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# COMPLETION CLOSED ALGEBRAS AND MODELS OF PEANO ARITHMETIC

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Abstract: Each model M of PA determines two completion closed algebras: the system of all sets of natural numbers parametrically and nonparametrically coded in M respectively. This paper is devoted to the study of models for which these two systems coincide. They are called thrifty models. A general existence theorem is proved which implies e.g. that in each countable nonstandard model of PA the class of all initial segments that are models of PA is symbiotic with the class of all models that are thrifty models of PA.

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§ 1. Introduction; basic notions and known facts. Saying "model" we shall always mean a non-standard countable model of PA; M, $M_1$, etc. vary over such models. N denotes the set of natural numbers and, at the same time, the standard model. A set $X \subseteq N$ is parametrically coded in M if there is a formula $\varphi(x,y)$ of $L_{PA}$ and $a \in M$ such that $X = \{n \in N; M \models \varphi(\bar{n},a)\}$. X is nonparametrically coded in M if there is a $\varphi(x) \in L_{PA}$ such that $X = \{n \in N; M \models \varphi(\bar{n})\}$. SS(M) and $SS_0(M)$ denote the system of all sets of natural numbers parametrically and nonparametrically coded in M respectively. (Standard system and o-standard system; note that $SS_0(M)$ = Rep(Th(M)), i.e. the system of all sets representable (binumerable) in Th(M).) The system SS(M) may be alternatively defined using the natural M-definable enumeration $D_0, D_1 \ldots$ of M-finite sets of elements of $M: SS(M) = \{D_a \cap \cap N; a \in M\}$.

We have the following easy Fact:

Fact 1. (i) $SS_0(M) \subseteq SS(M)$, (ii) $M_1 \equiv M_2$ implies $SS_0(M_1) = SS_0(M_2)$. (iii) $M_1 \subseteq_e M_2$ implies $SS(M_1) = SS(M_2)$ where $M_1 \subseteq_e M_2$ means that $M_1$ is an initial segment of $M_2$.

A set $\mathfrak{X}$ of subsets of N is c-closed (completion closed) iff (1) $\mathfrak{X}$ is countable, (2) $\mathfrak{X}$ is closed under Turing reducibility, i.e. if Y is recursive in some elements $X_1, \ldots, X_n$ of $\mathfrak{X}$ then $Y \in \mathfrak{X}$ and (3) for each $X \in \mathfrak{X}$ which is (codes) an infinite dyadic tree there is a $Y \in \mathfrak{X}$ which is an infinite path through X (cf. [3, 6, 7]). Scott [6] was the first to investigate c-closed algebras. The situation may be summarized as follows:

Fact 2 (Scott, cf. [7] 3.3). Let $\mathfrak{X}$ be a set of subsets of natural numbers. Then the following are equivalent:

(i) $\mathfrak{X}$ is c-closed

(ii) there is a model M such that $\mathfrak{X} = SS(M)$

(iii) there is a model M such that $\mathfrak{X} = SS_0(M)$

In more details, we have the following

Fact 3. (1) (Scott) If T is a consistent axiomatizable extension of PA in the same language and if $\mathfrak{X}$ is c-closed then there is a model $M \models T$ such that $\mathfrak{X} = SS_0(M)$.

(2) (Friedman, modified by Guaspari ): If $\mathfrak{X}$ is c-closed and T is a consistent extension of PA such that $T \in \mathfrak{X}$ then there is an $M \models T$ such that $\mathfrak{X} = SS(M)$.

The following well known theorem of Friedman shows the importance of the notion of a standard system.

<u>Fact 4</u> (Friedman). (1) $M_1$ is isomorphic to a submodel of $M_2$ iff $SS(M_1) \subseteq SS(M_2)$ and $Th_{\Sigma_1}(M_1) \subseteq Th_{\Sigma_1}(M_2$. (2) $M_1$ is isomorphic to an initial segment of $M_2$ iff $SS(M_1) = SS(M_2)$ and $Th_{\Sigma_1}(M_1) \subseteq Th_{\Sigma_1}(M_2)$. (Here $Th_{\Sigma_1}(M_1) = \{\varphi \in \Sigma_1 ; M_1 \models \models \varphi\}$ .)

