COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

22,2 (1981)

MONOTONIC VALUATIONS AND VALUATIONS OF TRIADS OF HIGHER TYPES

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Abstract: This paper is a contribution to the mathematics in the alternative set theory. In [M2], we introduced the problems of valuations of special structures, so called triads. E.g. we can deal with equivalences, ideals and filtres as triads.

Our problems consist in finding a simple representation (valuation) of triads in special, "numerical" ones. The key role is played here by the theorem on valuations of $\sigma^{\mathfrak{M}}$- and $\pi^{\mathfrak{M}}$-triads, which has been presented in [M2]. In this paper, we define a stronger variant of the notion of valuation (the so called monotonic valuation) and we state the basic theorems on monotonic valuation of $\sigma^{\mathfrak{M}}$- and $\pi^{\mathfrak{M}}$-triads. Furthermore, we define triads of higher types, i.e. $\sigma\pi^{\mathfrak{M}}$- and $\pi\sigma^{\mathfrak{M}}$-triads, and present some results about their valuation.

Key words: Alternative set theory, valuation, monotonic valuation, matrix of classes, $\sigma\pi^{\mathfrak{M}}$-class, $\pi\sigma^{\mathfrak{M}}$-class.

Classification: 02K10, 02K99, 08A05

Introduction. In § 1, we study monotonic valuations of triads. We present there some consequences for ideals and filters. In § 2, matrices of classes and $\sigma\pi^{\mathfrak{M}}$- and $\pi\sigma^{\mathfrak{M}}$-classes are defined and their basic properties stated. Results concerning valuations of triads of higher types are presented in § 3.

§ 0. Preliminaries

0.0.0. We use usual definitions and notations of the alternative set theory. The class of natural numbers (finite natural numers resp.) is denoted by N (FN resp.). We use $\alpha, \beta, \gamma, \delta, \xi, \vartheta$ (m,n,i,j,k resp.) as variables ranging over natural (finite natural numbers resp.). We shall use lower-case letters to denote sets. RN is the class of rational numbers and we put $RN(\geq 0) = \{x \in RN; x \geq 0\}$, $RN(>0) = \{x \in RN; x > 0\}$. $(0,1]$ denotes the interval $\{x \in RN; 0 < x \leq 1\}$. The identity mapping is designated by Id.

0.0.1. A codable class $\mathfrak{M}$ is called a standard system iff (1) $V \subseteq \mathfrak{M}$, (2) let $\varphi(x)$ be a normal formula of the language $FL_{\mathfrak{M}}$. (For $FL_{\mathfrak{M}}$, see 0.1.0. in [M1].) Then $\{x; \varphi(x)\} \in \mathfrak{M}$. (3) Let $X \in \mathfrak{M}$ be a class such that $0 \neq X \subseteq N$. Then there exists the least element of X. Throughout this paper let $\mathfrak{M}$ denote a standard system. A string is a relation R with $dom(R) \in N$. A string R is a σ-string iff $R''\{\alpha\} \subseteq R''\{\alpha+1\}$ holds for each $\alpha+1 \in dom(R)$. A class X is called $\sigma^{\mathfrak{M}}$- ($\pi^{\mathfrak{M}}$-resp.) class iff there is a string $R \in \mathfrak{M}$ such that $X = \bigcup_n R''\{n\}$ ($X = \bigcap_n R''\{n\}$ resp.). X is a σ- (π- resp.) class iff X is a union (intersection resp.) of countable sequence of set-theoretically definable classes. X is a $\sigma\pi$- ($\pi\sigma$- resp.) class iff X is a union (intersection resp.) of countable sequence of π-classes (σ-classes resp.).

0.0.2. An e-structure is a structure $\mathcal{A} = \langle A, F, E \rangle$ where F is a binary function, E is a unary function and we have (1) F is associative on A, (2) $E \circ E = Id$ and (3) $F(E(x),E(y)) = E(F(x,y))$ holds for each $x,y \in A$ or $F(E(x),E(y)) = E(F(y,x))$

holds for each $x,y \in A$. We define the canonical relation $\lhd_a$ of $\mathcal{A}: x \lhd_a y \equiv (\exists z \in A)(F(x,z)=y)$. If there is no danger of confusion, we shall write simply $\lhd$ instead of $\lhd_a$. The relation $\lhd$ is transitive on A.

Example. We define the mapping $F^0 : (V^2 \cup \{0\})^2 \longrightarrow V^2 \cup \{0\}$ as follows: $F^0(\langle x,y\rangle,\langle u,v\rangle) = \langle x,v\rangle$ (0 resp.) iff $y = u$ ($y \neq u$ resp.) and $F^0(w,0) = F^0(0,w) = 0$ for each $w \in V^2 \cup \{0\}$. $\langle V^2 \cup \{0\}, F^0, Id\rangle$ is an e-structure.

0.0.3. Let $\mathcal{A} = \langle A,F,E\rangle$ be an e-structure and let $Q,B \subseteq A$ be classes closed under F and E. We denote by $\mathcal{A}|Q$ the restriction of the structure $\mathcal{A}$ on Q. The triple $\langle \mathcal{A}, \mathcal{A}|Q, \mathcal{A}|B\rangle$ is called a triad over $\mathcal{A}$. Let $\mathcal{A}(Q,B)$ designate this triad. A $\sigma^{\mathfrak{M}}$- ($\pi^{\mathfrak{M}}$- resp.) triad is a triad $\mathcal{A}(Q,B)$ such that $\mathcal{A}$, $B \in \mathfrak{M}$ and Q is a $\sigma^{\mathfrak{M}}$-class ($\pi^{\mathfrak{M}}$-class resp.).

Convention. We shall write σ^0 (π^0 resp.) instead of σ^{Sd_V} (π^{Sd_V} resp.) where Sd_V is the standard system of all set-theoretically definable classes.

Examples. $\langle N,+,Id\rangle$ (FN,$\{0\}$) is a σ^0-triad, $\langle RN(\geq 0),+, Id\rangle$ ($[\geq 0]$,$\{0\}$) is a π^0-triad. (We put $[\geq 0] = \{x \in RN(\geq 0); x \doteq 0\}$ where $x \doteq y \equiv (\forall n)\ (|x-y| < \frac{1}{n} \vee (x > n \,\&\, y > n) \vee (x < -n \,\&\, y < -n))$.)

0.0.4. Let $\mathcal{A} = \langle A,F,E\rangle$, $\tilde{\mathcal{A}} = \langle \tilde{A},\tilde{F},\tilde{E}\rangle$ be e-structures. A mapping $H: A \longrightarrow \tilde{A}$ is called valuation of $\mathcal{A}$ in $\tilde{\mathcal{A}}$ iff for each $x,y \in A$ holds: $H(F(x,y)) \lhd_{\tilde{a}} \tilde{F}(H(x),H(y))$ and $H(E(x)) = \tilde{E}(H(x))$. Let $\mathcal{A}(Q,B)$, $\tilde{\mathcal{A}}(\tilde{Q},\tilde{B})$ be triads. A mapping $H: A \longrightarrow \tilde{A}$ is called valuation of $\mathcal{A}(Q,B)$ in $\tilde{\mathcal{A}}(\tilde{Q},\tilde{B})$ iff H is a valuation of $\mathcal{A}$ in $\tilde{\mathcal{A}}$ and we have for each $x \in A: x \in Q \equiv H(x) \in \tilde{Q}$ and $x \in B \equiv H(x) \in \tilde{B}$.

0.1.0. Let $k \in FN$. Let, for each $i \leq k$, R_i be an $a(i)+1$-ary relation, $a(i) \in FN$. We denote by $[\![R_1]\!]_k(X,Y)$ the formula

$R_0''X^{a(0)} \subseteq Y \& \ldots \& R_k''X^{a(k)} \subseteq Y$.

