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CONSTANT AND VARIABLE DROP THEOREMS ON METRIZABLE LOCALLY CONVEX SPACES

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Abstract: A maximality principle on quasi-ordered quasi-metrizable uniform spaces, appearing as a common extension of both "uniform" Bróndsted's and "abstract" Brézis-Browder's ones is used to obtain a number of constant as well as variable drop theorems on metrizable locally convex spaces.

Key words: Quasi-ordered quasi-metrizable uniform space, maximal element, closed mapping, constant drop, support theorem, variable drop, mapping theorem.

Classification: Primary 54E35, 54C10, 46A05, 52A07
Secondary 54C08, 54H25, 47H17

Let X be a finite or infinite dimensional Banach space. For any y in X and $r > o$, let $S(y,r)$ denote the closed sphere with center y and radius r. Given $x,y \in X$ and $r > o$ (respectively, given $x \in X$ and $o \leq q < 1$) let $K(x;y,r)$ $(V(x,q))$ indicate the subset of all combinations $\lambda x+(1-\lambda)z$, $o \leq \lambda \leq 1$, $z \in S(y,r)(S(o,q\|x\|))$ and call them the constant (variable) drop generated by x, y and r (x and q). The following results established by Daneš [12] (cf. also Bróndsted [5]) and, respectively, by Turinici [28] must be mentioned as a start point of our developments.

Theorem 1. Let Y be a closed subset of X and let $y \in X$,

$r > o$ be such that Y is disjoint from $S(y,r)$. Then, to every $x \in Y$ there corresponds a $z \in bd(Y) \cap K(x;y,r)$ (here, bd indicates the boundary) with the property $K(z;y,r) = \{z\}$.

Theorem 2. Let X_1 be a closed subset of X and suppose $q \in \in [o,1)$ is such that, for any $x \neq o$ in X_1 the subset $X_1 \cap V(x,q)$ contains more than one point. Then, we necessarily have $o \in X_1$.

As already pointed out by Brézis and Browder [4] (see also Ursescu [29]), the first result - appearing as a non-convex extension of the famous Bishop-Phelps´ support theorem [3] - represents a very appropriate instrument of the normal solvability theory as developed by Pohozhayev [23], Browder [8], as well as by Zabreiko and Krasnoselskii [31]. On the other hand, as indicated in the above quoted author´s paper, the second result may be viewed as an abstract variant of a very interesting mapping theorem established by Altman [1] and having some useful applications to nonlinear programming [2]. Taking into account these facts, a metrizable locally convex generalization of these contributions may therefore be of interest. It is precisely our main aim to state and prove such a couple of extended variants of the above results, the basic tool of our investigations being a maximality principle on quasi-ordered quasi-metrizable uniform spaces appearing as a common extension of both "uniform" Bróndsted´s and "abstract" Brézis-Browder´s ones. As applications, a metrizable locally convex version of the above quoted Bishop-Phelps´ support theorem and, respectively, Altman´s mapping theorems will be given.

Let X be a nonempty set and let $D = (d_i; i \in N)$ be a denumerable family of quasi-metrics on X. It is well known that, by the construction

$$d = \sum_{i \in N} (1/2^i) d_i/(1 + d_i)$$

the structure (X,d) appears as a quasi-metric space (respectively, a metric space in case D is a sufficient family ($d_i(x,y) = o$, all $i \in N$ imply $x = y$)); for this reason, (X,D) will be generally termed a quasi-metrizable (respectively, metrizable) uniform space. We shall say the sequence $(x_n; n \in N)$ is a D-Cauchy one provided that it is d_i-Cauchy for any $i \in N$, and D-convergent to x when $d_i(x_n,x) \longrightarrow o$ as $n \longrightarrow \infty$ for all $i \in N$ (in which case we write $x_n \xrightarrow{D} x$). Also, $\leq$ being a quasi-ordering (that is, a reflexive and transitive relation) on X, let us say the sequence $(x_n; n \in N)$ is monotone if $x_i \leq x_j$ whenever $i \leq j$, and bounded from above provided that $x_n \leq y$, all $n \in N$, for some y in X called in this context an upper bound of the considered sequence. Finally, the element z of X will be said to be D-maximal when $z \leq y$ implies $d_i(z,y) = o$, all $i \in N$.