The importance of $SS_0$ is illustrated by the following theorems:

<u>Fact 5</u> (Jensen-Ehrenfeucht; Guaspari; cf. [7] 3.4). Let $\mathfrak{X}$ be a c-closed algebra and let $M_0$ be a model. There is an $M \equiv M_0$ such that $SS(M) = \mathfrak{X}$ iff $SS_0(M_0)$.

<u>Fact 6</u> (see [8] 3.14). Let $M_1$, $M_2$ be models. $M_2$ has an initial segment elementarily equivalent to $M_1$ iff $SS_0(M_1) \subseteq \subseteq SS(M_2)$ and $Th_{\Sigma_1}(M_k) \subseteq Th_{\Sigma_1}(M_2)$.

We shall investigate the question under what conditioms the two standard systems - parametrical and nonparametrical - coincide. Let us call a model $M$ such that $SS(M) = SS_0(M)$ a <u>thrifty</u> model. We can give some immediate examples.

<u>Examples</u>. (1) Each minimal model of a complete extension of PA (term model) is thrifty. Thus each model has an elementary thrifty submodel. Each elementary end extension of a thrifty model is thrifty

(2) Each recursively saturated model is non-thrifty (since if <u>M</u> is rec. saturated then $Th(M) \in SS(M) - SS_0(M)$). Thus each model has an elementary extension which is non-thrifty. Each elementary end extension of a non-thrifty model is non-thrifty.

In the next section we shall prove a general existence theorem for thrifty models; in section 3 we shall prove some corollaries.

§ 2. An existence theorem. Let $\varphi^1$ be $\varphi$ and $\varphi^0$ be $\neg\varphi$. A formula $\varphi(x)$ with exactly one free variable is flexible over T if for each $g:N \to N$, $T \cup \{\varphi(\bar{n})^{g(n)}; n \in N\}$ is consistent. $\varphi(x)$ is $\Sigma_k$-flexible over T if for each set S of $\Sigma_k$ sentences such $T \cup S$ is consistent $\varphi(x)$ is flexible over $T \cup S$.

Lemma (Jensen-Ehrenfeucht [5]). For each axiomatizable consistent theory $T \supseteq PA$ in the language of PA and for each k there is a formula $\varphi(x)$ $\Sigma_k$-flexible over T.

Theorem. Let $T \supseteq PA$ be a consistent extension of PA (in the language of PA), let $T = T_0 \cup T_1$ where $T_0$ is axiomatizable and $T_1 \subseteq \Sigma_k$ for some k. Let $\mathfrak{X}$ be c-closed and such that $T \in \mathfrak{X}$.

Then there is an $M \models T$ such that $SS_0(M) = SS(M) = \mathfrak{X}$ and such that M is not an end extension of its minimal elementary submodel.

Proof. Clearly, T is incomplete and hence we may assume that $T \cup Th(N)$ is inconsistent and that there is a formula $\chi(x)$ such that $T \cup (\exists!x)\chi(x)$ and for each $n \in N$, $T \models \neg\chi(\bar{n})$. Our proof will consist in defining a completion of T. Let
$d, c_0, c_1, c_2, \ldots$ be new constants,
$\mathfrak{X} = Y_0, Y_1, Y_2, \ldots$
$\varphi_0, \varphi_1, \varphi_2, \ldots$ be a natural recursive enumeration of all formulas in $L_{PA}(d, c_0, c_1, \ldots)$. For each formula $\alpha(x)$ of $L_{PA}$ with just one free variable let $Def_{\alpha}(d)$ be the formula saying that

d is the unique x satisfying $\alpha(x)$. Put

$T_0 = T \cup \{(\forall y \leq d) \neg \chi(y)\} \cup \{\neg \mathrm{Def}_\alpha(d);\ \alpha(x) \in L_{PA}\}$.