0.1.1. Let $\mathcal{A} = \langle A,F,E\rangle$ be an e-structure. We define the mapping $F_3 : A^3 \longrightarrow A$ as follows: $F_3(x,y,z) = F(F(x,y),z)$. A σ-string R is called <u>σ-string in $\mathcal{A}$ over</u> B iff $R''\{0\} = B$, $R''\{dom(R)-1\} = A$ and $[\![F,F_3]\!](R''\{\alpha\}, R''\{\alpha+1\})$, $E''R''\{\alpha\} \subseteq R''\{\alpha\}$ holds for each $\alpha \in dom(R)-1$.

0.1.2. An e-structure $\langle A,F,E\rangle$ <u>is commutative</u> iff F is a commutative mapping on A.

§ 1. <u>Monotonic valuations</u>

1.0.0. Let $\mathcal{A}$ be an e-structure and let $Q \subseteq A$. Q is <u>closed in $\mathcal{A}$</u> iff

$$(\forall x \in A)(\forall y \in Q)(x \lhd_a y \longrightarrow x \in Q).$$

A triad $\mathcal{A}(Q,B)$ is called <u>closed triad</u> iff Q, B are closed in $\mathcal{A}$.

<u>Examples</u>. (1) Let $\mathcal{A} = \langle A^2 \cup \{0\}, F^0, G\rangle$ be an e-structure and let $0 \neq R \subseteq A^2$. R is closed in $\mathcal{A}$ iff $R = rng(R) \times A$.

(2) Suppose, moreover, that A has at least two points. Then no relation R, $0 \neq R \subseteq A^2$, is a closed in $\mathcal{A}$ and closed under F^0.

(3) Let $Q \subseteq P(a)$ be an ideal. Then Q is a closed universe in $\langle P(a), \cup, Id\rangle$.

(4) $\langle N,+,Id\rangle$ $(FN,\{0\})$, $\langle RN(\geq 0),+,Id\rangle$ $([\geq 0],\{0\})$ are closed triads.

1.0.1. A valuation H of an e-structure $\mathcal{A}$ in an e-structure $\tilde{\mathcal{A}}$ is called <u>monotonic valuation of $\mathcal{A}$ in $\tilde{\mathcal{A}}$</u> iff we have $(\forall x,y \in A)(x \lhd_a y \longrightarrow H(x) \lhd_{\tilde{a}} H(y))$. A valuation of a triad $\mathcal{A}(Q,B)$ in a triad $\tilde{\mathcal{A}}(\tilde{Q},\tilde{B})$ is called <u>monotonic</u> iff it is a monotonic valuation of $\mathcal{A}$ in $\tilde{\mathcal{A}}$.

Examples. (1) We put, for each $\alpha \in N$, $G(\alpha) = 2^{\alpha}$. G is a monotonic valuation of $\langle N,+,Id\rangle$ $(FN,\{0\})$ in $\langle RN(>0),\cdot, Id\rangle$ $(BRN(>0),\{1\})$.

(2) We put, for each $x \in RN(\geq 0)$, $G(x) = 2^{-x}$. We have $G: :RN(\geq 0) \rightarrow (0,1]$ and G is a monotonic valuation of $\langle RN(\geq 0), +,Id\rangle$ $(BRN(\geq 0),\{0\})$ in $\langle (0,1],\cdot,Id\rangle$ $((0,1],\{1\})$.

Proposition. Let G be a valuation of a triad $\mathcal{T}$ in a triad $\tilde{\mathcal{T}}$ and let H be a monotonic valuation of $\tilde{\mathcal{T}}$ in a triad $\hat{\mathcal{T}}$. Then $H \circ G$ is a valuation of $\mathcal{T}$ in $\hat{\mathcal{T}}$. If, moreover, G is a monotonic valuation then $H \circ G$ is a monotonic one.

Proof follows immediately from the definitions.

Proposition. Let H be a monotonic valuation of a triad $\mathcal{T}$ in a closed triad $\hat{\mathcal{T}}$. Then $\hat{\mathcal{T}}$ is a closed triad.

Proof. Let $\mathcal{T} = \mathcal{A}(Q,B)$, $\hat{\mathcal{T}} = \hat{\mathcal{A}}(\hat{Q},\hat{B})$. Assume that $y \in Q$ and $x \in A$, $x \lhd y$. Then $H(x) \lhd H(y)$ and $H(y) \in \hat{Q}$ hold. Thus $H(x) \in \hat{Q}$ and, consequently, $x \in Q$.

1.1.0. Let $\mathcal{A} = \langle A,F,E\rangle$ be an e-structure. A structure $\langle A,F,E,G\rangle$ is called a u-expansion of $\mathcal{A}$ iff G is a binary function and we have for each $x,y,z \in A$: (1) $F(G(x,z),G(y,z)) = G(F(x,y),z)$ (distributivity)

(2) $x \lhd y \rightarrow G(y,x) = x$

(3) $G(x,y) \lhd x$.

Example. $\langle P(a),\cup,Id,\cap\rangle$ is a u-expansion of $\langle P(a),\cup,Id\rangle$.

Theorem. (On monotonic valuation of $\sigma^{\mathfrak{M}}$- and $\pi^{\mathfrak{M}}$-triads.) Let $\mathcal{A}$ be a commutative e-structure, $\mathcal{A} \in \mathfrak{M}$, and suppose that $\mathcal{A}$ has a u-expansion in $\mathfrak{M}$.

(1) Let $\mathcal{T}$ be a closed $\sigma^{\mathfrak{M}}$-triad over $\mathcal{A}$. Then there is a monotonic valuation H of $\mathcal{T}$ in $\langle N,+,Id\rangle$ (FN,$\{0\}$) and $H \in \mathfrak{M}$.

(2) Let $\mathcal{T}$ be a closed $\pi^{\mathfrak{M}}$-triad over $\mathcal{A}$. Then there is a monotonic valuation H of $\mathcal{T}$ in $\langle RN(\geq 0),+,Id\rangle([\geq 0]$, $\{0\})$ and $H \in \mathfrak{M}$.

At first we shall prove one lemma. Note yet that writing $y \trianglelefteq x$ we mean the relation $y \lhd x \vee y = x$.

<u>Lemma</u>. Let $\mathcal{A} = \langle A,F,E\rangle$ be a commutative e-structure and let $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq A$ be classes such that $[\![F,E]\!](A_i A_{i+1})$ and $\lhd'' A_i \subseteq A_{i+1}$ hold for i = 0,1,2. We put, for i = 1,2, $P_i = \trianglelefteq''(A_i \cap E''A_i)$. Then we have $P_1 \subseteq P_2$, $F''P_1^2 \subseteq P_2$ and, for i = = 0,1,

$$A_1 \subseteq P_{i+1} \subseteq A_{i+1},\ E''P_{i+1} \subseteq P_{i+1},\ \trianglelefteq''P_{i+1} \subseteq P_{i+1}.$$

Proof. We deduce from the commutativity of $\mathcal{A}$ that the following relations hold: $x \trianglelefteq y \rightarrow E(x) \trianglelefteq E(y)$, $x \trianglelefteq \bar{x} \,\&\, y \trianglelefteq \bar{y} \rightarrow F(x,y) \trianglelefteq F(\bar{x},\bar{y})$. Thus, we have $E''P_{i+1} \subseteq P_{i+1}$, i=0,1. By using the transitivity of $\trianglelefteq$ we deduce that $\trianglelefteq''P_{i+1} \subseteq P_{i+1}$, i = = 0,1. The relation $P_1 \subseteq P_2$ is clear. We put, for i=0,1,2,3, $Q_i = A_i \cap E''A_i$. We have, for i=0,1,2: $Q_i \subseteq A_i \subseteq Q_{i+1} \subseteq A_{i+1}$, $F''Q_i^2 \subseteq Q_{i+1}$ and $E''Q_i \subseteq Q_i$. We shall prove that $\lhd''Q_i \subseteq Q_{i+1}$ holds for i=0,1,2. Assume $y \lhd x$ and $x \in Q_i$ (i=0,1,2). We have $\lhd''A_i \subseteq A_{i+1}$ and, consequently, $y \in A_{i+1}$ holds. We have $E(x) \in Q_i$ and, obviously, $E(x) \in A_i$. We obtain from this that $E(y) \in A_{i+1}$ and, finally, $y \in A_{i+1} \cap E''A_{i+1}$. We deduce from the relations above that $A_i \subseteq Q_i \subseteq P_i \subseteq Q_{i+1} \subseteq A_{i+1}$ holds for i=0,1. We shall prove $F''P_1^2 \subseteq P_2$. Assume that $x,y \in P_1$, $\bar{x},\bar{y} \in Q_1$ and $x \trianglelefteq \bar{x}$, $y \trianglelefteq \bar{y}$ hold. We have $F(\bar{x},\bar{y}) \in Q_2$ and $F(x,y) \trianglelefteq F(\bar{x},\bar{y})$.

