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ON A QUESTION OF K. LEEB

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Abstract: We prove a theorem on common properties of embeddings of two finite lattices in finite partition lattices.

Key words: Finite lattice, embedding, partition lattice.

Classification: 06A20, 05C99

K. Leeb posed the following question:

For which bijections $f:L_1 \longrightarrow L_2$ between finite lattices L_1 and L_2 there exist embeddings $\varphi_i:L_i \longrightarrow \Pi(A)$, $i = 1,2$, of L_i in a finite partition lattice $\Pi(A)$, such that for every $x \in L_1$ the partitions $\varphi_1(x)$ and $\varphi_2(f(x))$ are isomorphic?

We can completely settle the question using the strong theorem on symmetric embeddings contained in [1].

We call a partition π of a finite lattice L a partition on non-crossing antichains iff the following conditions hold:

a. each block of π is an antichain in L,

b. for every $x,y,u,v \in L$, if $x \pi y$, $u \pi v$ and $x < u$, then $y \not> v$.

If $x,y \in L_1$, we say that a mapping $f:L_1 \longrightarrow L_2$ converts order of x, y, if $x < y$ and $f(x) > f(y)$.

The following theorem is the main result of [1].

<u>Theorem 1</u>: Let L be a finite lattice and π a partition of L on non-crossing antichains. Then there exists a finite set A and an embedding $\varphi : L \longrightarrow \Pi(A)$ such that $x \pi y$ iff the partitions $\varphi(x)$ and $\varphi(y)$ are isomorphic.

<u>Proof</u>: See [1].

<u>Theorem 2</u>: Let $f: L_1 \longrightarrow L_2$ be a bijection between finite lattices L_1, L_2. Then there exists a finite set A and embeddings $\varphi_i : L_i \longrightarrow \Pi(A)$, $i = 1,2$, such that $\varphi_1(x)$ and $\varphi_2(f(x))$ are isomorphic partitions for every $x \in L_1$ iff f does not convert order of any couple $x, y \in L_1$.

<u>Proof</u>: Suppose that f does not convert order of any couple. Then f maps the least element of L_1 to the least element of L_2 and the same holds for the greatest elements.

Let L be "the parallel join" of L_1 and L_2, i.e. the lattice obtained from L_1 and L_2 by identifying the least and the greatest elements resp., and letting all remaining pairs $x \in L_1$ and $y \in L_2$ be non-comparable.

Now let π be the partition of L with blocks $\{x, f(x)\}$ for all $x \in L_1$. Since f does not convert order of any couple, π is a partition of L on non-crossing antichains. By Theorem 1 there exists an embedding $\varphi : L \longrightarrow \Pi(A)$ with $\varphi(x)$ and $\varphi(f(x))$ isomorphic for all $x \in L_1$. The restrictions of φ on L_1 and L_2 are the desired embeddings $\varphi_1 : L_1 \longrightarrow \Pi(A)$ and $\varphi_2 : L_2 \longrightarrow \Pi(A)$.

If f converts order of a couple $x, y \in L_1$, we have $x < y$ and

$f(x) > f(y$. Now one can easily check that for any embeddings $\varphi_i : L_i \longrightarrow \Pi(A)$, $i = 1,2$, the requirements $\varphi_1(x)$ is isomorphic to $\varphi_2(f(x))$ and $\varphi_1(y)$ is isomorphic to $\varphi_2(f(y))$ are mutually exclusive.

R e f e r e n c e

[1] P. PUDLÁK: Symmetric embeddings of finite lattices into finite partition lattices, Comment. Math. Univ. Carolinae 20(1979), 183-187.

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