(The formula $(\forall y \leq d) \neg \chi(y)$ says that d is less than the unique element satisfying $\chi$ and $\neg \mathrm{Def}_\alpha(d)$ guarantees that d is not defined by $\alpha$ in any model of $T^o$.)

Let $T^n$ be given; we define $T^{n+1}$. The construction of $T^{n+1}$ guarantees that

(1) $\varphi_n$ is decided and witnessed;

(2) $Y_n \in SS_o(\underline{M})$ for each $\underline{M} \models T^{n+1}$;

(3) $Dc_n \cap N \in \mathcal{X}$ for each $\underline{M} \models T^{n+1}$.

Assume that $T^n \in \mathcal{X}$ and has the form

$$T_0 \cup \{\neg \mathrm{Def}_\alpha(d); \alpha\} \cup A$$

where A is a set of formulas in $L_{PA}(d, c_o, \ldots, c_p)$ for a certain p and $A \subseteq \Sigma_q(d, c_o, \ldots, c_p)$ for a certain q. (Each element of A has the form $\psi(c_o, \ldots, c_p, d)$.) We may assume that A is closed under conjunction.

Point (1) is handled in the usual manner: at most two formulas are added to $T^n$ and we obtain a theory $T'$ of the form

$$T_0 \cup \{\neg \mathrm{Def}_\alpha(d); \alpha\} \cup A'$$

where $A'$ satisfies the same conditions as A did (with possibly other values of p, q; the constants $c_i$ are used as witnessing constants).

Concerning (2): We want to use a $\Sigma_q$-flexible formula; but the trouble is that $T'$ is a theory in $L_{PA}(d, c_o, \ldots, c_p)$, not in $L_{PA}$. Form the following theory T":

$$T'' = T_0 \cup (\exists^m y)(\exists x_o, \ldots, x_p)\, \psi(x_o, \ldots, x_p, y);$$
$$\psi(c_o, \ldots, c_p, d) \in A',\ m \in N\}.$$

Here $\exists^m$ is the quantifier "there are at least m elements such that...". Certainly, $T' \vdash T''$ (if $M$ is a model in which there are at most m elements y satisfying $(\exists \underline{x}) \psi(\underline{x}, y)$ and at the same time $(\exists \underline{x}) \psi(\underline{x}, d)$ then d is definable in $\underline{M}$).

Claim. $T'$ is a conservative extension of $T''$.

Let $\underline{M} \models T''$; it suffices to find a $\underline{M}_1 \succ \underline{M}$ expandable to a model $\underline{M}'$ of $T'$. To show this it is sufficient to observe that, by compactness, $\mathrm{diag}(\underline{M}) \cup T'$ is consistent. Indeed, finitely many axioms $\psi_i(\underline{c}, d)$ may be combined into one, $\psi(\underline{c}, d)$; and if $\alpha_1, \ldots, \alpha_m \in L_{PA}$ then using $T'' \vdash (\exists^{m+1} y)(\exists \underline{x}) \psi(\underline{x}, y)$ we may expand each model of $T''$ to a model of

$$T'' \cup \psi(\underline{c}, d) \cup \bigwedge_{i=1}^{m} \neg \mathrm{Def}_{\alpha_i}(d).$$

This proves the claim.

To complete the point (2), let $\sigma(x)$ be a $L_{PA}$-formula $\Sigma_{q+1}$-flexible w.r.t. $T_0$. Let $B = \{\sigma(\overline{m}); m \in Y_n\} \cup \{\neg \sigma(\overline{m}); m \notin Y_n\}$. B is consistent with $T''$ and therefore with $T'$; put $T^+ = T' \cup B$. Then for each $\underline{M} \models T^+$ we have $Y_n \in SS_0(\underline{M})$. This completes the point (2); observe that $T^+ \in \mathfrak{X}$.

