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CONSTRUCTIONS OF ENDOMORPHIC UNIVERSES AND SIMILARITIES

Alena VENCOVSKÁ

Abstract: In this paper we investigate properties of endomorphic universes and similarities in the alternative set theory. We describe conditions on similarities to be extendable to automorphisms. Further we show how specially located endomorphic universes A can be constructed for which there is a set d satisfying $A[d] = V$.

Key words: Alternative set theory, similarity, automorphism, endomorphic universe, fully revealed, definable.

Classification: 03E70, 03H20

We shall briefly recall some notions from alternative set theory which we frequently use.

A function F is a similarity (see sec. 1, ch. 5, [V]) iff for each set formula $\varphi(z_1,\dots,z_n)$ of the language FL and for each $x_1,\dots,x_n \in dom(F)$ we have

$$\varphi(x_1,\dots,x_n) \equiv \varphi(F(x_1),\dots,F(x_n)).$$

If F is a function and φ a formula of the language $FL_{dom(F)}$ then φ^F is the formula resulting from φ by replacing all parameters by their images in the function F.

If F and H are functions then $F \cup H$ is a similarity iff for each set formula $\varphi(z_1,\dots,z_n)$ of the language $FL_{dom(H)}$ and for each $x_1,\dots,x_n \in dom(F)$ we have

$$\varphi(x_1,\dots,x_n) \equiv \varphi^{H}(F(x_1),\dots,F(x_n)).$$

A similarity whose domain equals V is called endomorphism. A similarity whose domain and range equal V is called automorphism. Classes X, Y are similar iff there is a similarity F such that dom(F) = X and rng(F) = Y.

A class A is endomorphic universe iff it is similar to V. For a class A and a set d the class A[d] is defined as $\{f(d); f \in A \;\&\; d \in dom(f)\}$.

If A is an endomorphic universe and $d \in \bigcup A$ then A[d] is the smallest endomorphic universe subclass of which is the class $A \cup \{d\}$. X is a Sd_T-class, $Sd_T(X)$ iff there is a set formula $\varphi(z)$ of the language FL_T such that $X = \{x; \varphi(x)\}$.

Sd(X) is used instead of $Sd_V(X)$.

$\sigma_T(X)$ and $\pi_T(X)$ will denote that there are countably many Sd_T-classes such that X is their union or intersection respectively. Again we omit writing V and speak about σ- or π-classes. Fin(X) denotes that X is a finite class.

A class X is revealed, Rev(X), iff for each countable $Y \subseteq X$ there is a set u such that $Y \subseteq u \subseteq X$.

X is fully revealed iff for each normal formula $\varphi(z,Z)$ of the language FL the class $\{x; \varphi(z,Z)\}$ is revealed. Each Sd-class is fully revealed.

It can be proved that if X is fully revealed then for each normal formula $\varphi(z,Z)$ even of the language FL_V the class $\{x; \varphi(x,X)\}$ is revealed (see § 2, [S-V 1]).

Each countable descending sequence of non-empty revealed classes has non-empty intersection (see sec. 5, ch. 2, [V]).

Thus if a revealed class X is a subclass of the union of an ascending sequence $\{X_n; n \in FN\}$ of Sd-classes then $X \subseteq X_n$ for some $n \in FN$.

Def_X denotes the class of all sets definable by a set formula of FL_X.

Through the whole paper, G denotes a one-one mapping of V onto N which is a Sd_0-class. Such a mapping has been constructed in sec. 1, ch. 2, [V].

In a natural way, G induces a linear ordering on V which is referred to by saying G-smaller, G-greater. Each Sd_T-class has the G-first element and this element belongs to Def_T.

1. A class R is said to be closed on subsets if $r \in R$ and $r_1 \subseteq r$ imply that $r_1 \in R$.

Definition. Let R be a class closed on subsets. Let $\mathcal{T}$ be a codable system of pairs $\langle Q, r\rangle$ such that $\langle Q, r\rangle \in \mathcal{T}$ implies that $r \in R \cap Fin$ and Q is a non-empty class. Suppose that for $r \in R \cap Fin$ and $r_1 \subseteq r$ the inclusion $\mathcal{T}''\{r_1\} \subseteq \mathcal{T}''\{r\}$ holds. Then $\mathcal{T}$ is called a system over R.

Note that for $S \subseteq R \cap Fin$, $\mathcal{T}''S$ denotes the system of all Q such that there is $s \in S$ with $\langle Q, s\rangle \in \mathcal{T}$.

The system $\mathcal{T}''(R \cap Fin)$ is called the field of $\mathcal{T}$ and denoted $\mathcal{F}(\mathcal{T})$.

Definition. Let $\mathcal{T}$ be a system over R. A class $M \subseteq \cup R$ is satiate with $\mathcal{T}$ on R iff $P_{Fin}(M) \subseteq R$ and for each decreasing sequence $\mathcal{K} = \{Q_n; n \in FN\} \subseteq \mathcal{T}'' P_{Fin}(M)$ the intersection $\bigcap \mathcal{K} \cap M$ is non-empty.

Specially, for M satiate with $\mathcal{T}$ on R we have $M \cap Q \neq 0$ whenever there is $s \in P_{Fin}(M)$ with $\langle Q, s\rangle \in \mathcal{T}$.

The following three special systems will be useful.