The following maximality principle will play a central role in the sequel.

Theorem 3. Let the quasi-ordered quasi-metrizable uniform space $(X,D,\leq)$ be such that

(i) any monotone sequence in X is both D-Cauchy and bounded from above.

Then, to every x in X there corresponds a D-maximal element z in X with $x \leq z$.

Proof. Of course, without any restriction we may suppose D is an increasing family ($d_i \leq d_j$ whenever $i \leq j$). We claim the following property holds at every x in X

(1) for any $i \in N$ and $\varepsilon > o$ there exists $y = y(i, \varepsilon) \geq x$ such that $d_i(y,z) < \varepsilon$, all $z \geq y$.

Indeed, assume by contradiction (1) were not valid. Then, there must be a couple $i \in N$, $\varepsilon > o$ such that, for any $y \geq x$, a $z \geq y$ may be found with $d_i(y,z) \geq \varepsilon$. It immediately follows a monotone sequence $(y_n; n \in N)$ in X may be chosen with $d_i(y_n, y_{n+1}) \geq \varepsilon$, all $n \in N$, contradicting the first part of (i) and proving our claim. In such a case, given x in X, it is not hard to construct a monotone sequence $(x_n; n \in N)$ in X with $x \leq x_n$, all $n \in N$, and

(2) $n \in N$, $y \geq x_n$ imply $d_n(x_n,y) < 1/2^n$.

By the second part of (i), $x_n \leq z$, all $n \in N$ (so, by (2), $x_n \xrightarrow{D} z$) for some z in X. Clearly, $x \leq z$; moreover, again by (2), $z \leq y$ implies $x_n \xrightarrow{D} y$ that is, $d_i(z,y) = o$, all $i \in N$, and the proof is complete. Q.E.D.

A partial indication about the power of this maximality principle follows from the considerations below. Let $(X, \leq)$ be a quasi-ordered set, $(X, e, \leq)$ a quasi-ordered metric space and $(\varphi_i; i \in N)$ a denumerable family of mappings from X into Y. As a first application of Theorem 3, the following "combined" maximality principle may be formulated.

<u>Theorem 4</u>. Suppose that, for any $i \in N$

(ii) φ_i is increasing

(iii) every monotone sequence in $\varphi_i(X)$ is e-Cauchy

Then, the following conclusions are - respectively - valid.

A). Under the assumption: there is a uniformity $\mathcal{U}$ on X with

(iv) any monotone $\mathcal{U}$-Cauchy sequence $(x_n; n \in N)$ in X converges to some x in X with $x_n \leq x$, all $n \in N$

(v) for every U in $\mathcal{U}$ there exists $i \in N$ and $\varepsilon > o$ such that $x \leq y$ and $e(\varphi_i(x), \varphi_i(y)) < \varepsilon$ imply $(x,y) \in U$

given any x in X there exists z in X with $x \leq z$ and, in addition, $z \leq y$ implies $(z,y) \in U$, all U in $\mathcal{U}$.

B). Under the supplementary hypothesis

(vi) any monotone sequence in X has an upper bound

to every x in X there corresponds an element z in X with $x \leq z$ and, in addition, $z \leq y$ implies $\varphi_i(z) = \varphi_i(y)$, all $i \in N$.

Proof. Let us define a family of quasi-metrics $D = (d_i; i \in N)$ on X by

$$d_i(x,y) = e(\varphi_i(x), \varphi_i(y)), \text{ all } x,y \in X, i \in N$$

and let $(x_n; n \in N)$ be a monotone sequence in X. By (ii) + (iii), the first part of (i) will be established. It remains only to prove (iv) + (v) lead us to the second part of (i) (because, by (vi), this assertion is trivial). To this end, let U in $\mathcal{U}$ be arbitrary fixed and let $i \in N$, $\varepsilon > o$ be introduced by (v). From the above conclusion about our sequence, there exists $n = n(i, \varepsilon) \in N$ such that $d_i(x_p, x_q) < \varepsilon$, all $p,q \in N$, $n \leq p \leq q$ so (again invoking (v)) $(x_p, x_q) \in U$, all $p,q \in N$, $n \leq p \leq q$, proving $(x_n; n \in N)$ is a monotone $\mathcal{U}$-Cauchy sequence and completing, by (iv), our argument. Consequently, in either case Theorem 3 applies. Q.E.D.

