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COMPLEMENTS IN THE LATTICE OF UNIFORMITIES

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Abstract: The complementation in the lattice of all uniformities is investigated. E.g. the characterization of countable uniform spaces with a complement is given.

Key words: Lattice of uniformities on a given set, metric space, complements in a lattice.

Classification: 54E15, 54E35

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0. Introduction. We shall investigate complements in the lattice $\nu(X)$ of all uniformities on a given set X. We show that $\nu(X)$ is complemented iff X is finite. When X is infinite, there are many uniformities which admit a complement. The set of all complements to a given uniformity can be very extensive. The main result is a characterization of metrizable uniformities on a countable set which possess complements. Using a simple argument, this is generalized to metrizable uniformities whose topology is separable. The results have been announced in [5].

Other properties of $\nu(X)$, especially atoms in $\nu(X)$, have been investigated in [3],[4],[6].

1. Preliminaries and general remarks.

1.1. Let $\nu(X)$ be the set of all uniformities on a set X. If $\mathcal{U}$, $\mathcal{V} \in \nu(X)$, write $\mathcal{U} \prec \mathcal{V}$ if $\mathcal{U}$ is finer than $\mathcal{V}$, equivalently, if the identity map id:$X \longrightarrow X$ is uniformly continuous from $(X,\mathcal{U})$ to $(X,\mathcal{V})$. Then $\prec$ is an ordering making $\nu(X)$ a complete lattice. The smallest element of $\nu(X)$ will be denoted by $\underline{0}$; it is the uniformly discrete uniformity. The largest element, denoted by $\underline{1}$, is the indiscrete uniformity. Recall that, as in any lattice, $\mathcal{U}$ is a complement to $\mathcal{V}$ if $\mathcal{U} \wedge \mathcal{V} = \underline{0}$ and $\mathcal{U} \vee \mathcal{V} = \underline{1}$.

Notice that $\nu(X)$ is often regarded with the order $\subset$ (inclusion) which is just opposite to $\prec$: from the point of view of complementation, the difference between $\subset$ and $\prec$ is, of course, irrelevant.

1.2. In what follows, we shall use the following obvious facts.

a) If $\mathcal{U}$, $\mathcal{V} \in \nu(X)$, then $\mathcal{U} \vee \mathcal{V} = \underline{1}$ iff the only pseudometric on X which is uniformly continuous with respect to both $\mathcal{U}$ and $\mathcal{V}$ is $\rho = 0$ (i.e. $\rho(x,y) = 0$ for all $x,y \in X$).

b) Further, $\mathcal{U} \wedge \mathcal{V} = \underline{0}$ iff there is a $\mathcal{U}$-uniform cover $\{U_i\}$ and a $\mathcal{V}$-uniform cover $\{V_j\}$ such that the cover $\{U_i \cap V_j\}$ consists of singletons and this happens iff there are pseudometrics ρ, σ , uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$ respectively, such that $\rho(x,y) + \sigma(x,y) \geq K$ for some positive K and all $x \neq y$.

1.3. If X is finite then $\nu(X)$ can be identified with the lattice of partitions of X which is obviously complemented.

1.4. Remark. If X is infinite then $\nu(X)$ is not complemented.

Indeed, if $\mathcal{U} \neq \underline{0}$ is a proximally discrete uniformity on X (one inducing the discrete proximity, that is, one such that all finite covers of X are $\mathcal{U}$-uniform) then $\mathcal{U}$ has no complement because for any $\mathcal{V} \neq \mathbf{1}$,

$$\mathcal{U} \vee \mathcal{V} \prec p\,\mathcal{U} \vee p\,\mathcal{V} = p\,\mathcal{V} \neq \underline{1}$$

where p denotes the precompact modification (i.e. $p\,\mathcal{U}$ is generated by all finite covers which are uniform with respect to $\mathcal{U}$).

1.5. Also, a uniformity which is not proximally discrete need not admit a complement:

<u>Example</u>. <u>The uniformity</u> $\mathcal{S}$ <u>of a Cauchy sequence</u> (i.e. the uniformity on $\{1/n; n \in N\}$ induced by the usual metric on the reals) <u>has no complement</u>. To be proved later.