We deduce from this that $F(x,y) \in P_2$.

Proof of the theorem. We have $[\![F,E,\lhd]\!](Q,Q)$ and $[\![F,E,\lhd]\!](B,B)$. Let S be a σ-string of Q, $S \in \mathfrak{M}$, such that $dom(S) \notin FN$ and $B \subseteq S(0) \subseteq S(\alpha) \subseteq A$, $[\![F,E,\lhd]\!](S(\alpha), S(\alpha+1))$ hold for each $\alpha+1 \in dom(S)$. The existence of the S follows from 2.1.0 in [M1]. We put, for each $\alpha \in dom(S)$, $\alpha \geq 1$: $P_\alpha = \unlhd''(S(\alpha) \cap E''S(\alpha))$. We deduce from the previous lemma that the following hold for each $\alpha < dom(S)-2$:

$$S(\alpha) \subseteq P_{\alpha+1} \subseteq S(\alpha+2),\ F''P^2_{\alpha+1} \subseteq P_{\alpha+2},\ E''P_{\alpha+1} \subseteq P_{\alpha+1},\ \unlhd''P_{\alpha+1} \subseteq P_{\alpha+1}.$$

Thus $\cup P_n = Q$ holds. Let $\vartheta \in \frac{1}{4}dom(S)-FN$. We put

$$\langle x,\alpha\rangle \in M \equiv (\alpha=0 \,\&\, x \in B) \vee (1 \leq \alpha < \vartheta \,\&\, x \in \unlhd''(S(2\alpha) \cap E''S(2\alpha))) \vee (\alpha = \vartheta \,\&\, x \in A).$$

We have $M \in \mathfrak{M}$ and, moreover, $M''\{\alpha\} = P_{2\alpha}$, $1 \leq \alpha < \vartheta$, and $M''\{0\} = B$, $M''\{\vartheta\} = A$. Thus the following propositions hold: (i) $M \in \mathfrak{M}$ is a σ-string in $\mathcal{A}$ over B, (ii) $\lhd''M(\alpha) \subseteq M(\alpha)$ for each $\alpha \leq \vartheta$, (iii) $\bigcup_n M(n) = Q$.

A path in A is a function t such that $dom(t) \in N$ and $rng(t) \subseteq A$. Let t be a path in A. We put $\mathcal{V}_M(t) = \sum\{G^*(x); x \in rng(t)\}$, where $G^*: A \to N$ is the function defined as follows: $G^*(x) = 0$ iff $x \in B$ and $G^*(x) = 2^{\min\{\alpha \leq \vartheta; x \in M(\alpha)\}}$ iff $x \in A - B$. We define the function [F] with the domain $\cup\{\{t\} \times \{\langle\alpha,\beta\rangle; \alpha \leq \beta \,\&\, \beta \in dom(t)\}; t \text{ is a path in } A\}$ by induction over N: $[F](t,\langle\alpha,\alpha\rangle) = t(\alpha)$ and $[F](t,\langle\alpha,\beta+1\rangle) = F([F](t,\langle\alpha,\beta\rangle), t(\beta+1))$. Writing [F](t) we mean $[F](t,\langle 0, dom(t)-1\rangle)$. We have proved in [M2], 3.0.2 that the function $H: A \to N$ such that $H(x) = \min\{\mathcal{V}_M(t); [F](t) = x\}$ is a valuation of $\mathcal{A}(Q,B)$ in $\langle N,+,Id\rangle(FN,\{0\})$ and $H \in \mathfrak{M}$. We shall prove that H is a monotonic valuation of $\mathcal{A}$ in $\langle N,+,Id\rangle$. Let $\langle A,F,E,G\rangle$ be a u-expansion of $\mathcal{A}$ in $\mathfrak{M}$. Let $x \lhd y$ and suppose that t is a path in A such that $[F](t) = y$. We define the path t^x in A with $dom(t^x) = dom(t)$: $t^x(\alpha) = G(t(\alpha),x)$ for each $\alpha \in dom(t)$. The following holds:

(i) $\alpha \in dom(t) \longrightarrow [F](t^x,0,\alpha) = G([F](t,0,\alpha),x)$.

Remark. We write here and $[F](t,\alpha,\beta)$ instead of $[F](t,\langle\alpha,\beta\rangle)$.

Proof of (i). By induction. $t^x(0) = G(t^x(0),x)$. Thus $[F](t,0,0) = G([F](t,0,0),x)$ holds. Suppose that the statement holds for α and let $\alpha+1 \in dom(t)$. We have

$[F](t^x,0,\alpha+1) = F([F](t^x,0,\alpha),t^x(\alpha+1)) = F([F](t^x,0,\alpha),$
$G(t(\alpha+1),x)) = F(G([F](t,0,\alpha),x),G(t(\alpha+1),x)) =$
$= G(F([F](t,0,\alpha),t(\alpha+1)),x) = G([F](t,0,\alpha+1),x)$.

(ii) $[F](t^x) = x$.

Proof: $[F](t^x) = G([F](t),x) = G(y,x) = x$.

(iii) $(\forall\alpha \in dom(t))(t^x(\alpha) \lhd t(\alpha))$.

Proof: We have $t^x(\alpha) = G(t(\alpha),x)$ and $G(t(\alpha),x) \lhd t(\alpha)$.

Only the following must be proved: $x \lhd y \longrightarrow H(x) \leq H(y)$. Suppose that $[F](t) = y$ and let $\mathcal{V}_M(t) = H(y)$. Let $\alpha \in dom(t)$ and let $\gamma \leq$ be such that $t(\alpha) \in M(\gamma)$. We have $\lhd''M(\gamma) \subseteq M(\gamma)$. We obtain from this and by using (iii) that $t^x(\alpha) \in M(\gamma)$. We deduce from this that $\mathcal{V}_M(t^x) \leq \mathcal{V}_M(t)$. Thus we have $H(x) \leq \mathcal{V}_M(t^x) \leq \mathcal{V}_M(t) = H(y)$.

The statement (2) can be proved similarly.

1.1.1. We say that a function $H:P(a) \longrightarrow RN(\geq 0)$ ($H: P(a) \longrightarrow (0,1]$ resp.) is an <u>additive</u> (<u>multiplicative</u> resp.) <u>function on</u> a iff we have for each $x,y \in P(a)$: (1) $H(x \cup y) \leq H(x) + H(y)$, (2) $x \subseteq y \longrightarrow H(x) \leq H(y)$, ((1) $H(x \cap y) \geq$

$\geq H(x)\cdot H(y)$, (2) $x \leq y \rightarrow H(x) \leq H(y)$ resp.).