Suppose X is a class and R a non-empty class closed on subsets. Define the system $\mathcal{T}_1$ over R:
$\langle Q,r\rangle \in \mathcal{T}_1$ iff $r \in R \cap Fin$ and $Q = Q_\varphi = \{x; \varphi(x)\}$ for a set formula $\varphi(z)$ of the language $FL_{X\cup r}$ such that there is a set x satisfying $\varphi(x)$.
We shall show that if M is satiate with $\mathcal{T}_1$ on R then M is an endomorphic universe and $X \subseteq M \subseteq \bigcup R$.
As $P_{Fin}(M) \subseteq R$, we have $M \subseteq \bigcup R$.
Let $x \in X$. Then $\{x\} = Q_\varphi$ for the set formula $\varphi(z) = (z=x)$ of the language FL_X and therefore $\langle \{x\},0\rangle \in \mathcal{T}_1$. It follows that $\{x\} \cap M \neq 0$ and $X \subseteq M$.
Let $\{\varphi_n(z); n \in FN\}$ be a sequence of set formulas of the language FL_M such that $(\exists x)(\forall n)\varphi_n(x)$ holds. Defining $Q_n =$ $= \{x; (\forall k \leq n)\varphi_k(x)\}$ and $\mathcal{K} = \{Q_n; n \in FN\}$, we get a descending sequence $\mathcal{K} \subseteq \mathcal{T}_1$ "$P_{Fin}(M)$ for which the intersection $\bigcap \mathcal{K} \cap M$ must be non-empty. It follows that $(\exists x \in M)(\forall n)\varphi_n(x)$ holds. By the fourth part of the first theorem in [S-V 1], M is an endomorphic universe.

Let X, R be as above and let d be a set. The system $\mathcal{T}_2$ contains all pairs belonging to $\mathcal{T}_1$ and moreover all pairs $\langle Q,r\rangle$ where $r \in R \cap Fin$ and $Q = Q_w = \{f; f(d) = w\}$ with $w \in V$. As in the previous case we can show that a class M satiate with $\mathcal{T}_2$ on R is an endomorphic universe such that $X \subseteq M \subseteq \bigcup R$. For each $w \in V$ we have $\langle Q_w, 0\rangle \in \mathcal{T}_2$ and therefore $M \cap Q_w \neq 0$, i.e. there is $f \in M$ with $f(d) = w$. Consequently $M[d] = V$.

Let R be a non-empty class closed on subsets. Define the system $\mathcal{T}_3$ over R: $\langle Q,r\rangle \in \mathcal{T}_3$ iff $r \in R \cap Fin$ and either

$Q = V \times \{w\}$ or $Q = \{w\} \times V$ where $w \in V$. Suppose that F is a similarity such that $r \in R$ implies that $F \cup r$ is a similarity. We shall show that if M is satiate with $\mathcal{T}_3$ on R then M is an automorphism and $M \supseteq F$.
For each finite $f \subseteq M$ the class $F \cup f$ is a similarity as $P_{Fin}(M) \subseteq R$. Therefore $F \cup M$ is a similarity. For each $w \in V$ the classes $\{w\} \times V$ and $V \times \{w\}$ belong to $\mathcal{T}_3" \{0\}$ and therefore have non-empty intersections with M, i.e. M is an automorphism. The facts that $M \cup F$ is a similarity and $dom(F) \subseteq dom(M)$ imply that $M \supseteq F$.

Note that the fields of $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ consist of Sd-classes i.e. of revealed classes only.

We shall investigate the conditions under which there exists a class satiate with a given system $\mathcal{T}$.

Definition. Let R be a class closed on subsets. Let $r \in R$. The class $\{z; r \cup \{z\} \in R\}$ is called the supply of r in R and denoted $Sp_R(r)$.

Obviously, if $r \in R$ and $r_1 \subseteq r$ then $Sp_R(r_1) \supseteq Sp_R(r)$. If the class R is revealed and $r \in R$ then $Sp_R(r)$ is revealed, too. In order to prove it, let us consider a sequence $\{z_n; n \in FN\} \subseteq \subseteq Sp_R(r)$. As R is revealed and $\{r \cup \{z_n\}; n \in FN\} \subseteq R$, there is a set $\{r_\alpha; \alpha \leq \alpha_0\} \subseteq R$ such that $\alpha_0 \notin FN$ and $r_n = r \cup \{z_n\}$ for each $n \in FN$. Let $\{z_\alpha: \alpha \leq \gamma_0\}$ be a set-prolongation of $\{z_n; n \in \in FN\}$. There is $\beta_0 \notin FN$, $\beta_0 \leq \alpha_0$, γ_0 such that for each $\alpha \leq \beta_0$ we have $r_\alpha = r \cup \{z_\alpha\}$. Therefore $\{z_\alpha; \alpha \leq \beta_0\} \subseteq \subseteq Sp_R(r)$ and $Sp_R(r)$ is revealed.

Definition. Let R be a class closed on subsets, $\mathcal{T}$ a system over R. $\mathcal{T}$ is said to be available iff for each $\langle Q,r\rangle \in \mathcal{T}$ the intersection $Q \cap Sp_R(r)$ is non-empty.

Now our theorem can be stated.

<u>Theorem.</u> Let R be a revealed non-empty class closed on subsets. Let $\mathcal{T}$ be an available system over R such that the field of $\mathcal{T}$ contains revealed classes only. Then there is a class M satiate with $\mathcal{T}$ on R.

<u>Proof.</u> Let us begin with an observation.
If W is a class such that $P_{Fin}(W) \subseteq R$ and if $Q \in \mathcal{T}''P_{Fin}(W)$ then for any $r \in P_{Fin}(W)$ the intersection $Q \cap Sp_R(r)$ is non-empty. For if r_0 an element of $P_{Fin}(W)$ such that $\langle Q, r_0\rangle \in \mathcal{T}$ then for any $r \in P_{Fin}(W)$ we have $r \cup r_0 \in P_{Fin}(W)$ and therefore $r \cup r_0 \in$ $\in R \cap Fin$. It follows that $\langle Q, r \cup r_0\rangle \in \mathcal{T}$ and $Q \cap Sp_R(r \cup r_0)$ is a non-empty class. As $Sp_R(r) \supseteq Sp_R(r \cup r_0)$, the intersection $Q \cap Sp_R(r)$ is non-empty, too.