1.6. As we shall see, there are many uniformities which have complement. The class of these uniformities is closed under the following operations.

<u>Claim a</u>. If X, Y <u>are disjoint</u>, $\mathcal{U}$ <u>has a complement in</u> $\nu(X)$, $\mathcal{V}$ <u>has a complement in</u> $\nu(Y)$ <u>then the sum of</u> $\mathcal{U}$ <u>and</u> $\mathcal{V}$ <u>has a complement in</u> $\nu(X \cup Y)$.

Proof. Let $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ be complements to $\mathcal{U}$ and $\mathcal{V}$, resp.; let $\mathcal{W}$ be the uniformity on $X \cup Y$ generated by all pseudometrics ρ such that

(1) the restriction of ρ to X is uniformly continuous with respect to $\widetilde{\mathcal{U}}$ and the restriction of ρ to Y is uniformly continuous with respect to $\widetilde{\mathcal{V}}$.

(2) if $x \in X$ and $y \in Y$ then $\rho(x,y) = \rho(x,x_0) + \rho(y,y_0)$ where x_0, y_0 are arbitrary but fixed points, $x_0 \in X$, $y_0 \in Y$. Then $\mathcal{W}$ is clearly a complement to the sum of $\mathcal{U}$ and $\mathcal{V}$.

Claim b. Let $\mathcal{U}$ have a complement in $\nu(X)$ and let $\mathcal{V}$ have a complement in $\nu(Y)$. Then the product uniformity $(X,\mathcal{U})\times(Y,\mathcal{V})$ has a complement in $\nu(X\times Y)$.

Proof. Let $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$ be complements to $\mathcal{U}$, $\mathcal{V}$, respectively. Let $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ be the product uniformities $(X,\mathcal{U})\times$ $\times(Y,\mathcal{V})$ and $(X,\widetilde{\mathcal{U}})\times(Y,\widetilde{\mathcal{V}})$, respectively. We shall prove that $\widetilde{\mathcal{W}}$ is a complement to $\mathcal{W}$.

Let ρ be a pseudometric which is uniformly continuous with respect to both $\mathcal{W}$ and $\widetilde{\mathcal{W}}$. Each subspace $\{x\}\times Y$ of $(X\times Y,\mathcal{W})$ can be identified with $(Y,\mathcal{V})$. In the sense of this dentification, the restriction of ρ to $\{x\}\times Y$ is uniformly continuous with respect to $\mathcal{V}$. Analogously for $\widetilde{\mathcal{V}}$. As $\mathcal{V}\vee\widetilde{\mathcal{V}}=\underline{1}$ in $\nu(Y)$, $\rho=0$ on $\{x\}\times Y$. Using the same argument we prove that $\rho=0$ on each $X\times\{y\}$. Then $\rho=0$ on all of $X\times Y$ and so $\mathcal{W}\vee\widetilde{\mathcal{W}}=\underline{1}$ in $\nu(X\times Y)$ by 1.2 a.

To prove $\mathcal{W}\wedge\widetilde{\mathcal{W}}=\underline{0}$, use the fact that $\mathcal{U}\wedge\widetilde{\mathcal{U}}=\underline{0}$, $\mathcal{V}\wedge\widetilde{\mathcal{V}}=\underline{0}$ to find covers $\{U_i\}$, $\{\widetilde{U}_j\}$, $\{V_k\}$, $\{\widetilde{V}_h\}$ which are uniform with respect to $\mathcal{U}$, $\widetilde{\mathcal{U}}$, $\mathcal{V}$, $\widetilde{\mathcal{V}}$, respectively, such that the covers $\{U_i\cap\widetilde{U}_j\}$, $\{V_k\cap\widetilde{V}_h\}$ consist of singletons, see 1.2 b. Then the same holds for $\{(U_i\cap\widetilde{U}_j)\times(V_k\cap\widetilde{V}_h)\}=$ $=\{(U_i\cap V_k)\times(\widetilde{U}_j\cap\widetilde{V}_h)\}$. As the cover $\{U_i\times V_k\}$ is $\mathcal{W}$-uniform and the cover $\{\widetilde{U}_j\times\widetilde{V}_h\}$ is $\widetilde{\mathcal{W}}$-uniform, we have $\mathcal{W}\wedge\widetilde{\mathcal{W}}=\underline{0}$ by 1.2 b.

