1. Proposition. $a\ell \circ ar = ar \circ a\ell$.

Proof. Let $G \in A$, $a,b,c \in G$ and $(a,b) \in s(G)$, $(b,c) \in r(G)$. Then $ab = b = bc$, $ba = a$, $cb = c$, $a.ca = ba.ca = bc.a = ba = a$, $ca.a = ca.ba = cb.a = ca$, $c.ca = cb.ca = c.ba = ca$, $ca.c = ca.cb = c.ab = cb = c$, $(a,ca) \in r(G)$, $(ca,c) \in s(G)$. We have proved that $s(G) \circ r(G) \subseteq r(G) \circ s(G)$ and the rest is clear.

7.3. Example. Denote by M the class of medial groupoids. By 6.2, we have a radical m_M satisfying (A) and (B). By 5.1, $\bar{m}_M$ is an idempotent radical.

7.3.1. Proposition. Every finite groupoid from A is $\bar{m}_M$-torsionfree.

Proof. It is well known that every simple distributive groupoid is medial. Hence $m_M \subseteq fr$ (see 6.3) and the result easily follows.

7.4. Example. For every $G \in A$, define a relation $j(G)$ by $(a,b) \in j(G)$ iff the subgroupoid generated by a,b,c,d is medial for all $c,d \in G$. Denote by $md(G)$ the least congruence of G containing $j(G)$. Then md is a semipreradical satisfying (A), (B) and (D).

8. Examples. Let A designate the class of regular distributive idempotent groupoids. Then both p and q (see 7.1) are hereditary preradicals satisfying (B) and $\ell(G,p) \leq o$, $\ell(G,q) \leq o$ for every $G \in A$, o being the first infinite ordinal (see [2]). Further, both $\hat{p}$ and $\hat{q}$ are hereditary radicals, $\hat{p}:\hat{q} = \hat{q}:\hat{p}$, $p = ar$, $q = a\ell$, $\hat{p} = \widehat{ar}$ and $\hat{q} = \widehat{a\ell}$.

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