Let (X,D) be a quasi-metrizable uniform space. A function $\varphi: X \longrightarrow R$ will be said to be D-lsc (usc) provided that, for any sequence $(x_n; n \in N)$ in X and any couple $x \in X$, $t \in R$, relations $x_n \xrightarrow{D} x$ and $\varphi(x_n) \leq t (\geq t)$, all $n \in N$, imply $\varphi(x) \leq t (\geq t)$.

Also, (X',D') being another quasi-metrizable uniform space, we shall say the mapping $T:X \longrightarrow X'$ is closed when $x_n \xrightarrow{\mathcal{D}} x$ and $Tx_n \xrightarrow{\mathcal{D}'} x'$ imply $Tx = x'$. Suppose in what follows (X,D) and (X',D') are complete quasi-metrizable uniform spaces and $T: :X \longrightarrow X'$ is a closed mapping from X into X'. Let us introduce a new denumerable family of quasi-metrics $E = (e_i; i \in N)$ on X by the convention

$$e_i(x,y) = \max\,(d_i(x,y), d_i'(Tx,Ty)),\ x,y \in X,\ i \in N$$

In this case, as a second application of Theorem 3, the following "operator" maximality principle may be formulated

<u>Theorem 5</u>. Let the denumerable families $(\varphi_i; i \in N)$ and $(\psi_i; i \in N)$ of functions from X into R be such that

(vii) φ_i and ψ_i are E-lsc and bounded from below, for all $i \in N$.

Then, to every x in X there corresponds an element z in X such that (a) $d_i(x,z) \leq \varphi_i(x) - \varphi_i(z)$, $d_i'(Tx,Tz) \leq \psi_i(x) - \psi_i(z)$, $i \in N$, (b) for any y in X with $d_i(z,y) \leq \varphi_i(z) - \varphi_i(y)$, $d_i'(Tz,Ty) \leq \psi_i(z) - \psi_i(y)$, $i \in N$, we necessarily have $d_i(z,y) = = o$, $d_i'(Tz,Ty) = o$, all $i \in N$.

<u>Proof</u>. Let us define a quasi-ordering $\leq$ on X by

$x \leq y$ if and only if $d_i(x,y) \leq \varphi_i(x) - \varphi_i(y)$, and $d_i'(Tx,Ty) \leq \psi_i(x) - \psi_i(y)$, all $i \in N$

and let $(x_n; n \in N)$ be a monotone sequence in X, that is

$d_i(x_n,x_m) \leq \varphi_i(x_n) - \varphi_i(x_m)$, $d_i'(Tx_n,Tx_m) \leq \psi_i(x_n) - - \psi_i(x_m)$, all $n,m \in N$, $n \leq m$, all $i \in N$.

Firstly, as $(\varphi_i(x_n); n \in N)$ and $(\psi_i(x_n); n \in N)$ are decreasing sequences (hence, by the second part of (vii), Cauchy sequences) in R for all $i \in N$, it immediately follows that $(x_n; n \in N)$

and $(Tx_n; n \in N)$ are D (D')-Cauchy sequences in X (X') or, in other words, that $(x_n; n \in N)$ is an E-Cauchy sequence in X. Secondly, by completeness hypotheses, $x_n \xrightarrow{D} x$ and $Tx_n \xrightarrow{D'} x'$ for some $x \in X$, $x' \in X'$ and this gives (by closedness hypothesis) $Tx = x'$ that is, $x_n \xrightarrow{E} x$ in which case, from the preceding relation we get (by a limit process combined with the first part of (vii))

$$d_i(x_n,x) \leq \varphi_i(x_n) - \varphi_i(x),\ d_i'(Tx_n,Tx) \leq \psi_i(x_n) - \psi_i(x),$$

all $n \in N$, $i \in N$

proving $x_n \leq x$, all $n \in N$. Consequently, Theorem 3 again applies (with D replaced by E) and the proof is finished. Q.E.D.