1.7. Proposition. If $(X,\mathcal{U})$ is dense in $(Y,\mathcal{V})$ and if $\mathcal{U}$ has a complement in $\nu(X)$ then $\mathcal{V}$ has a complement in $\nu(Y)$.

Proof. Let $\widetilde{\mathcal{U}}$ be a complement to $\mathcal{U}$. Let $\widetilde{\mathcal{V}}$ be the uniformity on Y generated by all pseudometrics ρ such that the restriction of ρ to X is uniformly continuous with respect to $\widetilde{\mathcal{U}}$, $\rho(x,y) \leq 1$ for all $x,y \in Y$ and $\rho(x,y) = 1$ if the points x,y are distinct and at least one of them lies in Y - X. Then $\widetilde{\mathcal{V}}$ is a complement to $\mathcal{V}$.

1.8. Remark. Let a $\mathcal{U} \in \nu(X)$ admit a complement. Then it admits a pseudometrizable complement.

Proof. Let $\mathcal{V}$ be a complement to $\mathcal{U}$. Choose pseudometrics ρ , σ as in 1.2 b. Let $\mathcal{V}'$ be the uniformity induced by σ . Then $\mathcal{U} \wedge \mathcal{V}' = \underline{0}$ by 1.2 b and $\mathcal{U} \vee \mathcal{V}' = \underline{1}$ because $\mathcal{V} \prec \mathcal{V}'$.

2. Variety of complements. In this section, we show that the class of all complements to a given uniformity can be very extensive.

2.1. Consider the uniformity $\mathcal{D}$ of two adjacent sequences, that is, $\mathcal{D}$ is a uniformity on N induced by the metric

$\sigma(2i-1,2i) = 1/2^i$, $\sigma(x,y) = 1$ otherwise $(x \neq y)$. Further, denote by $\mathcal{D}'$ the uniformity on N induced by the pseudometric

$\sigma(2i-1,2i) = 0$, $\sigma(x,y) = 1$ otherwise $(x \neq y)$.

2.2. The uniformity $\mathcal{D}$ is topologically discrete (all points are isolated). Thus, if $\mathcal{V}$ is a complement to $\mathcal{D}$ then $\mathcal{V}$ must not contain any isolated point because such a point would be isolated in $\mathcal{V} \vee \mathcal{D}$. The following proposition shows that the non-existence of isolated points is sufficient for a uniformity to be a copy of a complement to $\mathcal{D}$.

Proposition. Let $\mathcal{U}$ be a metrizable uniformity without isolated points on N. Then there exists a complement $\mathcal{V}$ to $\mathcal{D}$ in $\nu(N)$ such that $(N,\mathcal{V})$ is uniformly homeomorphic to $(N,\mathcal{U})$

Proof. It is enough to show that there is a complement to $\mathcal{U}$ in $\nu(N)$ which is uniformly homeomorphic to $\mathcal{D}$. Let $\mathcal{U}$ be induced by a metric ρ. We may assume that

(1) there are $a,b \in N$ with $\rho(a,b) > 2$.

It follows from (1) and from the fact that ρ admits no isolated points that there exists a sequence $\{x_i'\}$ with

(2) a) $\{x_i' : i \in N\} = N$,

b) $\{(x_{2i-1}', x_{2i}') ; i \in N\} = \{(x,y) \in N \times N;\ \rho(x,y) > 1\}$.

Define a new sequence $\{x_i\}$ by induction as follows. Put $x_0 = x_0'$. Let $X_k = \{x_0, \dots, x_{k-1}\}$ have been defined. If $x_k' \notin X_k$, put $x_k = x_k'$. If $x_k' \in X_k$, choose an $x_k \in N - X_k$ such that

(3) $\rho(x_k, x_k') < 1/2^k$,

(4) $\rho(x_{2i-1}, x_{2i}') > 1$ whenever k is odd, $k = 2i-1$, and

(5) $\rho(x_{2i-1}, x_{2i}) > 1$

whenever k is even, $k = 2i$; this is possible because X_k is finite and because of the lack of isolated points.

