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A CHARACTERIZATION OF REALCOMPACTNESS IN TERMS OF THE TOPOLOGY OF POINTWISE CONVERGENCE ON THE FUNCTION SPACE

V. V. USPENSKII

Abstract: We prove a theorem of which the following two statements are immediate corollaries: (1) if $C_p(X)$ and $C_p(Y)$ are homeomorphic and X is realcompact, then Y is realcompact; (2) let k be a non-measurable cardinal and $f: R^k \longrightarrow R$ be such a function that its restriction to every countable subset of R^k is continuous, then f is continuous.

Key words: Realcompact spaces, function spaces.

Classification: 54A25, 54C35, 54D60

When $C_p(X)$ - the space of all realvalued continuous functions defined on X, with the topology of pointwise convergence - is realcompact? A sufficient condition was found by A.V. Arhangel'skii [1]: X is normal, and every $\aleph_0$-continuous function $f: X \longrightarrow R$ is continuous. A function $f: X \longrightarrow Y$ is called k-continuous if its restriction to every subset $A \subset X$ of power $\leq k$ is continuous. Chigogidze proved later that for a normal space X the above condition is also necessary for $C_p(X)$ to be realcompact. Finally, using a slight modification of the concept of k-continuity, Arhangel'skii gave a complete answer to the posed question [2]. Call a function $f: X \longrightarrow R$ strictly k-continuous if for every $A \subset X$ with $|A| \leq k$ there exists a continuous function $g: X \longrightarrow R$ such that $f|_A = g|_A$. Now for a Tychonoff space X the following two conditions are equivalent:

(1) $C_p(X)$ is realcompact; (2) every strictly $\aleph_0$-continuous function $f:X \longrightarrow R$ is continuous (for a normal X "strictly" can be omitted), [2]. In order to state this theorem more generally, consider the cardinal functions q, t_o, t_m ("the Hewitt number", "the functional tightness" and "the modified functional tightness", respectively) defined as follows, [2] (all spaces are Tychonoff):

$q(X) = \min\{k:$ for every $x \in \beta X \setminus X$ there exists a family γ of open subsets of βX such that $x \in \bigcap \gamma \subset \beta X \setminus X$ and $|\gamma| \leq \leq k\}$;

$t_o(X) = \min\{k:$ every k-continuous function $f:X \longrightarrow R$ is continuous$\}$;

$t_m(X) = \min\{k:$ every strictly k-continuous function $f:X \longrightarrow R$ is continuous$\}$.

Then $q(X) = \aleph_0$ iff X is realcompact. The theorem of Arhangel'skii asserts that the equality $t_m(X) = q(C_p(X))$ holds. The aim of the present paper is to prove the "dual" equality $t_m(C_p(X)) = q(X)$. The inequality " $\geq$ " is due to A.V. Arhangel'skii [2, Corollary 6], but the opposite inequality $t_m(C_p(X)) \leq q(X)$ is new.

Theorem 1. For every Tychonoff X,

$t_m(C_p(X)) = t_o(C_p(X)) = q(X) = q(C_p(C_p(X)))$.

Proof: X can be embedded as a closed subspace in $C_p(C_p(X))$, so $q(X) \leq q(C_p(C_p(X)))$. Applying the equality $t_m(X) = q(C_p(X))$ to $C_p(X)$ instead of X, we see that $q(C_p(C_o(X))) = t_m(C_p(X))$. Let $k = q(X)$. As $t_m(C_p(X)) \leq t_o(C_p(X))$, it is enough to prove that $t_o(C_p(X)) \leq k$.

Lemma. Let $\varphi: Y \longrightarrow Z$ be a continuous surjection. If $t_o(Y) \leq k$, B is a base of open sets in Y and for every $G \in B$

there exists an open subset $H \subset Z$ such that $\varphi(G) \subset H \subset \bigcup\{\overline{A}^Z : A \subset \varphi(G)$ and $|A| \leq k\}$ $(*)$, then $t_o(Z) \leq k$.

<u>Proof</u>: Let $f: Z \longrightarrow R$ be k-continuous. The mapping $f \circ \varphi : Y \longrightarrow R$ is continuous, for it is k-continuous and $t_o(Y) \leq k$. Let $z_o \in Z$ and $\varepsilon > 0$. Choose $y_o \in Y$, and $G \in B$ so that $\varphi(y_o) = z_o$, $y_o \in G$ and $f \circ \varphi(G) \subset [f(z_o) - \varepsilon , f(z_o) + \varepsilon]$. If $H \subset Z$ satisfies the condition $(*)$, then $z_o \in \varphi(G) \subset H$, and k-continuity of f implies $f(H) \subset f(\bigcup\{\overline{A} : A \subset \varphi(G)$ and $|A| \leq k\}) \subset \bigcup\{\overline{f(A)} : A \subset \varphi(G)$ and $|A| \leq k\} \subset [f(z_o) - \varepsilon , f(z_o) + \varepsilon]$, which means that f is continuous.