Concerning (3): To handle $c_n$, let

$S = \{$s dyadic word; there is no proof of contradiction of length $\leq \mathrm{lh}(s)$ in $T^+ \cup \{m \in Dc_n; (s)_m = 1 \,\&\, m \quad \mathrm{lh}(s)\} \cup$

$\cup \{m \notin Dc_n; (s)_m = 0 \,\&\, m \quad \mathrm{lh}(s)\}$.

Then S is recursive in $T^+$ and hence in $\mathfrak{X}$ ; if Y is an infinite path through S belonging to $\mathfrak{X}$ then we put

$T^{n+1} = T^+ \cup \{m \in Dc_n; (Y)_m = 1\} \cup \{m \notin Dc_n; (Y)_m = 0\}$.

Then $T^{n+1} \in \mathfrak{X}$ and has the desired form.

Let $T^\infty = \bigcup_n T^n$; then $T^\infty$ satisfies (1),(2),(3) for all n and thus the model $\underline{M}$ of $T^\infty$ formed by the witnessing constants $c_i$ in the usual manner satisfies $SS(\underline{M}) = \mathfrak{X} = SS_{,}(M)$.

The reduct of $\underline{M}$ to $L_{PA}$ is a thrifty model of T and is not an end extension of its minimal elementary submodel since $d_M$ is an element of M which is not definable in $\underline{M}$ but is smaller than the least element satisfying $\chi(x)$.

Remark. Using many $d'$s one can obtain a model $\underline{M}$ as above such that non-definable elements are co-initial with M - N.

§ 3. Corollaries and remarks

Corollary 1. (a) For each model $\underline{M}$ and each n, there is an initial segment $I \subseteq_e M$ such that $I \models PA$, $I \equiv_{\Sigma_n} \underline{M}$ and I is thrifty.

(b) For each model M and each n, there is an end-extension $K \supseteq_e M$ such that $K \models PA$, $K \equiv_{\Sigma_n} M$ and K is thrifty.

Proof. Put $T_0 = PA$, $T_1 = Th_{\Sigma_{n+1}}(M)$, $\mathfrak{X} = SS(M)$; let $M'$ be the model guaranteed by our theorem. Then, by Fact 4, $M'$ is isomorphic both to a segment I of M and to an end-extension K of M.

Corollary 2. Let M be a model. Segments $I \subseteq_e M$ that are models of PA are symbiotic with the segments $J \subseteq_e M$ such that J is a thrifty model of PA.

Proof. Let $a,b \in M$, $I \subseteq_e M$, $a < I < b$, $I \models PA$. Let $I' \subseteq_e I$ be a model such that $I' \equiv_{\Sigma_2} I$, $I'$ thrifty. Using theorem 10 from [4] we can map $I'$ isomorphically onto an $I'' \subseteq_e M$, $a < I'' < b$. (Let $a < I_1 < I < I_2 < b$, $I_1, I_2 \models PA$; there is an isomorphism $\wp$ of $I'$ onto a segment $I''$ of $I_2$ such that $\wp$ is identical on $I_1$.)

Remarks. (1) One can show similarly that the class of all models of PA is symbiotic with the class of all segments $K \subseteq_e M$ such that (1) $K \models PA$, (2) K is thrifty and (3) K is a strong cut in M: given a, I, b, first extend I to an $I'$ as above and then extend $I'$ to $I'' \subseteq_e I'$ such that $I'$ is strong in $I''$ (and $I' \equiv I''$).

(2) Clearly, in the above corollaries we do not use the fact that the model obtained from our Theorem is not an end extension of its minimal elementary submodel; to obtain the corollaries, an obvious simplification (omitting everything concerning d) would suffice.

(3) Corollary 1 cannot be strengthened to $I \equiv M$: elementary equivalence preserves $SS_0(M)$ and $I \subseteq_e M$ guarantees $SS(I) = SS(M)$.