Let $\{\mathcal{K}_\alpha ; \alpha \in \Omega\}$ be a sequence of countable descending sequences $\mathcal{K} = \{Q_n; n \in FN\} \subseteq \mathcal{F}(\mathcal{T})$ such that each descending countable $\mathcal{K} \subseteq \mathcal{F}(\mathcal{T})$ occurs uncountably many times in $\{\mathcal{K}_\alpha ; \alpha \in \Omega\}$. We shall construct an ascending sequence $\{M_\alpha ; \alpha \in \Omega\}$ considering successively the sequences $\mathcal{K}_\alpha$ for $\alpha \in \Omega$.

M_α^- will denote the class $\bigcup \{M_\gamma ; \gamma \in \alpha \cap \Omega\}$.
Let $\beta \in \Omega$. Suppose the ascending sequence $\{M_\alpha ; \alpha \in \beta \cap \Omega\}$ has been constructed so that for each $\alpha \in \beta \cap \Omega$ the following conditions holds:

(*) M_α is at most countable, $P_{Fin}(M_\alpha) \subseteq R$ and $M_\alpha \cap \bigcap \mathcal{K}_\alpha$ is non-empty provided that $\mathcal{K}_\alpha \subseteq \mathcal{T}''P_{Fin}(M_\alpha^-)$.

Then M_β^- is at most countable and $P_{Fin}(M_\beta^-) \subseteq R$ since

$P_{Fin}(M^-_\beta) = \bigcup\{P_{Fin}(M_\alpha); \alpha \in \beta \cap \Omega\}$.
If $\mathcal{K}_\beta \nsubseteq \mathcal{T}''P_{Fin}(M^-_\beta)$, we define $M_\beta = M^-_\beta$.
Suppose $\mathcal{K}_\beta \subseteq \mathcal{T}''P_{Fin}(M^-_\beta)$, $\mathcal{K}_\beta = \{Q_n; n \in FN\}$.
M^-_β is either countable or finite. In each case we can order its elements to a sequence $\{x_k; k \in FN\}$ or $\{x_k; k < k_0\}$ respectively. For each $k \in FN$ $(k \leq k_0)$ the set $\{x_1, \dots, x_k\} \in R$ as $P_{Fin}(M^-_\beta) \subseteq R$. Let us fix k and consider the sequence $\{Q_n \cap Sp_R(\{x_1, \dots, x_k\}); n \in FN\}$.
As R is revealed, the class $Sp_R(\{x_1, \dots, x_k\})$ is revealed. By the assumption on $\mathcal{F}(\mathcal{T})$ the classes Q_n are revealed. The above observation implies that $Q_n \cap Sp_R(\{x_1, \dots, x_k\})$ are non-empty classes. It follows that the considered sequence is a countable descending sequence of non-empty revealed classes and as such it has non-empty revealed intersection which equals

$\bigcap \mathcal{K}_\beta \cap Sp_R(\{x_1, \dots, x_k\})$.

Therefore also the sequence $\{\bigcap \mathcal{K}_\beta \cap Sp_R(\{x_1, \dots, x_k\}); k \in FN\}$ (or the corresponding finite one) is a descending sequence of non-empty revealed classes and has non-empty intersection. Choose an element x from this intersection and define $M_\beta = M^-_\beta \cup \{x\}$.

If $r \in P_{Fin}(M_\beta)$ then $r \subseteq \{x, x_1, \dots, x_k\}$ for some $k \in FN$. As $x \in Sp_R(\{x_1, \dots, x_k\})$ we have $\{x, x_1, \dots, x_k\} \in R$ and thence $r \in R$. Consequently $P_{Fin}(M_\beta) \subseteq R$. We have defined M_β satisfying $(*)$. Using the theorem on definition by transfinite recursion (cf. sec. 3, ch. 2, [V]) we can define an ascending sequence of classes M_α satisfying $(*)$ for all $\alpha \in \Omega$.
Put $M = \bigcup\{M_\alpha; \alpha \in \Omega\}$. M is satiate with $\mathcal{T}$ on R: $P_{Fin}(M) = \bigcup\{P_{Fin}(M_\alpha); \alpha \in \Omega\} \subseteq R$ and for each countable des

cending sequence $\mathcal{K} \subseteq \mathcal{T}"P_{Fin}(M)$ there is $\beta \in \Omega$ with $\mathcal{K} \subseteq \mathcal{T}"P_{Fin}(M_\beta)$ and $\alpha \in \Omega$, $\alpha > \beta$ with $\mathcal{K}_\alpha = \mathcal{K}$. By the construction, $\bigcap \mathcal{K}_\alpha \cap M$ is non-empty, i.e. $\bigcap \mathcal{K} \cap M$ is non-empty.

We introduce the following concept.

<u>Definition</u>. A class X is σ-fully revealed iff there is an ascending sequence $\{X_n; n \in FN\}$ of fully revealed classes such that $X = \bigcup \{X_n; n \in FN\}$.

It can easily be seen that a pair of classes $\langle X,Y \rangle$ (see [S-1] for the formal definition) is σ-fully revealed iff there are ascending sequences $\{X_n; n \in FN\}$ and $\{Y_n; n \in FN\}$ such that the pair $\langle X_n, Y_n \rangle$ is fully revealed for each n and $X = \bigcup \{X_n; n \in FN\}$ and $Y = \bigcup \{Y_n; n \in FN\}$.