Concerning the first of these applications, it must be noted that, in case $Y = R$, e = the usual distance in R and $\leq$ the usual dual ordering on R, Theorem 4(B) - reductible to a previous author's result [26] - appears as a sequential version of Brézis-Browder's ordering principle [4], while Theorem 4(A) as a sequential extension of a similar Bróndsted's maximality principle [5]. At the same time, the second of these applications - refining Theorem 2 of the above quoted author's paper - may be viewed as a "denumerable" variant of a related Downing-Kirk's result [13] (see also Turinici [27]) as well as (under the assumption T is the identity mapping) of a variational type Ekeland's result [14, 15, 16] or, equivalently, - after Bróndsted's pattern [6] - of the fixed point Caristi-Kirk's theorem [10, 19] (see in this direction Kasahara [18], Browder [9], Wong [30] , Pasicki [22], Siegel [24], Turinici [25], Bróndsted [7] for a number of interesting new viewpoints concerning this problem) so that, our initial maximality principle extends all these contributions.

In what follows, a precise statement of the results announced in the introductory part of the note will be performed. Let X be a metrizable locally convex space whose topology is generated by the denumerable sufficient family of seminorms $D = (|\cdot|_i; i \in N)$. For any y in X and any $r = (r_i; i \in N)$ in R^N with $r_i \geq o$, $i \in N$, let $B(y,r)$ denote the subset of all z in X with $|y-z|_i \leq r_i$, $i \in N$; also, given any x in X, let $K(x;y,r)$ indicate the subset of all combinations $\lambda x+(1-\lambda)z$, $o \leq \lambda \leq 1$, $z \in B(y,r)$, and call it the (constant) drop generated by x, y and r. Clearly, $B(y,r)$ is a closed convex subset of X and so is $K(x;y,r)$; indeed, let $(u_n = \lambda_n x + (1-\lambda_n)v_n; n \in N)$ - for some $(\lambda_n; n \in N)$ in $[o,1]$ and $(v_n; n \in N)$ in $B(y,r)$ - be such that $u_n \xrightarrow{D} u$ for some u in X then (observing that, without loss of generality one may suppose $\lambda_n \neq 1$, $n \in N$ and $\lambda_n \longrightarrow \lambda \neq 1$) $v_n = (u_n - \lambda_n x)/(1-\lambda_n) \xrightarrow{D} (u-\lambda x)/(1-\lambda) \in B(y,r)$ proving our assertion. Suppose further (X,D) is a complete metrizable locally convex space. Then, as an interesting application of our initial maximality principle, the following (constant) drop theorem can be derived.

__Theorem 6.__ Let the closed subset Y of X, the element y in X and the vector $r = (r_i; i \in N)$ in R^N with $r_i > o$, $i \in N$, be such that Y is disjoint from $B(y,r)$. Then, to any x in Y and any $s = (s_i; i \in N)$ in R^N with $o \leq s_i < r_i$, $i \in N$, there corresponds a $z = z(x,s)$ in $bd(Y) \cap K(x;y,s)$ with $Y \cap K(z;y,s) = \{z\}$.

__Proof.__ Let $\preceq$ denote the ordering on Y defined by

$u \preceq v$ if and only if $v \in K(u;y,s)$

(the fact that $\preceq$ is actually an ordering is an immediate consequence of our conventions). Given x in Y arbitrary fixed, let us put $\beta_i = |x-y|_i$, $i \in N$; also, denote by α_i the $|\cdot|_i$-distance between y and Y (clearly, $\alpha_i \geq r_i$), for all $i \in N$. Now, let u, v in Y be such that $x \leq u \leq v$. As $u,v \in K(x;y,s)$, it clearly follows $|u-y|_i$, $|v-y|_i \leq \beta_i + s_i$, all $i \in N$. On the other hand, as $u \leq v$ means $v = \lambda u+(1-\lambda)w$ for some $o \leq \lambda \leq 1$, $w \in B(y,s)$, one has

$$|v-y|_i \leq \lambda|u-y|_i + (1-\lambda)\ |w-y|_i \leq \lambda|u-y|_i + (1-\lambda)s_i, i \in N$$

and this immediately gives (by the above relations)

$$(1-\lambda)(\alpha_i-s_i) \leq (1-\lambda)(|u-y|_i-s_i) \leq |u-y|_i - |v-y|_i,\ i \in N$$

Finally, again from the relation between u and v

$$|u-v|_i \leq (1-\lambda)|u-w|_i \leq (1-\lambda)(\beta_i + 2s_i),\ i \in N$$

so, combining with the preceding one

$$|u-v|_i \leq ((\beta_i+2s_i)/(\alpha_i-s_i))(|u-y|_i - |v-y|_i),\ i \in N$$

proving condition (i) of Theorem 3 will be satisfied (with X replaced by Y) and completing the argument. Q.E.D.

