Martin Markl

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REGULAR FUNCTIONS OVER CONFORMAL QUATERNIONIC MANIFOLDS

M. MARKL

Abstract: In the paper the notion of regular functions (in Fueter sense) defined on conformal quaternionic manifold is introduced.

Key words: Quaternions, regular quaternionic functions, conformal quaternionic manifolds, fiber bundle.

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§ 1. Introduction. The aim of the paper is to define a notion analogical to the notion of a holomorphic function on complex manifold in the quaternionic case.

We take so called regular functions (defined by Fueter, see definition 1) as the local model for quaternionic analogue of holomorphic functions.

The fact that the composition of two such regular functions need not be regular again gives rise to two problems. We have to define a notion of quaternionic manifold (using quaternionic charts and a pseudogroup of mappings) and we have to create a notion of a regular function on such a manifold.

We can solve both problems at the same time, if we introduce a notion of conformal quaternionic manifold (see definition 6) and if we define a "regular function" as a section of a special canonical fiber bundle (see definition 8 and definition 9).

§ 2. Regular functions

Definition 1 (see[1]). Let U be an open subset of H, where H is the algebra of quaternions. The real differentiable function $f:U \longrightarrow H$ is called regular on U, if

$$\frac{\partial f}{\partial q_0} + i\frac{\partial f}{\partial q_1} + j\frac{\partial f}{\partial q_2} + k\frac{\partial f}{\partial q_3} = 0$$

in each point $q = q_0 + iq_1 + jq_2 + kq_3$ of U. The set of all regular functions over U will be denoted by O(U).

Definition 2 (see [2]). Let us denote by G the conformal group of H, it means the group of all mappings of the form

$$(a + b\cdot q)\ (c + d\cdot q)^{-1}$$

where $a,b,c,d \in H$ and $a\cdot d - c\cdot b \neq 0$.

Definition 3. Let $f(q) = (a + b\cdot q)\ (c + d\cdot q)^{-1}$ be an element of G. Let us denote by J_f the function

$$J_f(q) = (c + d\cdot q)^{-1}\cdot |c + d\cdot q|^{-2}$$

Theorem 4. Let $f \in G$, U be an open set in H and let us suppose that f is continuous on U. Then the function $F:H \longrightarrow H$ is regular on U if and only if the function

$$J_f(q)\cdot F \circ f(q)$$

is regular on $f^{-1}(U)$.

Proof: See [1].

The following theorem is in fact the main theorem of this paper. It contains the "chain law" for the functions J.

Theorem 5. If $f,g \in G$, then $J_{f\circ g}(q) = J_g(q)\cdot J_f(g(q))$.

Proof: The proof is straightforward, but the necessary calculation is long. We take two functions $f,g \in G$ in the form

$$f = (a_1 + b_1 q)(c_1 + d_1 q)^{-1}$$

$$g = (a_2 + b_2 q)(c_2 + d_2 q)^{-1}$$

and we denote $h = f \circ g$. Then the formulae

$$J_h(q) = ((c_1c_2 + d_1a_2) + (c_1d_2 + d_1b_2)q)^{-1} \cdot$$
$$\cdot |(c_1c_2 + d_1a_2) + (c_1d_2 + d_1b_2)q|^{-2}$$
$$J_f(g(q)) = (c_1 + d_1(a_2 + b_2q)(c_2 + d_2q)^{-1})^{-1} \cdot$$
$$\cdot |c_1 + d_1(a_2 + b_2q)^{-1}|^{-2}$$

hold and the theorem follows by direct calculation.

<u>Definition 6</u> (see [2]). We say that the real four-dimensional manifold M is a <u>conformal quaternionic</u> (<u>one-dimensional</u>) <u>manifold</u>, if and only if there exists the atlas on M such that the transition functions belong to G.

<u>Example 7</u> (quaternionic projective space). Take $H \times H - (0,0)$ with the relation

$(q_1,q_2) \sim (q_1',q_2') \equiv$ there exists $c \in H$ such that $q_i = q_i' \cdot c$ for $i = 1,2$. Denote $P(H) = H^2 - (0,0)/\sim$.

Then P(H) is a conformal manifold with the trivialisation

$$U_i = \{(q_1,q_2) \in H^2 : q_i \neq 0\} \text{ for } i = 1,2$$

$$p_i : U_i \to H \qquad p_1(q_1,q_2) = q_2q_1^{-1} \qquad p_2(q_1,q_2) = q_1q_2^{-1}$$

The transition functions are $p_{12}(q) = q^{-1}$, $p_{21}(q) = q^{-1}$. Clearly p_{12} and p_{21} belong to G.

P(H) is an example of compact manifold. It is isomorphic with the one-point compactification of H. There are other conformal manifolds, for example the torus $T = H/Z^4$.

§ 3. <u>The fiber bundle</u> A(M). In this section we define a line fiber bundle A(M) as the suitable space for regular sections over conformal manifold M.

Definition 8. Consider the trivialisation (U_i, p_i) of a conformal manifold M. Denote $p_{ij} = p_j \circ p_i^{-1}$. Over each U_i we define A(M) to be trivial, isomorphic to $U_i \times H$. The transition functions are the following ones

$$U_i \times H \longrightarrow U_j \times H$$
$$(x, q_i) \longmapsto (x, q_j)$$
$$q_j = J^{-1}_{p_{ij}}(p_i(x)) \cdot q_i$$

It can be shown by direct calculation that the transition functions satisfy the chain rule, i.e. that A(M) is well defined.

Let $(U_i, p_i), i = 1,2,3$ be the trivialisations of M. Let $x \in U_1 \cap U_2 \cap U_3 \neq 0$. Let us write for simplicity $p_1(x) = q$, $J_{p_{ij}} = J_{ij}$.

We obtain from the definition

$$q_3 = J_{13}^{-1}(q) \cdot q_1$$

If we calculate gradually the transitions functions from U_1 to U_2 and from U_2 to U_3, we obtain

$$q_2 = J_{12}^{-1}(q) \cdot q_1$$

$$q_3 = J_{23}^{-1}(p_2(x)) \cdot q_2 = J_{23}^{-1}(p_2(x)) \cdot J_{12}^{-1}(q) \cdot q_1 =$$
$$= (J_{12}(q) \cdot J_{23}(p_{12}(q)))^{-1} \cdot q_1 .$$

But from definition 3 and theorem 5 $J_{12}(q) \cdot J_{23}(p_{12}(q))$ is equal to $J_{12}(q)$.

Now we define the notion of regular section of the fiber bundle A(M).

Definition 9. We say that a section $u: M \longrightarrow A(M)$ is regular, if for each trivialisation (U_i, p_i) of M the function

$u_i \circ p_i^{-1}$, where u_i is the trivialisation of u over U_i, belongs to O(U).

Now we must show that this definition is correct. Let $u_1(p_1^{-1})$ be regular at the point q. From the definition 8 for the trivialisation u_2 over (U_2, p_2) it holds

$$u_2(p_2^{-1}(q)) = J_{12}^{-1}(p_{12}(q)) \cdot u_1(p_2^{-1}(q)).$$

From the theorem 4 is $u_2(p_2^{-1})$ regular.

References

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Matematicko-fyzikální fakulta, Universita Karlova, Sokolovská 83, 18600 Praha 8, Československo

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