Each Sd-class and each σ-class obviously is σ-fully revealed.

2. Now we shall apply the theorem to constructions of endomorphic universes with special properties.

Let X, Y be classes. We denote by R(X,Y) the class $\{x; Def_{X \cup x} \cap Y = 0\}$. x) The following assertions can easily be verified.

a) R(X,Y) is closed on subsets.

b) R(X,Y) is non-empty iff $Def_X \cap Y = 0$.

c) $\bigcup R(X,Y) \subseteq V-Y$

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x) In [Ve 1] there is introduced the notion of the reserve of X with respect to Y, $Rsv(X,Y) = \{z; Def_{X \cup \{z\}} \cap Y = 0\}$. Thus $Sp_{R(X,Y)}(r) = Rsv(X \cup r, Y)$ for $r \in R$.

d) If $r \in R(X,Y)$, then $Sp_{R(X,Y)}(r) \supseteq Def_{X \cup r}$.

e) If X and Y are the unions of ascending sequences $\{X_n; n \in FN\}$ and $\{Y_n; n \in FN\}$ respectively then $R(X,Y) = \bigcap \{R(X_n,Y_n); n \in FN\}$.

We shall show that

f) If the pair $\langle X,Y \rangle$ is σ-fully revealed then $R(X,Y)$ is revealed.

Let X and Y be the unions of ascending sequences $\{X_n; n \in FN\}$ and $\{Y_n; n \in FN\}$ of classes X_n and Y_n respectively such that the pairs $\langle X_n, Y_n \rangle$ are fully revealed. By the definition of $R(X_n,Y_n)$, this class is the intersection of all classes

$\{u; \neg (\exists x_1,\dots,x_k \in X_n \cup u)(\exists y \in Y_n)((\exists ! w) \psi(\vec{x},w) \ \& \ \psi(\vec{x},y))\}$

where $\psi(z_1,\dots,z_k,z)$ is a set formula of the language FL ($\vec{x}$ abbreviates $x_1,\dots,x_k$). There are countably many of such classes as FL is countable, and each of them is revealed as the pair $\langle X_n,Y_n \rangle$ is fully revealed.

Using this observation and the assertion e) we see that $R(X,Y)$ is an intersection of countably many revealed classes and therefore it is revealed , too.

The following theorem closely resembles a result from [Ve 1]. It is proved here as a simple application of the first theorem.

<u>Theorem</u>. Let $\langle X,Y \rangle$ be a σ-fully revealed pair of classes such that $Def_X \cap Y = 0$. Then there is an endomorphic universe A with $A \cap Y = 0$ and $A \supseteq X$.

<u>Proof</u>. Put $R = R(X,Y)$. R is non-empty, closed on subsets and revealed. In the preceding section there was defined the

system $\mathcal{T}_1$ over R. Let $\langle Q,r\rangle \in \mathcal{T}_1$, $Q = Q_\varphi$. By the theorem 1 in [Ve 1] there is $x_0 \in Def_{X\cup r}$ such that $\varphi(x_0)$ holds. By the assertion d) it follows that $Sp_R(r)\cap Q \neq 0$, i.e. $\mathcal{T}_1$ is available. By the first theorem, there is a class M satiate with $\mathcal{T}_1$ on R. M is an endomorphic universe and $X \subseteq M \subseteq \cup R$. By the assertion c), $\cup R \subseteq V-Y$, i.e. M = A has the desired properties.

For certain purposes (cf. [S-Ve]) we need a theorem analogous to the preceding one, claiming moreover that there exists a set d with A[d] = V.

It will be convenient to introduce the following definition.

<u>Definition</u>. Let $\psi(Z)$ be a property of classes, C and D classes. We say that C helps approximate ψ in classes σ-depending on D, $Apr(\psi,C,D)$ iff for all classes S, L and X we have

$$(\psi(X)\ \&\ \sigma_S(L)\ \&\ X\subseteq L''D) \Longrightarrow (\exists Y \in Sd_{C\cup S})\ (X\subseteq Y\subseteq L''D).$$

<u>Lemma</u>. Let D be a σ_C-class. Then $Apr(Rev,C,D)$.

<u>Proof</u>. Let X be a revealed class, L a σ_S-class for some S and $X\subseteq L''D$. There are ascending sequences $\{L_n; n\in FN\}$ and $\{D_n; n\in FN\}$ of Sd_S-classes and Sd_C-classes respectively such that $L = \cup\{L_n; n\in FN\}$ and $D = \cup\{D_n; n\in FN\}$. The class $L''D$ is the union of the ascending sequence of $Sd_{S\cup C}$-classes $L_n''D_n$. As X is revealed and $X\subseteq L''D$, there is $n\in FN$ such that $X\subseteq L_n''D_n$, i.e. $Y = L_n''D_n$ is a $Sd_{S\cup C}$-class satisfying $X\subseteq Y\subseteq L''D$.

Consequently, if D is a σ_C-class then $Apr(Sd,C,D)$ and $Apr(\pi,C,D)$ hold as $\psi_1(Z) \Longrightarrow \psi_2(Z)$ implies $Apr(\psi_1,C,D) \Longrightarrow$ $\Longrightarrow Apr(\psi_2,C,D)$.

Let X^{FN} denote the class $\bigcup\{X^k; k\in FN\}$, i.e. the class of all ordered k-tuples of elements of X, $k\in FN$.

<u>Theorem</u>. Let $\langle X,Y\rangle$ be a σ-fully revealed pair of classes and d a set such that $Def_X\cap(Y\cup\{d\}) = 0$ and $d\in\bigcup Def_X$. Suppose $Apr(Sd, X\cup\{d\}, X^{FN}\times Y)$ holds.
Then there is an endomorphic universe A such that $A\supseteq X$, $A\cap Y =$ $= 0$, $d\notin A$ and $A[d] = V$.