It follows by the construction and by (2) that $\{x_i ; i \in N\} = N$ and that the points x_i are pairwise distinct. Thus we can define a metric σ on N by

(6) $\sigma(x_{2i-1}, x_{2i}) = 1/2^i$, $\sigma(x,y) = 1$ otherwise $(x \neq y)$.

Let $\mathcal{W}$ be the uniformity induced by σ. Clearly $\mathcal{W} \simeq \mathcal{D}$.

Following (5),(6), $\rho + \sigma > 1$ and so $\mathcal{U} \wedge \mathcal{W} = \underline{0}$ by 1.2 b. It remains to prove that $\mathcal{U} \vee \mathcal{W} = \underline{1}$. Let $x,y \in N$. By (2) there exists $z \in N$ with $\rho(x,z) > 1 < \rho(z,y)$. As x, y are not isolated, one can choose i, j such that, for a given $\varepsilon > 0$,

(7) $\rho(x, x_{2i-1}') < \varepsilon/10$, $\rho(x_{2i}', z) < \varepsilon/10$,

$\rho(z, x_{2j-1}') < \varepsilon/10$, $\rho(x_{2j}', y) < \varepsilon/10$

and such that i, j are so large that

$$(8) \quad 1/2^i < \varepsilon/10, \quad 1/2^j < \varepsilon/10.$$

Then

$$(9) \quad \rho(x,x'_{2i-1}) + \rho(x'_{2i-1},x_{2i-1}) + \sigma(x_{2i-1},x_{2i}) + \\ + \rho(x_{2i},x'_{2i}) + \rho(x'_{2i},z) + \rho(z,x'_{2j-1}) + \\ + \rho(x'_{2j-1},x_{2j-1}) + \sigma(x_{2j-1},x_{2j}) + \rho(x_{2j},x'_{2j}) + \\ + \rho(x'_{2j},y) < \varepsilon .$$

Indeed, use

(7) to the first, the fifth, the sixth, the tenth summand,

(3) and (8) to the second, the fourth, the seventh, the ninth summand,

(6) to the third and the eighth summand.

Now $\mathcal{U} \vee \mathcal{W} = \underline{1}$ by (9) and by the following lemma.

2.3. Lemma. Let $\mathcal{U}, \mathcal{W}$ be uniformities on X induced by pseudometrics ρ, σ, respectively. If there exists an integer n such that for every $\varepsilon > 0$ and every pair $x,y \in X$ there is a chain $x = x_0, x_1, \dots, x_n = y$ with

$$\sum_{i=1}^{n} \min\left(\rho(x_{i-1},x_i), \sigma(x_{i-1},x_i)\right) < \varepsilon ,$$

then $\mathcal{U} \vee \mathcal{W} = \underline{1}$ in $\nu(X)$.

Proof. Let π be a pseudometric which is uniformly continuous with respect to both $\mathcal{U}$ and $\mathcal{W}$. We have to prove $\pi = 0$; then $\mathcal{U} \vee \mathcal{W} = \underline{1}$ by 1.2 a. Suppose $\pi(x,y) > 0$ for some $x,y \in X$. There exists $\delta > 0$ such that

$$(*) \quad a,b \in X, \ \rho(a,b) < \delta \ \text{or} \ \sigma(a,b) < \delta \Longrightarrow \pi(a,b) < \frac{\pi(x,y)}{n}.$$

Choose $x_0, x_1, \dots, x_n$ as in Lemma for x, y and $\varepsilon = \delta$. Then for every i, $\min(\rho(x_{i-1},x_i), \sigma(x_{i-1},x_i)) < \delta$ and so, by $(*)$, $\pi(x_{i-1},x_i) < \pi(x,y)/n$. Hence

$$\sum_{i=1}^{m} \pi(x_{i-1},x_i) < \pi(x,y),$$

a contradiction with the triangle inequality.