Instead of $C_p(X)$ we shall consider its subspace $Z = \{f \in C_p(X) : f(X) \subset (0,1)\}$, which is homeomorphic to $C_p(X)$. For $f \in Z$ denote by $\tilde{f}$ the extension of f to βX, and put $Y = \{\tilde{f} : f \in Z\} = \{g \in C_p(\beta X) : 0 \leq g \leq 1$ and $g^{-1}(0) \cup g^{-1}(1) \subset \beta X \setminus X\} \subset C_p(\beta X)$. The tightness of $C_p(\beta X)$ is countable, so $t_o(Y) \leq t(Y) \leq t(C_p(\beta X)) = \aleph_0 \leq k$. Let $\varphi : Y \longrightarrow Z$ be the restriction, $\varphi(g) = g|_X$ for $g \in Y$. By the lemma, the proof will be complete when we check the condition $(*)$. Let B be the standard base in Y, i.e. elements of B are the sets $G = \bigcap\{M(x, O_x) : x \in E\}$, where E is a finite subset of βX, $\{O_x : x \in E\}$ is a family of nonempty open subsets of the closed interval $[0,1]$ and $M(x, O_x) = \{g \in Y : g(x) \in O_x\}$. We claim that if $G \in B$ is as above, then for $H = \{f \in Z : f(x) \in O_x$ for every $x \in E \cap X\} = \varphi(\bigcap\{M(x, O_x) : x \in E \cap X\})$ the condition $(*)$ is satisfied. The inclusion $\varphi(G) \subset H$ is obvious. It remains to show that for every $f \in H$ there exists a set $A \subset G$ such that $|A| \leq k$ and $f \in \overline{\varphi(A)}$. To this end, put $E_1 = E \cap X$, $E_2 = E \cap (\beta X \setminus X)$, and choose a family γ of open subsets of βX such that $E_2 \subset \bigcap \gamma \subset \beta X \setminus X$ and $|\gamma| \leq k$. This is possible by the definition of $q(X)$. We may suppose that γ is closed under finite intersections and

$(\bigcup\gamma)\cap E_1 = \emptyset$. For each $U\in\gamma$ let $g_U: \beta X \to [0,1]$ be a function such that $g_U(\beta X\setminus U)\subset\{0\}$ and $g_U(E_2)\subset\{1\}$. Choose a function $t\in G$ which does not assume values 0 and 1, and define $h_U = \tilde{f}\cdot(1-g_U) + t\cdot g_U \in C(\beta X)$. Clearly $h_U\in Y$. Moreover, $h_U\in G$: since $h_U|_{E_1} = f|_{E_1}$ and $h_U|_{E_2} = t|_{E_2}$, we have $h_U(x)\in O_x$ for each $x\in E_1\cup E_2 = E$. Let $A = \{h_U : U\in\gamma\}$. Then $A\subset G$ and $|A|\le|\gamma|\le k$. For every finite subset $F\subset X$ there exists a set $U\in\gamma$ which has an empty intersection with F. The corresponding function h_U coincides with f on the set F. Consequently, $f\in\overline{\varphi(A)}$. The theorem is proved.

Corollary 1. X is realcompact iff $C_pC_p(X)$ is realcompact.

Corollary 2. Suppose $C_p(X)$ and $C_p(Y)$ are homeomorphic. If X is realcompact, then Y is realcompact.

The same conclusion was known to be true under the assumption that $C_p(X)$ and $C_p(Y)$ are isomorphic as topological vector spaces.

Corollary 3. If a cardinal k is nonmeasurable, then $t_o(R^k) = \aleph_o$; in other words, every $\aleph_o$-continuous function $f: R^k\to R$ is continuous.

Proof: Let $D(k)$ be a discrete space of power k. When k is nonmeasurable, $D(k)$ is realcompact. Apply the theorem to $X = D(k)$ and note that $R^k = C_p(D(k))$.

If k is a measurable cardinal, there exists a discontinuous function $f: R^k\to R$ which is n-continuous for every nonmeasurable cardinal n. To construct such a function, choose a nontrivial two-valued measure m on $D(k)$. Every $g\in C(D(k))$ coincides with a constant almost everywhere relative to m. Let $f(g)$ be this constant.

Corollary 3 solves a problem posed in [1, ch. 4, § 2]. It can be generalized as follows:

Theorem 2. Let $\{X_\alpha : \alpha \in A\}$ be a family of first countable spaces and $X = \prod\{X_\alpha : \alpha \in A\}$. If $|A|$ is nonmeasureable, then every $\aleph_0$-continuous function $f:X \longrightarrow R$ is continuous.

Proof. Let $\mathcal{D} = \mathcal{D}(2)$. Arguing as above and applying the lemma to the natural continuous bijection $C_p(\beta\mathcal{D}(k), \mathcal{D}) \longrightarrow \mathcal{D}^k$, one shows that $t_o(\mathcal{D}^k) = \aleph_0$ for every nonmeasurable cardinal k. Our theorem now follows by Theorems 1.1 and 2.4 of [3].

Theorem 2 should be compared with the Noble's result [3, Theorem 5.1]: if $\{X_\alpha : \alpha \in A\}$ is a family of first countable spaces, $X = \prod\{X_\alpha : \alpha \in A\}$ and the cardinal $|A|$ is not sequential, then every sequentially continuous function $f:X \longrightarrow R$ is continuous. A cardinal k is sequential iff there exists a sequentially continuous function $f: \mathcal{D}^k \longrightarrow R$ which is not continuous. The first sequential cardinal is regular limit [4] and does not exceed the first real-valued measurable cardinal. Under the Martin's Axiom MA a cardinal k is sequential iff it is real-valued measurable iff it is Ulam measurable [5, 6]. So if MA is added to the assumptions of Theorem 2, its conclusion can be refined by writing "sequentially continuous" instead of " $\aleph_0$-continuous.

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