<u>Proof</u>. Put $R = R(X, Y\cup\{d\})$. R is non-empty, closed on subsets and revealed as the pair $\langle X,Y\rangle\cup\{d\}\rangle$ is σ-fully revealed. We shall show that the system $\mathcal{T}_2$ over R is available. Let $\langle Q,r\rangle\in\mathcal{T}_2$. For $Q = Q_\varphi$ we see exactly as in the previous theorem that $Sp_R(r)\cap Q\neq 0$. Suppose that $Q = Q_w$ and $Sp_R(r)\cap Q_w =$ $= 0$, i.e. $Q_w\subseteq V - Sp_R(r)$.
For each set formula $\psi(z_1,\dots,z_{k+2})$ of the language FL_r define

$$C_\psi = \{\langle x,\langle x_1,\dots,x_k\rangle,y\rangle; (\exists! z)\psi(x,\vec{x},z) \ \&\ \psi(x,\vec{x},y)\}$$

($\vec{x}$ abbreviates $x_1,\dots,x_k$). Recall that $\langle x,y,z\rangle = \langle x,\langle y,z\rangle\rangle$.
Define C as the union of all C_ψ.
Each C_ψ is a Sd_r-class. As r is finite, FL_r is countable. It follows that C is a σ_r-class.
The class $V - Sp_R(r)$ is the union of all classes C_ψ"$(X^k\times Y)$, and therefore $V - Sp_R(r) = C$"$(X^{FN}\times Y)$. We have

$$Sd(Q_w)\ \&\ \sigma_r(C)\ \&\ Q_w\subseteq C"(X^{FN}\times Y).$$

By the property $Apr(Sd, X\cup\{d\}, X^{FN}\times Y)$ there is a class W such that $W\in Sd_{X\cup\{d\}\cup r}$ and $Q_w\subseteq W\subseteq V - Sp_R(r)$.
Let w_0 be the G-first v satisfying $Q_v\subseteq W$. Then $w_0\in Def_{X\cup\{d\}\cup r}$.
Let $\varsigma(z_1,z_2)$ be the set formula of the language $FL_{X\cup r}$ such that $(\exists! z)\varsigma(d,z)\ \&\ \varsigma(d,w_0)$ holds.
Let u be an element of Def_X such that $d\in u$ and let f be the

function with $dom(f) = u$ which assigns to each $x \in u$ the G-first v satisfying $\varphi(x,v)$ if such a set v exists and 0 otherwise. Then $f \in Def_{X \cup r}$ and therefore $f \in Sp_R(r)$ as $Sp_R(r) \supseteq Def_{X \cup r}$. On the other hand $f(d) = w_0$, i.e. $f \in Q_{w_0}$. This is a contradiction because $Q_{w_0} \subseteq W \subseteq V - Sp_R(r)$. It follows that the system $\mathcal{T}_2$ is available.

The first theorem guarantees the existence of a class M satiate with $\mathcal{T}_2$ on R. $M = A$ has all desired properties.

<u>Corollary</u>. Let X, Y be $\sigma_{X \cup \{d\}}$-classes, $Def_X \cap Y = 0$ and let d be an element of $(\bigcup Def_X) - Def_X$.
Then there is an endomorphic universe A, $A \supseteq X$, $A \cap Y = 0$, $d \notin A$ and $A[d] = V$.

<u>Proof</u>. It suffices to show that $Apr(Sd, X \cup \{d\}, X^{FN} \times Y)$ holds. It follows by the lemma stated above as $X^{FN} \times Y$ is a $\sigma_{X \cup \{d\}}$-class whenever X and Y are such.

3. Now we shall investigate how similarities can be prolonged.

Let us begin with set similarities. For a set similarity g there is naturally and uniquely determined function Ug such that the fact that Ug is a similarity is a necessary condition for g to be extendable to an automorphism. We shall show that this is also sufficient.

<u>Definition</u>. Let d be a set. For $n \in FN$ we define by recursion

$$\overline{P}(0,d) = d \qquad \overline{P}(n+1,d) = P(\overline{P}(n,d)) \cup \overline{P}(n,d)$$

If $P(d) \supseteq d$ then $\overline{P}(n,d) = P^n(d)$ where the symbol P^n denotes n-times iterated operation of power set.

For $m \leq n$ we have $\overline{P}(m,d) \subseteq \overline{P}(n,d)$.

If $x \in \overline{P}(n+1,d)$ then either $x \subseteq \overline{P}(n,d)$ or $x \in d$. Therefore the following definition is correct.

<u>Definition</u>. Let g be a set function, dom(g) = d. We define by recursion for $x \in \overline{P}(n,d)$, $n \in FN$

$Ug(x) = g(x)$ for $x \in d$,

$Ug(x) = Ug"x$ for $x \in \overline{P}(n+1,d) - d$.

Thus U_g is a σ-class, $dom(Ug) = \cup\{\overline{P}(n,d); n \in FN\}$ and $Ug \upharpoonright \overline{P}(n,d)$ is a set for each n. If $x \subseteq dom(Ug)$ then there is $n \in FN$ such that $x \subseteq \overline{P}(n,d)$ and therefore $x \in dom(Ug)$.

Similarly if $x_1,\dots,x_k \in dom(Ug)$ then $\langle x_1,\dots,x_k \rangle \in dom(Ug)$.

Suppose there is an autormorphism $H \supseteq g$. Then $Ug = H \upharpoonright dom(Ug)$ as for each set x we have $H(x) = H"x$ (cf. ch. 5, sec. 1, [V]). It follows that Ug is a similarity.