We say that an ideal Q (a filter Q resp.) on a is <u>determined by an additive</u> (<u>multiplicative</u> resp.) <u>function</u> H on a iff H is a monotonic valuation of $\langle P(a), \cup, Id\rangle$ (Q,0) in $\langle RN(\geq 0), +, Id\rangle$ ($[\geq 0]$,0) ($\langle P(a), \cap, Id\rangle$ (Q, 0) in $\langle (0,1], \cdot, Id\rangle$ ($[\leq 1]$,0) resp.). (We put $[\leq 1] = (0,1] \cap [1]$.) We deduce from the previous theorem that the following holds:

<u>Theorem</u>. (1) Let Q be an ideal on a which is a π-class. Then Q is determined by an additive set-function on a.

(2) Let Q be a filter on a which is a π-class. Then Q is determined by a multiplicative set-function on a.

<u>Remark</u>. The following assertion holds: let a be an infinite set. Then there is an ideal Q on a such that Q is a π-class and there is no function $v: a \rightarrow RN(\leq 0)$ so that $Q = \{u \subseteq a; \sum\{v(x); x \in u\} \doteq 0\}$.

§ 2. <u>Matrix of classes, $\sigma\pi^{\mathfrak{M}}$ - and $\pi\sigma^{\mathfrak{M}}$-classes</u>

2.0.0. R is called a <u>matrix in A of the type</u> ξ ($\xi \in N$) iff R is a relation with $dom(R) = \xi^2$ and we have for each $\alpha, \beta \in \xi$: $R''\{\langle\alpha,\beta\rangle\} \subseteq A$. We shall write simply $R(\alpha,\beta)$ instead of $R''\{\langle\alpha,\beta\rangle\}$.

Let R be a matrix in A of the type ξ. We put

$$\mathcal{H}_{\sigma\pi}(R) = \bigcup_m \bigcap_n R(m,n)$$

$$\mathcal{H}_{\pi\sigma}(R) = \bigcap_m \bigcup_n R(m,n)$$

$A \ominus R$ is a matrix in A of the type ξ so that $A \ominus R(\alpha,\beta) = A - R(\alpha,\beta)$ holds for each $\alpha, \beta \in \xi$. R is a <u>$\sigma\pi$-matrix</u> in A of the type ξ iff $R(\alpha+1,\beta) \supseteq R(\alpha,\beta)$ holds for each

$\alpha+1, \beta \in \xi$ and $R(\alpha,\beta) \supseteq R(\alpha,\beta+1)$ holds for each $\alpha, \beta+1 \in \xi$. R is a $\pi\sigma$ -matrix in A of the type ξ iff $A \ominus R$ is a $\sigma\pi$ -matrix in A of the type ξ.

2.0.1 Convention. Throughout this paper let ξ be an infinite natural number (i.e. $\xi \in N-FN$). By a matrix we mean a matrix of the type ξ.

Proposition. (1) Let R be a $\sigma\pi$ -matrix in A. Then $\mathcal{H}_{\sigma\pi}(R) = \bigcup_{\beta \in \xi - FN} \bigcup_{n} R(m,\beta)$.

(2) Let R be a $\pi\sigma$-matrix in A. Then $\mathcal{H}_{\pi\sigma}(R) = \bigcap_{\beta \in \xi - FN} \bigcap_{m} R(m,\beta)$.

2.0.2. Let R be a matrix in A. We denote by $R^{\sigma\pi}$ ($R^{\pi\sigma}$ resp.) a matrix in A so that the following holds for each $\alpha, \beta \in \xi$:

$\langle x,\langle\alpha,\beta\rangle\rangle \in R^{\sigma\pi} \equiv x \in \bigcup_{\gamma\le\alpha} \bigcap_{\delta\le\beta} R(\gamma,\delta)$ ($\langle x,\langle\alpha,\beta\rangle\rangle \in R^{\pi\sigma} \equiv$ $\equiv x \in \bigcap_{\gamma\le\alpha} \bigcup_{\delta\le\beta} R(\gamma,\delta)$ resp.).

Proposition. Let M be a matrix in A. (1) $(A \ominus M)^{\sigma\pi} = A \ominus M^{\pi\sigma}$, (2) $\mathcal{H}_{\sigma\pi}(M) = \mathcal{H}_{\sigma\pi}(M^{\sigma\pi})$, (3) $\mathcal{H}_{\pi\sigma}(M) = \mathcal{H}_{\pi\sigma}(M^{\pi\sigma})$.

Proof. (1) $(A \ominus M)^{\sigma\pi}(\alpha,\beta) = \bigcup_{\gamma\le\alpha} \bigcap_{\delta\in\beta} A - M(\gamma,\delta) = A - \bigcap_{\gamma\le\alpha} \bigcup_{\delta\le\beta} M(\gamma,\delta) = A - M^{\pi\sigma}(\alpha,\beta)$. (2) $M^{\sigma\pi}(\alpha,\beta) = \bigcup_{\gamma\le\alpha} \bigcap_{\delta\le\beta} M(\gamma,\delta)$. We put $\tilde{M}(\gamma,\beta) = \bigcap_{\delta\le\beta} M(\gamma,\delta)$. We have $\tilde{M}(\gamma,\delta+1) \subseteq \tilde{M}(\gamma,\delta)$, $\mathcal{H}_{\sigma\pi}(M) = \bigcup_{m} \bigcap_{n} \tilde{M}(m,n) = \mathcal{H}_{\sigma\pi}(\tilde{M})$ and $M^{\sigma\pi}(\alpha,\beta) \supseteq \tilde{M}(\alpha,\beta)$. Thus $\mathcal{H}_{\sigma\pi}(M^{\sigma\pi}) \supseteq \mathcal{H}_{\sigma\pi}(\tilde{M})$ holds. Suppose that $x \in \bigcap_{j} M^{\sigma\pi}(m,j)$. We have $x \in \bigcap_{j} \bigcup_{i\le m} M(i,j)$. Let F be a function, $F: FN \to m+1$, such that $x \in M(F(j),j)$ holds for each $j \in FN$. By using the axiom of prolongation we obtain a number $i_0 \le m$ so that $F(j) = i_0$ holds for infinitely many values of j. Thus $x \in M(i_0,j)$ holds for infinitely many values of j. We deduce from this that $x \in \bigcap_j M(i_0,j)$ and $x \in \mathcal{H}_{\sigma\pi}(M)$ holds. We obtain immediately that $\bigcap_j M^{\sigma\pi}(m,j) \subseteq \mathcal{H}_{\sigma\pi}(M)$ and, consequently, $\mathcal{H}_{\sigma\pi}(M^{\sigma\pi}) \subseteq \mathcal{H}_{\sigma\pi}(M)$ holds. (3) $\mathcal{H}_{\sigma\pi}(M) =$ $= \mathcal{H}_{\sigma\pi}(A \ominus M) = \mathcal{H}_{\sigma\pi}((A \ominus M)^{\sigma\pi}) = \mathcal{H}_{\sigma\pi}(A \ominus M^{\sigma\pi}) = \mathcal{H}_{\pi\sigma}(M^{\pi\sigma})$.

2.0.3. A <u>matrix</u> in A is <u>over</u> B iff $B \subseteq M(\alpha,\beta)$ holds for all $\alpha, \beta \in \xi$.

<u>Proposition</u>. Let M be a matrix in A over B, $M \in \mathfrak{M}$. (A1) $M^{\sigma\pi}$ is a $\sigma\pi$ -matrix in A over B, $M^{\sigma\pi} \in \mathfrak{M}$. (A2) If $\bigcap_n M(m,n) \subseteq \bigcap_n M(m+1,n)$ holds for all m then $\bigcap_n M^{\sigma\pi}(m,n) =$ $= \bigcap_n M(m,n)$ holds for all m. (B1) $M^{\pi\sigma}$ is a $\pi\sigma$-matrix in A over B, $M^{\pi\sigma} \in \mathfrak{M}$. (B2) If $\bigcup_n M(m,n) \supseteq \bigcup_n M(m+1,n)$ holds for all m then $\bigcup_n M^{\pi\sigma}(m,n) = \bigcup_n M(m,n)$ holds for all a.