§ 4. An ordering. Consider complete extensions T of PA (in the same language). Recall that, for each $M \models T$, $SS_0(M) = Rep(T)$.

Theorem 2. Let $T_1$, $T_2$ be complete extension of PA. The following are equivalent:

(i) Each $M_2 \models T_2$ has a submodel $M_1$ such that $M_1 \models T_1$.

(ii) Each $M_2 \models T_2$ has an initial segment such that $M_1 \models T_1$.

(iii) $\Sigma_1(T_1) \subseteq \Sigma_1(T_2)$ and $Rep(T_1) \subseteq Rep(T_2)$.

(iv) The minimal (term) model of $T_1$ is a submodel of the minimal model of $T_2$.

(v) There are thrifty models $M_1 \models T_1$, $M_2 \models T_2$ such that $M_1 \subseteq M_2$.

Proof. (i) $\Longleftrightarrow$ (ii) by Gaifman's theorem [2]. (i) $\Longleftrightarrow$ (iv) is obvious, as well as (iv) $\Longleftrightarrow$ (v). We prove (iii) $\Longrightarrow$ (iv) and (v) $\Longrightarrow$ (iii).

Let $M_2$ be the minimal model of $T_2$; assume (iii). By Fact 6, there is a model $M_1'$ of $T_1$ which is an initial segment of $M_2$. Let $M_1$ be the minimal model of $T_1$, then $M_1 \subseteq M_1' \subseteq M_2$. This proves (iv).

Let $M_1$, $M_2$ be as in (v); then $\Sigma_1(T_1) = Th_{\Sigma_1}(M_1) \subseteq$ $\subseteq Th_{\Sigma_1}(M_2) = \Sigma_1(T_2)$ and $Rep(T_1) = SS_0(M_1) = SS(M_1) \subseteq SS(M_2) =$ $= SS_0(M_2) = Rep(T_2)$. This proves (iii).

<u>Definition</u>. Put $T_1 \trianglelefteq^* T_2$ iff an of (i) - (v) holds.

<u>Fact</u>. $\trianglelefteq^*$ is a quasiordering (preorder) of all complete extensions of PA; there are continuum-many incomparable elements, ascending chains of the power $\aleph_1$, and $\trianglelefteq^*$ is not dense.

<u>Proof</u>. The existence of an antichain of power continuum follows from the existence of a $\Sigma_1$-formula flexible w.r.t. PA; this formula gives continuum many completions of PA with pairwise incomparable sets of $\Sigma_1$-consequences. A chain of power $\aleph_1$ is obtained by constructing a chain of $\aleph_1$ c-closed systems, which is easy.

We prove that $\trianglelefteq^*$ is not dense. (Cf. [4], 7(5).) Let S be a maximal set of $\Sigma_1$-sentences such that $(PA + Con_{PA} + S)$ is consistent. Then $PA + \neg Con_{PA} + S$ is also consistent; let R be a maximal set of $\Pi_1$-sentences such that $PA + \neg Con_{PA} +$ $+ S + R$ is consistent. Let $M_2$ be a thrifty model of $PA +$ $+ \neg Con_{PA} + S + R$ such that $S, R \in SS(M_2) = SS_0(M_2)$. Let $M_1$ be a segment of $M_2$ which is a model of $PA + Con_{PA} + S$. By Corollary 1, we may assume that $M_1$ is thrifty.

Put $T_1 = Th(M_1)$ and $T_2 = Th(M_2)$. If $T_1 \trianglelefteq T \trianglelefteq T_2$ then $Rep(T) = Rep(T_i) = SS(M_i) = SS_0(M_i)$ $(i = 1,2)$; if $T \models Con_{PA}$

then $\Sigma_1(T) = \Sigma_1(T_1)$ and if $T \models \neg Con_{PA}$ then $\Sigma_1(T) =$
$= \Sigma_1(T_2)$.

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