2.4. Remark. In Proposition 2.2, $\mathcal{D}$ can be replaced by $\mathcal{D}'$. The same proof can be used; one has only to replace $1/2^i$ in (6) by 0.

2.5. Denote $\mathcal{Q}$ the usual uniformity on the set Q of all rationals.

Proposition. Let $\mathcal{U}$ be a pseudometrizable uniformity on a countable set, say on N, which admits two disjoint uniformly discrete subsets which are proximal or, more generally, an arbitrary uniformity on N admitting a subspace whose uniformity is uniformly homeomorphic either to $\mathcal{D}$ or to $\mathcal{D}'$. Then there exists a complement to $\mathcal{Q}$ in $\nu(Q)$ which is uniformly homeomorphic to $\mathcal{U}$.

Proof. Let $A \subset N$ be the subset such that $\mathcal{U}/A \simeq \mathcal{D}$ (or $\mathcal{U}/A \simeq \mathcal{D}'$) (here "/" denotes a restriction of a uniformity to a subset and "$\simeq$" a uniform homeomorphism). Then there exists a $\mathcal{U}$-uniformly continuous pseudometric σ on N such that we can write $A = \{a_i\}$ to obtain

$$\sigma(a_{2i-1},a_{2i}) \leq 1/2^i, \quad \sigma(x,y) = 1 \text{ otherwise} \quad (x \neq y)$$

(see the definition of $\mathcal{D}$ and $\mathcal{D}'$). Then for each $x \in N-A$, the set $A_x = \{y \in A;\ \sigma(x,y) < 1/2\}$ contains at most two points, namely a_{2i-1}, a_{2i} for some i.

As the uniformity $\mathcal{Q}/Q-N$ has no isolated points, following 2.2 there exists a complement $\mathcal{V}$ to it in $\nu(Q-N)$ which admits a uniform homeomorphism $\alpha : (A, \mathcal{U}/A) \longrightarrow (Q-N, \mathcal{Q}/Q-N)$. Let us extend α to a mapping from N to Q as follows. Write N-A as a (finite or infinite) sequence, $N-A = \{x_0, x_1, x_2 \dots\}$

and define $\alpha(x_i)$ by induction in such a way that, for every i,

(10) $\alpha(x_i) \in N$, $\alpha(x_i) > \alpha(x_{i-1})$,

(11) $|\alpha(x_i) - \alpha(y)| \geq 1$ for every $y \in A_{x_i}$

(this is possible because the A_x's are finite). By (10), α is 1-i and (11) can be reformulated as

(12) $x \in N-A$, $y \in N$, $\sigma(x,y) < 1/2 \Rightarrow |\alpha(x) - \alpha(y)| \geq 1$.

Let $\mathcal{W}$ be the uniformity on $\alpha(N)$ such that α is a uniform homeomorphism from $(N,\mathcal{U})$ to $(\alpha(N),\mathcal{W})$. Then $\mathcal{W}/Q-N = \mathcal{V}$ and so $(\mathcal{W}/Q-N) \vee (\mathcal{Q}/Q-N) = \underline{1}$ in $\nu(Q-N)$. As Q-N is dense in $\alpha(N)$, also $\mathcal{W} \vee \mathcal{Q}/\alpha(N) = \underline{1}$ in $\nu(\alpha(N))$. Further $(\mathcal{W}/Q-N) \wedge (\mathcal{Q}/Q-N) = \underline{0}$ in $\nu(Q-N)$; it follows by (12) that $\mathcal{W} \wedge \mathcal{Q}/Q-N = \underline{1}$ in $\nu(\alpha(N))$. Hence $(\alpha(N), \mathcal{Q}/\alpha(N))$ admits a complement and so does $(Q,\mathcal{Q})$ because $\alpha(N)$ is dense in $(Q,\mathcal{Q})$, see 1.7.

3. Main results

3.1. Countable case. There arises a natural problem: Characterize those uniformities $\mathcal{U}$ on X, X countable, which have a complement in $\nu(X)$. A partial solution is given by the following theorem.