Proof. The assertions (A1),(B1) are easy, (B2) follows immediately from (A2). We shall prove (A2). Let $\tilde{M}$ be a matrix so that $\tilde{M}(\alpha,\beta) = \bigcap_{\delta \leq \beta} M(\alpha,\delta)$ holds for each $\alpha, \beta \in \xi$. We have $\bigcap_j \tilde{M}(\alpha,j) = \bigcap_j M(\alpha,j)$ and $\tilde{M}(\alpha,\beta+1) \subseteq \tilde{M}(\alpha,\beta)$. Thus $\bigcap_j \tilde{M}(\alpha,j) = \bigcup_{\beta \notin FN} \tilde{M}(\alpha,\beta)$ holds. By using the definition of $M^{\sigma\pi}$ we obtain $M^{\sigma\pi}(\alpha,\beta) = \bigcup_{\gamma \leq \alpha} \tilde{M}(\gamma,\beta)$. Finally, the following holds:

$$\bigcap_j M^{\sigma\pi}(m,j) = \bigcup_{j \notin FN} M^{\sigma\pi}(m,\beta) = \bigcup_{i \leq m} \bigcup_{\beta \notin FN} \tilde{M}(i,\beta) = \bigcup_{i \leq m} \bigcap_n \tilde{M}(i,j) = \bigcup_{i \leq m} \bigcap_n M(i,n) = \bigcap_n M(m,n).$$

2.0.4. Recall some notions from [M1] which we shall use in the following. Let $\mathcal{K}$ be a codable class. Writing $FL_{\mathcal{K}}$ we mean a language FL_K such that there is a relation S so that $\langle S,K \rangle$ is a coding pair which codes the class $\mathcal{K}$. A formula φ <u>is</u> $\langle X,Y \rangle$ <u>-hereditary</u> iff the general closure of the following formula holds:

$$X_1 \subseteq X_0 \& Y_0 \subseteq Y_1 \rightarrow (\varphi(\overset{X_0}{\overline{X}}, \overset{Y_0}{\overline{Y}}) \rightarrow \varphi(\overset{X_1}{\overline{X}}, \overset{Y_1}{\overline{Y}}))$$

(where $\varphi(\overset{X_0}{\overline{X}}, \overset{Y_0}{\overline{Y}})$ denotes the formula obtained from φ by replacing all free occurences of X, Y by X_0, Y_0 resp.).

Theorem. Let R be a $\sigma\pi$-matrix in A and let $A, R \in \mathfrak{M}$. Let $\varphi(X,Y)$ be a normal formula of the language $FL_{\mathfrak{M}}$ which is $\langle X,Y\rangle$-hereditary and suppose that $\varphi(A,A)$ holds. Suppose, moreover, that $\varphi(\bigcap_n R(m,n), \bigcap_n R(m+1,n))$ holds for each $m \in FN$. Then there is a $\sigma\pi$-matrix M in A, $M \in \mathfrak{M}$, and we have (1) $\alpha+1, \beta \in \xi \rightarrow \varphi(M(\alpha,\beta), M(\alpha+1,\beta))$, (2) $(\forall m)(\bigcap_n R(m,n) = \bigcap_n M(m,n))$.

Proof. Let P be a $\sigma\pi$-matrix in A so that: $P \in \mathfrak{M}$, $P(\xi-1,\beta) = A$ holds for each $\beta \in \xi$, $P(\alpha,\beta) = R(\alpha,\beta)$ holds for each $\alpha \in \xi$, $0 < \beta \in \xi$. We have for each $m \in FN$: $\bigcap_n R(m,n) = \bigcap_n P(m,n)$. We define functions $\tilde{t}$, t on $\xi \times \xi$: $t(0,\beta) = \beta$ for $\beta \in \xi$,

(i) $\tilde{t}(\alpha+1,\beta) = \max\{\gamma; \gamma \leq \min(t(\alpha,\beta),\beta) \& \\ \& \varphi(P(\alpha,t(\alpha,\beta)), P(\alpha+1,\gamma))\}$

(ii) $t(\alpha,\beta) = \min\{\tilde{t}(\alpha,\gamma); \gamma \geq \beta\}$.

We have $dom(t) = dom(\tilde{t}) = \xi \times \xi$, because $\varphi(A,A)$ holds. We deduce from the definitions that $t(\alpha,\beta) \leq \tilde{t}(\alpha,\beta) \leq \beta$ holds for all $\alpha,\beta \in \xi$. Thus $P(\alpha,\beta) \subseteq P(\alpha,t(\alpha,\beta))$ holds for all $\alpha, \beta \in \xi$. We shall prove that

(iii) $\varphi(P(\alpha,t(\alpha,\beta)), P(\alpha+1,t(\alpha+1,\beta)))$ is true for each $\alpha+1, \beta \in \xi$.

We deduce from the definition that $\varphi(P(\alpha,t(\alpha,\beta)), P(\alpha+1, \tilde{t}(\alpha+1,\beta)))$ holds for each $\alpha+1, \beta \in \xi$. We have $t(\alpha+1,\beta) \leq \tilde{t}(\alpha+1,\beta)$. We obtain from this and by using the facts that

P is a $\sigma\pi$-matrix and $\varphi(X,Y)$ is $\langle X,Y\rangle$-hereditary that (iii) holds.

We have $t(\alpha+1,\beta)\leq t(\alpha,\beta)$ for all $\alpha+1, \beta\in\xi$.

(*) $(t(\alpha+1,\beta)\leq\tilde{t}(\alpha+1,\beta)\leq t(\alpha,\beta))$.

We deduce from (ii) that

(**) $t(\alpha,\beta)\leq t(\alpha,\beta+1)$ holds for all α , $\beta-1\in\xi$.

Let M be a matrix in A with the following properties:

$$M(\alpha,\beta) = P(\alpha,t(\alpha,\beta)) \text{ for all } \alpha,\beta\in\xi.$$

(We have $t(\alpha,\beta)\leq\beta$ for each $\alpha,\beta\in\xi$, thus, the matrix M exists.) We have $M(\alpha,\beta)\subseteq P(\alpha,\beta)$ $(\alpha,\beta\in\xi)$ and $M(\alpha,\beta) = P(\alpha,t(\alpha,\beta))\subseteq P(\alpha+1,t(\alpha+1,\beta)) = M(\alpha+1,\beta)$ $(\alpha+1,\beta\in\xi)$. This follows from (*) and from the fact that P is a $\sigma\pi$-matrix. We deduce from (**) that $M(\alpha,\beta+1) = P(\alpha,t(\alpha,\beta+1))\subseteq P(\alpha,t(\alpha,\beta)) = M(\alpha,\beta)$ holds for all $\alpha,\beta+1\in\xi$.

The condition (1) of our theorem follows immediately from (iii) and from the definition of M. We shall prove (2). It is sufficient to prove that each function $t(m,\cdot)/FN$ (of the argument $\cdot$) is unbounded. (We deduce from the definition of t that each function $t(m,\cdot)$ is non-decreasing.) We shall prove it by induction on m. If m = 0 then $t(0,\alpha)=\alpha$ and the assertion holds. Suppose that the assertion holds for m. Let $k\in FN$. We shall prove that there is an $n\in FN$ so that $t(m+1,n)\geq k$. We suppose $\varphi(\bigcap_i P(m,i), \bigcap_i P(m+1,i)$. Thus $\varphi(\bigcap_i P(m,i), P(m+1,k))$ holds. By using the induction hypothesis we have $\bigcap_i P(m,i) = \bigcap_i P(m,t(m,i))$. We have $\varphi(P(m,\alpha)P(m+1,k))$ for all $\alpha\in\xi - FN$ and, consequently, there is a $j\in FN$ such that $\varphi(P(m, t(m,j), P(m+1,k))$ holds. The function $t(m,\cdot)\wedge FN$ is non-decreasing and unbounded. Thus, there is a $n\geq j$, $n\in FN$, with $k\leq\min(t(m,n)n)$. We have for all $i\geq n: t(m+1,i)\geq k$. Thus

$t(m+1,n)$ holds.