<u>Lemma</u>. Let Ug be a similarity. Then for each $x \subseteq dom(Ug)$ $Ug(x) = Ug"x$.

<u>Proof</u>. It holds by the definition for each $x \in dom(Ug) - d$, i.e. especially for each $\overline{P}(n,d)$, as $\overline{P}(n,d) \notin d$. Let $x \subseteq dom(Ug)$, $x \in d$. Let $n \in FN$ be such that $x \subseteq \overline{P}(n,d)$. As Ug is a similarity, we have

$Ug(x) \subseteq Ug(\overline{P}(n,d)) = Ug"\overline{P}(n,d)$ & $z \in x \equiv Ug(z) \in Ug(x)$

These two facts imply that Ug(x) does equal to Ug"x.

It follows that if Ug is a similarity then rng(Ug) is the class $\cup\{\overline{P}(n,rng(g)); n \in FN\}$ and $U(g^{-1}) = (Ug)^{-1}$.

<u>Lemma</u>. Let f, g be set functions such that f is finite

and $Ug \cup f$ is a similarity. Let $y \in V$. Then there are y' and y'' such that $Ug \cup f \cup \{\langle y', y\rangle\}$ and $Ug \cup f \cup \{\langle y, y''\rangle\}$ are similarities.

<u>Proof</u>. Let $\{\varphi_k(z_1,\dots,z_{m_k}); k \in FN\}$ be a sequence of all set formulas of the language $FL_{dom(f)}$. Let us define
$a_{k,n} = \{\langle 0, x_2,\dots,x_{m_k}\rangle ;\ x_2,\dots,x_{m_k} \in \overline{P}(n,d) \ \&\ \varphi_k(y, x_2,\dots,x_{m_k})\}$.
Then $a_{k,n} \subseteq dom(Ug)$ and therefore $a_{k,n} \in dom(Ug)$.
For each $n_0 \in FN$ the following holds:

$$(\exists x)(\forall k,n \leq n_0)(\forall x_2,\dots,x_{m_k} \in \overline{P}(n,d)_n)$$
$$(\varphi_k(x, x_2,\dots,x_{m_k}) \equiv \langle 0, x_2,\dots,x_{m_k}\rangle \in a_{k,n})$$

Namely, $x=y$ satisfies the above formula. As $Ug \cup f$ is a similarity, we have

$$(*) \quad \begin{array}{l}(\exists x)(\forall k,n \leq n_0)(\forall x_2,\dots,x_{m_k} \in Ug(\overline{P}(n,d))) \\ (\varphi_k^f(x, x_2,\dots,x_{m_k}) \equiv \langle 0, x_2,\dots,x_{m_k}\rangle \in Ug(a_{k,n}))\ .\end{array}$$

Considering the facts that for $n,k \in FN$

$$Ug(\overline{P}(n,d)) = Ug''\overline{P}(n,d),\ Ug(a_{k,n}) = Ug''a_{k,n},\ Ug(0) = 0$$

and that for $x_1,\dots,x_m \in dom(Ug)$

$$Ug(\langle x_1,\dots,x_m\rangle) = \langle Ug(x_1),\dots,Ug(x_m)\rangle$$

we can see that $(*)$ is equivalent to

$$(\exists x)(\forall k,n \leq n_0)(\forall x_2,\dots,x_{m_k} \in \overline{P}(n,d))$$
$$(\varphi_k^f(x, Ug(x_2),\dots,Ug(x_{m_k})) \equiv \langle 0, x_2,\dots,x_{m_k}\rangle \in a_{k,n})$$

By the axiom of prolongation there is y' satisfying

$$(\forall k,n \in FN)\ (\forall x_2,\dots,x_{m_k} \in \overline{P}(n,d))$$
$$(\varphi_k^f(y', Ug(x_2),\dots,Ug(x_{m_k})) \equiv \langle 0, x_2,\dots,x_{m_k}\rangle \in a_{k,n}).$$

We shall show that $Ug \cup f \cup \{\langle y', y\rangle\}$ is a similarity.

Denote $F = Ug \cup \{\langle y', y\rangle\}$. Let $\psi(z_1,\ldots,z_m)$ be a set formula of the language $FL_{dom(f)}$. We must verify that for any $x_1,\ldots,x_m \in dom(F)$

$$\psi(x_1,\ldots,x_m) \equiv \psi^f(F(x_1),\ldots,F(x_m))$$

holds. If there is not y among $x_1,\ldots,x_m$ then it is true because $Ug \cup f$ is a similarity. Otherwise we can suppose that $y=x_1$ and $x_2,\ldots,x_m \in dom(Ug)$. There is $k \in FN$ such that $\psi = \varphi_k$. Let $n \in FN$ be such that $x_2,\ldots,x_m \in \overline{P}(n,d)$. Then

$$\psi(y,x_2,\ldots,x_m) \equiv \varphi_k(y,x_2,\ldots,x_m) \equiv \langle 0,x_2,\ldots,x_m\rangle \in a_{k,n} \equiv$$
$$\equiv \varphi_k^f(y',Ug(x_2),\ldots,Ug(x_m)) \equiv \psi^f(y',Ug(x_2),\ldots,Ug(x_m))$$

which we have claimed.

As $Ug \cup f$ is a similarity, also its inverse, $(Ug)^{-1} \cup f^{-1} = U(g^{-1}) \cup f^{-1}$ is a similarity. By the above method y'' can be found such that $U(g^{-1}) \cup f^{-1} \cup \{\langle y'', y\rangle\}$ is a similarity. Therefore its inverse, $Ug \cup f \cup \{\langle y, y''\rangle\}$ is a similarity.