Let $\mathcal{U}$ be a metrizable uniformity on a countable set X. Let $(C,\mathcal{C})$ be the subspace of $(X,\mathcal{U})$ such that C is the set of all non-isolated points in $(X,\mathcal{U})$. Then we have:

Theorem. $\mathcal{U}$ has a complement iff at least one of the following conditions is fulfilled.

(i) $(X,\mathcal{U})$ admits two disjoint uniformly discrete subspaces which are proximal,

(ii) C <u>is infinite and</u> $\mathcal{C} \simeq \mathcal{S}$. (For $\mathcal{S}$ see 1.5.)

Proof. Sufficiency. Let $\mathcal{U}$ be induced by a metric ρ. If $(X,\mathcal{U})$ fulfils (i) then $\mathcal{U}$ has a complement by 2.5. Suppose (i) fails to be true. Let (ii) hold. Then there are two infinite subsets A, B of C, a point $z \in C$ and $\varepsilon > 0$ such that ε-neighborhoods of A, B, $\{z\}$ respectively are disjoint and, in addition, $X - (A \cup B \cup \{z\})$ is dense in $(X,\mathcal{U})$. Let $\{z_i\}$ be an 1-1 sequence converging to z which is contained in the ε-neighborhood of z. Let $\{(x_i,k_i)\}$ be the sequence of all couples (x,k) where $x \in A \cup B$ and k is a positive integer with $1/k < \varepsilon$. By means of induction, define an 1-1 sequence $\{z_i'\}$ in $X - (A \cup B)$ such that $\rho(z_i',x_i) < 1/k_i$. Further, let $\{a_j\}$ be a sequence of all points in $X - (A \cup B \cup \{z_i\})$. As the set $\{y \in A \cup B;\ \rho(a_j,y) > \varepsilon\}$ is infinite for every j, we can define an 1-1 sequence $\{b_j\}$ im $A \cup B$ such that

(1) $\rho(a_j,b_j) > \varepsilon$,

(2) if a_j is in the ε-neighborhood of A (of B) then $b_j \in B$ ($b_j \in A$, resp.).

Define a new pseudometric σ on X by

$\sigma(a_j,b_j) = 0$,

$\sigma(z_i,z_i') = 0$ (so that also $\sigma(b_j,z_i) = 0$ if $a_j = z_i'$),

$\sigma(x,y) = 1$ otherwise $(x \neq y)$.

Let $\mathcal{V}$ be the uniformity induced by σ . As $\rho(x,y) + \sigma(x,y) > \varepsilon$ for $x \neq y$, we have $\mathcal{U} \wedge \mathcal{V} = \underline{0}$. To prove $\mathcal{U} \vee \mathcal{V} = \underline{1}$, we use the lemma 2.3: if $x,y \in X$, there are $x',y' \in A \cup B$ with $\sigma(x,x') = \sigma(y,y') = 0$; consider chains $x,x',z_i',z_i,z_j,z_j',y',y$ where i, j are chosen such that $x_i = x'$, $x_j = y'$, k_i, k_j are sufficiently large and $\rho(z_i,z_j)$ sufficiently small.

Necessity. Suppose neither (i) nor (ii) holds. Let $\mathcal{U}$ have a complement $\mathcal{V}$; by 1.8 we may assume that $\mathcal{V}$ is induced by a pseudometric σ . Thus, there is $\varepsilon > 0$ such that $\rho(x,y) + \sigma(x,y) > 2\varepsilon$ for $x \neq y$. As C is finite or $\mathcal{C} \simeq \mathcal{S}$, there is $E \subset X$ with ρ-diameter $< \varepsilon$ such that C-E is finite and such that the set I of all points of E which are isolated in $(X, \mathcal{U})$ is infinite. If $x,y \in E$ then $\rho(x,y) < \varepsilon$ and so $\sigma(x,y) > \varepsilon$. Thus, denoting $C_x = \{z \in C;\ \sigma(x,z) = 0\}$ for $x \in I$, we have $C_x \subset C-E$ and $C_x \cap C_y = \emptyset$ for $x \neq y$. As C-E is finite, $C_x = \emptyset$ for all $x \in I$ but a finite number. This contradicts the following lemma.