2.0.5. Theorem. Let R be a $\pi\sigma$-matrix over B, $R,B \in \mathfrak{M}$. Let $\varphi(X,Y)$ be a normal formula of the language $FL_{\mathfrak{M}}$ which is $\langle X,Y\rangle$-hereditary and suppose $\varphi(B,B)$. Suppose, moreover, that $\varphi(\bigcup_n R(m+1,n), \bigcup_n R(m,n))$ holds for all $m \in FN$. Then there is a $\pi\sigma$-matrix M over $B, M \in \mathfrak{M}$, and we have: (1) $\alpha+1, \beta \in \xi \rightarrow \varphi(M(\alpha+1,\beta), M(\alpha,\beta))$, (2) $(\forall m)$ $(\bigcup_n R(m,n) = \bigcup_n M(m,n))$.

Proof. Let $P = V \ominus R$. P is a $\sigma\pi$-matrix in V - B, $P \in \mathfrak{M}$. The formula $\bar{\varphi}(X,Y) = (V-X, V-Y)$ is $\langle Y,X\rangle$-hereditary and $\bar{\varphi}(V-B,V-B) \equiv \varphi(B,B)$. We have for all $m \in FN$: $\varphi(\bigcup_n R(m+1,n), \bigcup_n R(m,n)) \equiv \bar{\varphi}(V-\bigcup_n R(m+1,n), V-\bigcup_n R(m,n)) \equiv \bar{\varphi}(\bigcap_n V-R(m+1,n), \bigcap_n V-R(m,n)) \equiv \bar{\varphi}(P(m+1,n), P(m,n))$. We deduce from the previous theorem that there exists a $\sigma\pi$-matrix S in V-B, $S \in \mathfrak{M}$, and the following holds: $\alpha+1, \beta \in \xi \rightarrow \bar{\varphi}(S(\alpha+1, \beta), S(\alpha,\beta))$, $(\forall m)(\bigcap_n S(m,n) = \bigcap_n P(m,n))$. Let $M = V \ominus S$. The M has the required properties.

2.1.0. X is a $\sigma\pi^{\mathfrak{M}}$- ($\pi\sigma^{\mathfrak{M}}$- resp.) class iff there is a matrix $M \in \mathfrak{M}$ so that $X = \mathcal{H}_{\sigma\pi}(M)$ ($X = \mathcal{H}_{\pi\sigma}(M)$) resp.).

We shall write $\sigma\pi^{0}$ ($\pi\sigma^{0}$ resp.) instead of the symbol $\sigma\pi^{Sd_V}$ ($\pi\sigma^{Sd_V}$ resp.). X is a $\sigma\pi^{\mathfrak{M}}$- ($\pi\sigma^{\mathfrak{M}}$-resp.) class iff there is a $\sigma\pi$ ($\pi\sigma$ resp.) matrix $M \in \mathfrak{M}$ and $X = \mathcal{H}_{\sigma\pi}(M)$ ($X = \mathcal{H}_{\pi\sigma}(M)$ resp.). This follows from 2.0.3.

2.1.1. A standard system $\mathfrak{M}$ is called saturated standard system (s.s.s. briefly) iff for every sequence $\{X_n\} \subseteq \mathfrak{M}$ there is a relation $R \in \mathfrak{M}$ with $(\forall m)(R''\{m\} = X_m)$.

For example, every Sd_V^{*} is a s.s.s., but Sd_V is a standard system which is not saturated.

<u>Proposition</u>. Let $\mathfrak{M}$ be s.s.s. (1) Q is a $\sigma^{\mathfrak{M}}$- ($\pi^{\mathfrak{M}}$-resp.) class iff there is a sequence $\{X_n\} \subseteq \mathfrak{M}$ and we have $Q = \bigcup_n X_n$ ($Q = \bigcap_n X_n$ resp.).

<u>Proposition</u>. Let $\mathfrak{M}$ be a s.s.s. (1) Q is a $\sigma\pi^{\mathfrak{M}}$ class iff $Q = \bigcup_m Q_m$, where each Q_m is a $\pi^{\mathfrak{M}}$-class. (2) Q is a $\pi\sigma^{\mathfrak{M}}$-class iff $Q = \bigcap_m Q_m$, where each Q_m is a $\sigma^{\mathfrak{M}}$-class.

Proof of this proposition follows immediately from the following

<u>Proposition</u>. (1) Let $\{Q_m\}$ be a sequence of $\pi^{\mathfrak{M}}$-classes and let $\mathfrak{M}$ be s.s.s. Then there is a matrix $M \in \mathfrak{M}$ with $(\forall m)(Q_m = \bigcap_n M(m,n))$. (2) Let $\{Q_m\}$ be a sequence of $\sigma^{\mathfrak{M}}$-classes and let $\mathfrak{M}$ be s.s.s. Then there is a matrix $M \in \mathfrak{M}$ with $(\forall m)(Q_m = \bigcup_n M(m,n))$.

Proof. Let $\{S_m\}$ be a sequence so that we have for each $m \in FN$: (i.e. $Q_m = \bigcup_n S_m(n)$)
a) $S_m \in \mathfrak{M}$, b) S_m is a σ-string of Q_m of the length ξ .
There is a relation $R \in \mathfrak{M}$ such that $(\forall m)(R''\{m\} = S_m)$ holds. We have for each m:$(\forall \alpha \leq m)(R''\{m\}$ is a σ-string of the length ξ .) Thus, there is a $\vartheta \in N-FN$ so that $(\forall \alpha \leq \vartheta)(R''\{\alpha\}$ is a σ-string of the length ξ). We can construct the required matrix M immediately from the relation $R \restriction \vartheta$. (2) follows from (1).

<u>Corollary</u>. Let $\mathfrak{M}$ be a s.s.s. Let X be a $\sigma\pi$ - ($\pi\sigma$ - resp.) class. Then X is a $\sigma\pi^{\mathfrak{M}}$ - ($\pi\sigma^{\mathfrak{M}}$-resp.) class. (For the notion of $\sigma\pi$ - ($\pi\sigma$ -resp.) class see 0.0.1 .)

2.1.2. We say that a sequence $\{X_n\}$ is a σ- (π-resp.) <u>sequence</u> iff we have $(\forall m)(X_m \subseteq X_{m+1})((\forall m)(X_{m+1} \subseteq X_m)$ resp.).

<u>Proposition</u>. Let $\varphi(X,Y)$ be a normal formula of the

language $FL_{\mathfrak{M}}$ which is $\langle X,Y\rangle$-hereditary. Let $\mathfrak{M}$ be a s.s.s. Let $A \in \mathfrak{M}$ and assume that $\varphi(A,A)$ holds. (A) Let $\{Q_m\}$ be a σ-sequence of $\pi^{\mathfrak{M}}$-classes so that $\varphi(Q_m,Q_{m+1}) \& Q_m \subseteq A$ holds for each $m \in FN$. Then there is a $\sigma\pi$-matrix $M \in \mathfrak{M}$ in A such that (1) $\bigcap_n M(m,n) = Q_m$ holds for all m, (2) we have $\varphi(M(\alpha,\beta), M(\alpha+1,\beta))$ for all $\alpha+1, \beta \in \xi$.