Let F be a function. Define $R(F) = \{f; F \cup f$ is a similarity$\}$. Obviously $R(F)$ is closed on subsets and it is non-empty iff F is a similarity.

Suppose F is a σ-fully revealed class, i.e. the union of an ascending sequence of fully revealed classes $\{F_n; n \in FN\}$. Then $R(F)$ is revealed because it is the intersection of all classes

$$\{f; (\forall x_1,\ldots,x_k \in dom(F_n \cup f))(\psi(x_1,\ldots,x_k) \equiv$$
$$\equiv \psi((F_n \cup f)(x_1),\ldots,(F_n \cup f)(x_k)))\}$$

where $n \in FN$ and $\psi(z_1,\ldots,z_k)$ is a set formula of the language FL.

<u>Theorem</u>. A set similarity g can be prolonged to an automorphism iff the function Ug is a similarity.

Proof. One part of the theorem has been already mentioned. Suppose that Ug is a similarity. Put $R = R(Ug)$. R is non-empty, closed on subsets and revealed as Ug is a σ-class. The system $\mathcal{T}_3$ defined in the first section is available over R by the previous lemma. The first theorem guarantees the existence of a class M satiate with $\mathcal{T}_3$ on R. M is the desired automorphism, $M \supseteq Ug \supseteq g$.

Let us make one simple observation about similarities.

Definition. Let F be a similarity. We denote by DF the class of all pairs $\langle x', x\rangle$ such that there is a set formula $\varphi(z)$ of the language $FL_{dom(F)}$ for which $(\exists ! z)\varphi(z)$ & $\varphi(x)$ & $\varphi^F(x')$ holds.

Theorem. Let F be a similarity. Then $dom(DF) = Def_{dom(F)}$ and DF is the unique similarity extending F to a similarity with the domain equal to $Def_{dom(F)}$.

The proof is easy. Note that if $dom(F) = rng(F)$ then $dom(DF) = rng(DF)$ and analogously for inclusions.

To prove our next theorem concerning prolongations of similarities to automorphisms we need classes defined as follows (recall that $FN^{(-)}$ denotes the class of all finite integers).

Let F be a function. $R^{\omega}(F)$ is the class of all functions f for which there exists a sequence $\{f_j; j \in FN^{(-)}\}$ such that

1) $f_0 = f$,

2) $dom(f_{j+1}) = rng(f_j)$ for all $j \in FN^{(-)}$,

3) $F \cup \bigcup\{f_j; j \in FN^{(-)}\}$ is a similarity.

Obviously, $R^{\omega}(F)$ is closed on subsets and is non-empty iff F is a similarity (then $\{\langle 0,0\rangle\} \in R^{\omega}(F)$).

Suppose F is a σ-fully revealed class, i.e. the union of an ascending sequence of fully revealed classes $\{F_n; n \in FN\}$. We shall show that then $R^{\omega}(F)$ is a revealed class.

Let $\{\varphi_k(z_1,\dots,z_{m_k}); k \in FN\}$ be a sequence of all set formulas of the language FL.

Define $C_{\alpha,n}$ as the class of all functions f for which there is a set sequence $\{f_\iota; -\alpha \leq \iota \leq \alpha\}$ such that

1) $f_0 = f$,

2) $\operatorname{dom} f_{\iota+1} = \operatorname{rng}(f_\iota)$ for all ι with $-\alpha \leq \iota < \alpha$,

3) setting $E = F_n \cup \bigcup \{f_\iota; -\alpha \leq \iota \leq \alpha\}$

the following holds:

$$(\forall k \leq n)(\forall x_1,\dots,x_{m_k} \in \operatorname{dom}(E))(\varphi_k(x_1,\dots,x_{m_k}) \equiv$$
$$\equiv \varphi_k(E(x_1),\dots,E(x_{m_k}))).$$

We claim that $R^{\omega}(F) = \bigcap \{C_{n,n}; n \in FN\}$.

Obviously $R^{\omega}(F) \subseteq \bigcap \{C_{n,n}; n \in FN\}$.

Let $f \in \bigcap \{C_{n,n}; n \in FN\}$. Let D_n be the class of all $\{f_\iota; -\alpha \leq \iota \leq \alpha\}$ satisfying the three conditions from the definition of $C_{\alpha,n}$ and such that $\alpha \geq n$.

The classes D_n are non-empty as $f \in C_{n,n}$ for each $n \in FN$, revealed because they are definable by a normal formula with the only class parameter F_n, and they form a descending sequence. Therefore their intersection is non-empty.

Let $\{f_\alpha; -\alpha \leq \iota \leq \alpha\}$ be an element of this intersection. Then $\alpha \notin FN$ and $F \cup \bigcup \{f_j; j \in FN^{(-)}\}$ is a similarity. Thence $f \in R^{\omega}(F)$ which proves the claim.

The classes $C_{n,n}$ are definable by a normal formula of the language FL with the only class parameter F_n and as such they are revealed. It follows that $R^{\omega}(F)$ is revealed.

Theorem. Let F be a similarity and a σ-fully revealed class. Suppose that $dom(F) = rng(F)$ and $Apr(\pi, dom(F), P_{Fin}(F))$ hold. Then there is an automorphism $\tilde{F}$, $\tilde{F} \supseteq F$.

Proof. Put $R = R^{\omega}(F)$. R is a non-empty revealed class closed on subsets. We shall show that the system $\mathcal{T}_3$ defined in the first section is available over R. Then by the first theorem there is a class M satiate with $\mathcal{T}_3$ on R which implies that M is an automorphism and $M \supseteq F$, i.e. $M = \tilde{F}$ has the desired properties.