Lemma. *Let ρ be a metric on X which induces a uniformity $\mathcal{U}$ such that $(X, \mathcal{U})$ does not admit two disjoint uniformly discrete subspaces which are proximal. Let σ be a pseudometric on X which induces a complement $\mathcal{V}$ to $\mathcal{U}$. Then for each point x which is isolated in $(X, \mathcal{U})$ there is a non-isolated point c in $(X, \mathcal{U})$ such that $\sigma(x,c) = 0$.*

Proof of Lemma. Let τ be the largest pseudometric on X with $\tau \leq \rho$ and $\tau \leq \sigma$. Thus $\tau(x,y)$ is the infimum of all numbers

$$\sum_{i=1}^{n} \min\,(\rho(x_{i-1},x_i),\ \sigma(x_{i-1},x_i))$$

where $x_0,\dots,x_n \in X$ and $x_0 = x$, $x_n = y$. As $\mathcal{U} \vee \mathcal{V} = \underline{1}$, we have $\tau = 0$. Hence if x is isolated in $(X, \mathcal{U})$ and $y \in X$ is such that $(x,y) > 0$, there are finite 1-1 sequences

$$s_0 = \{x_0^0,\dots,x_{k_0}^0\},$$

$$s_1 = \{x_0^1,\dots,x_{k_1}^1\},$$

$$\vdots$$

$$s_n = \{x_0^n,\dots,x_{k_n}^n\}$$

$$\vdots$$

where $x_0^n = x$ and $x_{k_n}^n = y$, $k_n > 1$, $x_i^n \neq x_{i+1}^n$ (for $i < k_n$) for every n, and, in addition,

$$\sigma(x_0^n, x_1^n) + \rho(x_1^n, x_2^n) + \sigma(x_2^n, x_3^n) + \ldots < 1/n, \quad n=0,1,\ldots .$$

Replacing $\{s_n\}$ by a suitable subsequences, if necessary, we may assume that the sequence $\{x_1^n\}$ is

(1) 1-1 and uniformly discrete in $(X, \mathcal{U})$, or

(2) 1-1 and Cauchy in $(X, \mathcal{U})$, or

(3) constant, say $x_1^n = c$ for every n.

The case (1) is impossible because the subspaces $\{x_1^n\}$, $\{x_2^n\}$ would be uniformly discrete and proximal in $(X, \mathcal{U})$. In case (2), the sequence $\{x_1^n\}$ would be Cauchy both with respect to $\mathcal{U}$ and to $\mathcal{V}$, a contradiction because $\mathcal{U} \wedge \mathcal{V} = \underline{0}$. Thus, case (3) takes place: we have $\sigma(x,c) = 0$ and c is non-isolated in $(X, \mathcal{U})$ because $\{x_2^n\}$ converges to c.

3.2. Larger cardinalities. Let us discuss the possibility of generalizations of the above results to larger cardinalities.

First observe that 1.6, Claim a can be easily generalized to infinite sums; this yields, for any set X, a certain class of uniformities in $\nu(X)$ which have complements.

Further, the proof of necessity in 3.1 did not use the fact that X is countable. Hence we have:

Proposition. If $(X, \mathcal{U})$ is an arbitrary uniform space such that $\mathcal{U}$ has a complement in $\nu(X)$ then either (i) or (ii) of 3.1 holds.

This, together with 1.7, implies:

Remark. Theorem 3.1 is valid for any separable metrizable uniformity $\mathcal{U}$.

Taking into account that an uncountable separable space has necessarily uncountably many non-isolated points, we get:

Corollary. Every separable metrizable uniformity on an uncountable set X has a complement in $\nu(X)$.

3.3. Remark. Let us notice that the usual functors from (uniformizable) topologies to uniformities and vice versa do not preserve complementation (e.g. the converging (Cauchy) sequence with its limit point does not have a uniform complement, nevertheless it has a topological complement which is even uniformizable). Moreover, constructions of topological complements lead often to non-uniformizable topologies, and, vice versa, uniform complements are often topologically trivial. Hence we have not discovered any reasonable relation between complementation in topologies and that in uniformities.

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