(B) Let $\{Q_m\}$ be a π-sequence of $\sigma^{\mathfrak{M}}$-classes so that $\varphi(Q_{m+1},Q_m) \& A \subseteq Q_m$ holds for all m. Then there is a $\pi\sigma$-matrix $M \in \mathfrak{M}$ over A such that (1) $\bigcup_n M(m,n) = Q_m$ for each m, (2) $\varphi(M(\alpha+1,\beta), M(\alpha,\beta))$ holds for each $\alpha+1, \beta \in \xi$.

Proof. (A) Let P be a matrix in A, $P \in \mathfrak{M}$, so that $(\forall m)(Q_m = \bigcap_n P(m,n)$ holds. (The existence follows from the previous proposition.) The matrix $P^{\sigma\pi}$ has the following properties: $P^{\sigma\pi}$ is a $\sigma\pi$-matrix in A, $P^{\sigma\pi} \in \mathfrak{M}$, and $(\forall m)(\bigcap_n P^{\sigma\pi}(m,n) = Q_m)$. The existence of the matrix in question follows from this and from 2.0.4.

(B) Let R be a matrix so that $R \in \mathfrak{M}$ and $(\forall m)(Q_m = \bigcup_n R(m,n))$ hold. Let P be a matrix so that $P(\alpha,\beta) = R(\alpha,\beta) \cup A$ $(\alpha,\beta \in \xi)$. We have $(\forall m)(Q_m = \bigcup_n P(m,n))$. The matrix $P^{\pi\sigma}$ is a $\pi\sigma$-matrix over A, $P^{\pi\sigma} \in \mathfrak{M}$ and $Q_m = \bigcup_n P^{\pi\sigma}(m,n)$ holds for each $m \in FN$. The proposition follows from this and by using 2.0.5.

2.2.0. Let, for each $i \leq k$, R_i be an a(i)+1-ary relation. We say that Q is a <u>universe</u> <u>w.r.t.</u> $\{R_i\}_k$ iff $[\![R_i]\!]_k (Q,Q)$ holds. Q is a $\sigma\pi^{\mathfrak{M}}$- ($\pi\sigma^{\mathfrak{M}}$-resp.) <u>universe</u> <u>w.r.t.</u> $\{R_i\}_k$ iff Q is a universe w.r.t. $\{R_i\}_k$ and, moreover, Q is a $\sigma\pi^{\mathfrak{M}}$-class ($\pi\sigma^{\mathfrak{M}}$-class resp.). (For $[\![\ ]\!]$ see 0.1.0.)

2.2.1. Proposition. Let $R_i \in \mathfrak{M}$, $i \leq k$, be as above. Let A,B,Q be **universes** w.r.t. $\{R_i\}_k$ and suppose that $B \subseteq Q \subseteq A$,

$B \in \mathfrak{M}$ and Q is a $\sigma\pi^{\mathfrak{M}}$-class. Then there is a $\sigma\pi$-matrix P in A over B, $P \in \mathfrak{M}$ and we have: (1) $\mathcal{H}_{\sigma\pi}(P) = Q$, (2) $\alpha+1, \beta \in \xi \rightarrow [\![R_i]\!]_k (P(\alpha,\beta) P(\alpha+1,\beta))$.

Proof. There is a $\sigma\pi$-matrix M in A so that $M \in \mathfrak{M}$ and we have $\mathcal{H}_{\sigma\pi}(M) = Q$. Let P be a matrix in A with $P(0,\beta) = M(0,\beta) \cup B$ for each $\beta \in \xi$ and $P(\alpha+1,\beta) = \bigcup_{i \leq k} R_i'' P^{a(i)} (\alpha,\beta) \cup M(\alpha+1,\beta) \cup P(\alpha,\beta) \cup B$ for each $\alpha+1, \beta \in \xi$. Clearly, P is a $\sigma\pi$-matrix in A over B, $P \in \mathfrak{M}$ and we have, for all $\alpha, \beta \in \xi$, $P(\alpha,\beta) \supseteq M(\alpha,\beta)$. Thus $Q \subseteq \mathcal{H}_{\sigma\pi}(P)$ holds. We have $\mathcal{H}_{\sigma\pi}(P) = \bigcup_m \bigcup_{\beta \notin FN} P(m,\beta)$. The relations $P(0,\beta) \subseteq M(0,\beta) \cup B \subseteq Q$ hold. Assume, for each $\beta \in \xi - FN$, $P(m,\beta) \subseteq Q$. We deduce from the definition that we have for each $\beta \in \xi - FN$, $P(m+1,\beta) \subseteq Q$. Thus, we have $\mathcal{H}_{\sigma\pi}(P)$ Q and the statement (1) is proved. Finally, the statement (2) follows immediately from the definition of P.

2.2.2. We say that a class Q is a <u>limit $\pi\sigma^{\mathfrak{M}}$-universe w.r.t.</u> $\{R_i\}_k$ iff there exists a matrix $M \in \mathfrak{M}$ such that (1) $\mathcal{H}_{\pi\sigma}(M) = Q$ and (2) we have for each $m \in FN$: $[\![R_i]\!]_k (\bigcup_n M(m+1,n), \bigcup_n M(m,n)) \,\&\, \bigcup_n M(m+1,n) \subseteq \bigcup_n M(m,n)$.

2.2.3. <u>Proposition</u>. Let $R_i \in \mathfrak{M}$ be an a(i)+1 ary relation, $i \leq k$. Let $A,B \in \mathfrak{M}$ be universes w.r.t. $\{R_i\}_k$ and let Q be a limit $\pi\sigma^{\mathfrak{M}}$-universe w.r.t. $\{R_i\}_k$, $B \subseteq Q \subseteq A$. Then there exists a $\pi\sigma$-matrix $M \in \mathfrak{M}$ in A over B so that (a) $\mathcal{H}_{\pi\sigma}(M) = Q$ and (b) $\alpha+1, \beta \in \xi \rightarrow [\![R_i]\!]_k (M(\alpha+1,\beta), M(\alpha,\beta))$.

Proof. There is a matrix $R \in \mathfrak{M}$ with $\mathcal{H}_{\pi\sigma}(R) = Q$ and $[\![R_i]\!]_k (R_{m+1}, R_m) \,\&\, R_{m+1}, R_m$ holds for each m. (We put $R_m = \bigcup_n R(m,n)$.) Let P be a matrix so that $P(\alpha,\beta) = R(\alpha,\beta) \cup B$ holds for each $\alpha, \beta \in \xi$. We have $P \in \mathfrak{M}$ and $B \subseteq \bigcap_n R_m$. We deduce from this that $R_m = P_m = \bigcup_n P(m,n)$ holds for each m.

$P^{\pi\sigma}$ is a $\pi\sigma$-matrix over B, $P^{\pi\sigma} \in \mathfrak{M}$, and $P_m^{\pi\sigma} = P_m^{\pi\sigma}$ holds for each m. By using 2.0.5 we obtain a matrix $\overline{P} \in \mathfrak{M}$ over B so that $\mathcal{H}_{\pi\sigma}(P) = Q$ and we have $\alpha+1, \beta \in \xi \rightarrow [\![R_i]\!]_k$ $(\overline{P}(\alpha+1, \beta), P(\alpha, \beta))$. Let M be a matrix so that $M(\alpha, \beta) = \overline{P}(\alpha, \beta) \cup A$ holds for each $\alpha, \beta \in \xi$. The M has the required properties.

§ 3. Valuations of $\sigma\pi^{\mathfrak{M}}$ - and $\pi\sigma^{\mathfrak{M}}$ -triads.

3.0.0. A triad $\mathcal{A}(Q,B)$ is a $\sigma\pi^{\mathfrak{M}}$ -triad (limit $\pi\sigma^{\mathfrak{M}}$-triad resp.) iff we have (1) $\mathcal{A} \in \mathfrak{M}$, $B \in \mathfrak{M}$, (2) Q is a $\sigma\pi^{\mathfrak{M}}$-class (limit $\pi\sigma^{\mathfrak{M}}$ universe w.r.t. $\{E,F\}$ resp.) (where $\mathcal{A} = \langle A,E,E \rangle$).