Suppose on the contrary that there is a finite $f \in R$ and a set $w \in V$ such that - let us say - $V \times \{w\} \subseteq V - Sp_R(f)$. Let $\{\varphi_k(z_1, \ldots, z_{m_k}); k \in FN\}$ be a sequence of all set formulas of the language FL. Let $\{f_j; j \in FN^{(-)}\}$ be a sequence satisfying the three conditions from the definition of $R^{\omega}(F)$.

Denote $H = \bigcup \{f_j; j \in FN^{(-)}\}$.

Let S be the class of all set sequences $s = \{s_\iota; -\alpha \leq \iota \leq \alpha\}$.

Call α the length of s. Set $\tilde{s} = \{\langle s_{\iota+1}, s_\iota \rangle; -\alpha \leq \iota < \alpha\}$.

Define C_n as the class of all pairs $\langle s, g \rangle$ such that $s \in S$ and ting $H_n = g \cup \tilde{s} \cup \bigcup \{f_j; -n \leq j \leq n\}$ the following holds:

$$\neg(\forall k \leq n)(\forall x_1, \ldots, x_{m_k} \in dom(H_n))$$

$$(\varphi_k(x_1, \ldots, x_{m_k}) \equiv \varphi_k(H_n(x_1), \ldots, H_n(x_{m_k}))).$$

Each class C_n is definable by a set formula of the language $FL_{\{f_j; -n \leq j \leq n\}}$. As the f_j are finite functions and $f_j \subseteq dom(H) \times dom(H)$, we have $\{f_j; -n \leq j \leq n\} \in Def_{dom(H)}$ and therefore C_n is a $Sd_{dom(h)}$-class for each n.

Let $C = \bigcup \{C_n; n \in FN\}$. C is a $\sigma_{dom(H)}$-class and the class $C''P_{Fin}(F)$ consists of all $s \in S$ such that $F \cup \tilde{s} \cup H$ is not a similarity.

For $v \in V$ let $S_n(v)$ be the class of those sequences s from S for which $s_0 = v$ and whose length is greater or equal to n and $S(v) = \bigcap \{ S_n(v); n \in FN \}$.

We claim that S(w) is a subclass of $C"P_{Fin}(F)$. Suppose on the contrary that there is $s \in S(w) - C"P_{Fin}(F)$. Then $\hat{f} = f \cup \{\langle s_1, s_0 \rangle\}$ is an element of $R^{\omega}(F)$ as can be seen by considering the sequence of functions $\hat{f}_n = f_n \cup \{\langle s_{n+1}, s_n \rangle\}$ for $n \in FN^{(-)}$; $\hat{f}_0 = \hat{f}$, $dom(\hat{f}_{n+1}) = rng(\hat{f}_n)$ for all $n \in FN^{(-)}$ and $F \cup \bigcup \{ \hat{f}_n; n \in FN^{(-)}\}$ is a subclass of $F \cup H \cup \tilde{s}$ and therefore a similarity. It means that $\langle s_1, s_0 \rangle \in Sp_R(f)$. But $s_0 = w$ and we have assumed that $V \times \{w\} \subseteq V - Sp_R(f)$. Our claim is justified. Thus we have

$$\pi(S(w)) \ \& \ \sigma_{dom(H)}(C) \ \& \ S(w) \subseteq C"P_{Fin}(F) .$$

By our assumption, $Apr(\pi, dom(F), P_{Fin}(F))$ holds. Therefore there is a $Sd_{dom(H) \cup dom(F)}$-class Y (i.e. $Sd_{dom(F \cup H)}$-class) such that $S(w) \subseteq Y \subseteq C"P_{Fin}(F)$. The class S(w) is the intersection of the descending sequence of Sd-classes $S_n(w)$; therefore the descending sequence $\{S_n(w) - Y; n \in FN\}$ of Sd-classes has empty intersection. Consequently there is $n \in FN$ such that $S_n(w) - Y$ is empty. i.e. $S_n(w) \subseteq Y$.

Let w_0 be the G-first set v such that $S_n(v) \subseteq Y$. Then w_0 is an element of $Def_{dom(F \cup H)}$. The function $F \cup H$ is a similarity and $dom(F \cup H) = rng(F \cup H)$; therefore also $D(F \cup H)$ is a similarity and $dom(D(F \cup H)) = rng(D(F \cup H)) = Def_{dom(F \cup H)}$. Consider the sequence $s = \{(D(F \cup H))^j(w_0); -n \leq j \leq n\}$. Obviously s belongs to $S_n(w_0)$ but not to $C"P_{Fin}(F)$ as $F \cup \tilde{s} \cup H$ is a subclass of $D(F \cup H)$ and therefore a similarity. This is a contradiction as $S_n(w_0) \subseteq \subseteq Y \subseteq C"P_{Fin}(F)$.

The theorem is proved.

Corollary. Let F be a $\sigma_{dom(F)}$-class and a similarity, dom(F) = rng(F). Then F can be prolonged to an automorphism.

Proof. It suffices to show that Apr(π,dom(F),P_{Fin}(F)) holds. Obviously P_{Fin}(F) is a $\sigma_{dom(F)}$-class as F is such and a previous lemma guarantees what is needed.

For example, F can be a similarity of the form $Id \upharpoonright u \cup \cup \{\langle u,u \rangle\} \cup H$, where u is a set and H is countable class satisfying dom(H) = rng(H).

If we replace in the above theorem the assumption dom(F) = rng(F) by dom(F) $\supseteq$ rng(F), we can get an endomorphism extending F. Without the assumption Apr(π,dom(F),P_{Fin}(F)) we can extend F to a similarity $\tilde{F}$ with dom($\tilde{F}$) = rng($\tilde{F}$) = A, where A is an endomorphic universe.

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