Again let Y be a closed subset of X, with a nonempty boundary bd(Y). We shall say $x \in bd(Y)$ is an essential point of Y provided that, given any neighborhood V of x there exists y in V and $r = (r_i;i \in N)$ in R^N with $r_i > o$, $i \in N$, such that $V \supset B(y,r)$ and $Y \cap B(y,r) = \emptyset$; the subset of all such points will be termed the essential boundary of Y and denoted by Bd(Y). Also, $z \in bd(Y)$ will be called a support point of Y when the element y in X and the vector $r = (r_i;i \in N)$ in R^N with $r_i > o$, $i \in N$, may be found with $Y \cap B(y,r) = \emptyset$ and $Y \cap K(z;y,r) = \{z\}$; the subset of all points having such a property will be denoted by Sp(Y). Now, as a direct consequence of the above result, the following "sequential" support theorem can be stated and proved.

<u>Theorem 7</u>. Let Y be a closed subset of X having a non-

empty essential boundary (hence, a nonempty boundary). Then, the subset of all support points is nonempty too, and dense in the essential boundary.

Proof. Let x be an arbitrary point of Bd(Y) and let V be a neighborhood of x (of course, without any loss one may suppose V is closed and convex). By the definition of the essential boundary, there exists y in X and $r = (r_i; i \in N)$ in R^N with $r_i > 0$, $i \in N$, such that $V \supset B(y,r)$ and $Y \cap B(y,r) = \emptyset$. By the above theorem, given $s = (s_i; i \in N)$ in R^N with $0 < s_i < r_i$, $i \in N$, a $z = z(x,s)$ in $bd(Y) \cap K(x;y,s)$ may be found with $Y \cap K(z;y,s) = \{z\}$. Clearly, $z \in Sp(Y)$; moreover, $V \supset B(y,r)$ implies $z \in V$ and this ends our argument. Q.E.D.

Regarding the elements involved into the above statements, some remarks are in order. Firstly, it is clear that, in case D reduces to a single element (that is, in case (X,D) becomes a Banach space) these results coincide with Theorem 1 and, respectively, the Bishop-Phelps´ support theorem quoted in the introductory part of the note. Secondly, as remarked by Holmes [17, ch. III, § 20] it is possible to construct closed subsets Y of X having no support points (hence no essential points) and this shows that, generally, the conclusion of Theorem 7 has a "relative" character (modulo the assumption Bd(Y) is not empty in case bd(Y) is such) in contrast to the "effective" character of the normed case (where Bd(Y) coincides with bd(Y)). Finally, it should be noted our statements may be put, without major changes into a "pure" metrizable uniform framework, by the use of a well-known Kuratowski´s embedding procedure [21, ch. II, § 15]; a detailed version of such a development will be given elsewhere.

Suppose in what follows Y is a complete metrizable locally convex space under the denumerable and sufficient family of seminorms $D' = (|\cdot|_i; i \in N)$. Given any x in Y, let $|x|$ denote the vector $(|x|_i; i \in N)$; also, letting $q = (q_i; i \in N)$ in R^N with $o \leq q_i < 1$, $i \in N$, let us put $V(x,q) = K(x;o,q|x|)$ and call it the variable drop generated by x and q. Now, as a useful application of the operator maximality principle we expressed before, the following variable drop theorem can be derived.

Theorem 8. Let Y_1 be a closed subset of Y having the property: there exists $q = (q_i; i \in N)$ in R^N with $o \leq q_i < 1$, $i \in N$, such that, for any y in Y_1 distinct from o, the intersection $Y_1 \cap V(y,q)$ contains more than one point. Then, we necessarily have $o \in Y_1$ (o is an element of Y_1).

Proof. Let u, v in Y_1 be such that $v \in V(u,q)$; then, $v = \lambda u + (1-\lambda)w$ for some $o \leq \lambda \leq 1$, $w \in B(o,q|u|)$ so that

$$|v|_i \leq \lambda |u|_i + (1-\lambda) q_i |u|_i, \ i \in N$$

or, equivalently,

$$(1-\lambda)(1-q_i)|u|_i \leq |u|_i - |v|_i, \ i \in N$$

At the same time, again from the relation between u and v

$$|u-v|_i \leq (1-\lambda)(1+q_i)|u|_i \ , \ i \in N$$

so, combining with the preceding one

$$|u-v|_i \leq ((1+q_i)/(1-q_i))(|u|_i - |v|_i), \ i \in N$$

proving all conditions of Theorem 5 hold (with $X = Y_1$, $D = D'$ and T = the identity mapping). Consequently, given x in Y_1, there exists z in Y_1 satisfying conclusions (a) + (b) of that result and this necessarily implies z = o because, otherwise, the hypothesis we accepted about the nonzero elements of Y_1

would contradict the conclusion (b). Q.E.D.

As an immediate consequence of this result, we have

Theorem 9. Let X be an abstract set and T a mapping from X into Y with T(X) closed in Y. Suppose there exists a vector $q = (q_i; i \in N)$ in R^N with $o \leq q_i < 1$, $i \in N$, such that, for any x in X with $Tx \neq o$, a $\bar{x} \in X$ may be found with $Tx \neq T\bar{x} \in V(Tx,q)$. Then, Tz = o for some z in X.

A simple inspection of this result shows the essential property of the mapping T we used here is the closedness of its range T(X). It would be interesting to know whether this condition may not be replaced by the closedness of its graph $G_T = ((x,Tx); x \in X)$ in case we suppose X is endowed with a quasi-metrizable uniform structure $D = (d_i; i \in N)$. In this direction, as a completion of the preceding statement, we have

Theorem 10. Let the complete quasi-metrizable uniform space (X,D) and the closed mapping $T: X \longrightarrow Y$ be such that a $q = (q_i; i \in N)$ in R^N with $o \leq q_i < 1$, $i \in N$ and a $r = (r_i; i \in N)$ in R^N with $r_i \geq o$, $i \in N$ may be found with the property: for any x in X with $Tx \neq o$ there exists $\bar{x}$ in X with $Tx \neq T\bar{x} \in V(Tx,q)$ and $d_i(x,\bar{x}) \leq r_i \,|Tx - T\bar{x}|_i$, $i \in N$. Then, the equation Tx = o has at least a solution in X.

Proof. By the above developments it follows that, x and $\bar{x}$ given as before

$$d_i(x,\bar{x}) \leq (r_i(1+q_i)/(1-q_i))(|Tx|_i - |T\bar{x}|_i),\ i \in N$$

$$|Tx-T\bar{x}|_i \leq ((1+q_i)/(1-q_i))(|Tx|_i - |T\bar{x}|_i),\ i \in N$$

As Theorem 5 again applies, it follows that, given x in X a corresponding z in X may be found with the properties (a) +

+ (b) of that result. Suppose $Tz \neq o$ then, by the hypotheses we adopted, there exists $\bar{z}$ in X with $Tz \neq T\bar{z} \in V(Tz,q)$ and $d_i(z,\bar{z}) \leq r_i \mid Tz-T\bar{z} \mid_i$, $i \in N$ so that, by the above relations, (b) will be contradicted. Therefore, necessarily, $Tz = o$ and the result follows. Q.E.D.

From a technical viewpoint, it is now evident that, in case (Y,D') reduces to a Banach space, Theorems 8 and 9 reduce to Theorem 2 and, respectively, Altman's mapping theorem [1] (see also Kirk and Caristi [20]); moreover, in case (X,D) reduces to a complete metric space, Theorem 10 may be identified with another Altman's mapping theorem (see the above reference as well as Downing and Kirk [13]). On the other hand, as pointed out by these authors, their contributions extend a similar Browder's one [8] so, the same conclusion may be formulated about our statements. Finally, it must be noted that, by the same procedure as that used here, one may state and prove a "denumerable" variant of some recent contributions in this direction due to Cramer and Ray [11] (see also Altman [2]); a development of these arguments will be done in a forthcoming paper.

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