Remark. We need not define limit $\sigma\pi^{\mathfrak{M}}$-triads because it follows from 2.2.1 that each $\sigma\pi^{\mathfrak{M}}$-triad would be such a limit $\sigma\pi^{\mathfrak{M}}$-triad.

Now, we shall construct some useful triads. We put, for each $f,g \in {}^{\xi}N$, $f \overset{\xi}{+} g = \{\langle f(\alpha)+g(\alpha), \alpha \rangle ; \alpha \in \xi\}$ (where we put ${}^{\xi}N = \{f \subseteq N \times \xi \text{ ; } f \text{ is a function } \& \operatorname{dom}(f) = \xi\}$).

Let $\mathcal{T}_{\sigma\pi}$ be the triad $\langle {}^{\xi}N, \overset{\xi}{+}, Id \rangle (Q_{\sigma\pi}, \{0\} \times \xi)$ where $Q_{\sigma\pi} = \{f \in {}^{\xi}N ; (\exists \delta \in \xi - FN)(\forall \alpha \in \delta)(f(\alpha) \in FN)\}$. The $\mathcal{T}_{\sigma\pi}$ is closed, $\sigma\pi^{0}$-triad.

Let $\mathcal{T}_{\pi\sigma}$ be the triad $\langle {}^{\xi}RN(\geq 0), \overset{\xi}{+}, Id \rangle (Q_{\pi\sigma}, \{0\} \times \xi)$ where $Q_{\pi\sigma} = \{f \in {}^{\xi}RN(\geq 0) ; (\forall \alpha \in \xi - FN), (f(\alpha) \doteq 0)\}$. (For $\doteq$ see 0.0.3.) The $\mathcal{T}_{\pi\sigma}$ is a closed, limit $\pi\sigma^{0}$-triad.

Let $\mathcal{T} = \mathcal{A}(Q,B)$ be a triad and let A be an universe in $\mathcal{A}$ (i.e. $\mathcal{A} | \tilde{A}$ is a substructure of $\mathcal{A}$). We designate the triad $\mathcal{A} | \tilde{A}(Q \cap \tilde{A}, B \cap \tilde{A})$ by $\mathcal{T} | \tilde{A}$.

We put $\check{\mathcal{T}}_{\sigma\pi} = \mathcal{T}_{\sigma\pi} | \{f \in {}^{\xi}N ; f$ is a non-increasing function} and $\hat{\mathcal{T}}_{\pi\sigma} = \mathcal{T}_{\pi\sigma} \mid \{f \in {}^{\xi}RN(\geq 0)$; f is a non-decreasing function}. $\check{\mathcal{T}}_{\sigma\pi}$ is closed, $\sigma\pi^{o}$-triad and $\hat{\mathcal{T}}_{\pi\sigma}$ is a closed, limit $\pi\sigma^{o}$-triad.

3.0.1. Theorem. Let $\mathcal{T}$ be a $\sigma\pi^{\mathfrak{M}}$-triad. Then there exists a valuation H of $\mathcal{T}$ in $\check{\mathcal{T}}_{\sigma\pi}$, $H \in \mathfrak{M}$.

3.0.2. Before we prove this theorem, remember some assertions from [M2].

There is a normal formula $\Phi(x,y,X,Y)$ of the language FL so that the following holds: let $\mathcal{A}$ be an e structure and let B be a universe in $\mathcal{A}$. Let S be a σ-string in $\mathcal{A}$ over B (see 0.1.1), $\mathcal{A}$,B, S $\in \mathfrak{M}$. Put $H^S = \{\langle x,y\rangle; \Phi(x,y,\mathcal{A},S)$. Then H^S is a valuation of $\mathcal{A}(B,B)$ in $\langle N,+,Id\rangle(\{0\},\{0\})$ such that $Q(\alpha) \subseteq \{x \in A; H^S(x) \leq 2^{\alpha}\} \subseteq Q(\alpha+1)$ holds for each $\alpha+1 \in dom(S)$ and $H^S \in \mathfrak{M}$. Suppose, moreover, that $R \in \mathfrak{M}$ is a σ-string in $\mathcal{A}$ over B, dom(R) = dom(S) and $S(\alpha) \subseteq R(\alpha)$ holds for each $\alpha \in dom(R)$. Then we have, for each $x \in A$, $H^R(x) \leq H^S(x)$. (See 3.0.3 in [M2].)

3.0.3. We now turn to the proof of the theorem. Let $\mathcal{T} = \mathcal{A}(Q,B)$. Fix a $\vartheta \in N$ such that $2 \cdot \xi \in \vartheta$. We deduce from 2.2.1 that there is a $\sigma\pi$-matrix P in A over B of the type ϑ, $P \in \mathfrak{M}$, and the following holds: $\mathcal{H}_{\sigma\pi}(P) = Q$ and $\alpha+1, \beta \in \vartheta \longrightarrow [\![F,E]\!](P(\alpha,\beta),P(\alpha+1,\beta))$. Let R be a matrix of the type ϑ such that $\alpha, \beta \in \vartheta \longrightarrow R(\alpha,\beta) = P(\alpha,\beta) \cap \cap E'' P(\alpha,\beta)$ holds. We have $B = E'' B \subseteq E'' P(\alpha,\beta)$ (for each $\alpha, \beta \in \vartheta$) and, consequently R is a $\sigma\pi$-matrix in A over B, $R \in \mathfrak{M}$. By using 2.0.4 in [M2] we obtain $\bigcup_m R(m,\beta) = \bigcup_m P(m,\beta)$ and $F'' R^2(\alpha,\beta) \subseteq R(\alpha+1,\beta)$, $E''R(\alpha,\beta) \subseteq R(\alpha,\beta)$. At first, $\mathcal{H}_{\sigma\pi}(R) = \mathcal{H}_{\sigma\pi}(P) = Q$ holds. Let S be a matrix such that we have: $S(0,\beta) = B$ for all $\beta \in \xi$, $S(\alpha,\beta) = R(2\alpha,\beta)$ for all $1 \leq \alpha < \xi - 1$, $\beta \in \xi$ and $S(\xi-1,\beta) = A$ for all $\beta \in \xi$. It is easy that S is a $\sigma\pi$-matrix in A over B, $S \in \mathfrak{M}$, and we have: $F"S^2(\alpha,\beta) \subseteq S(\alpha+1,\beta)$, $F_3S^3(\alpha,\beta) \subseteq S(\alpha+1,\beta)$ $(\alpha+1, \beta \in \xi)$ (because $F_3S^3(\alpha,\beta) = F_3R^3(2\alpha,\beta) \subseteq F"R^2(2\alpha+1,\beta) \subseteq R(2(\alpha+1),\beta) = S(\alpha+1,\beta)$ holds) and $E"S(\alpha,\beta) \subseteq S(\alpha,\beta)$. (For F_3 see 0.1.1.) We have also $\mathcal{H}_{\sigma\pi}(S) = Q = \bigcup_{m} \bigcup_{\beta \notin FN} S(m,\beta)$.

We designate by $S(\cdot,\beta)$ the relation $\{\langle x,\alpha\rangle ; x \in S(\alpha,\beta)\}$. Each $S(\cdot,\beta)$, $\beta \in \xi$, is a σ-string in $\mathcal{A}$ over B. We put

$$\langle\langle y, x\rangle, \beta\rangle \in H \equiv \Phi(y,x,\mathcal{A},S(\cdot,\beta)).$$

We have $H \in \mathfrak{M}$ and $H"\{\beta\}$ is a valuation of $\mathcal{A}(B,B)$ in $\langle N,+,Id\rangle$ $(\{0\},\{0\})$. We put, for each $x \in A$, $\langle\langle \alpha,\beta\rangle,x\rangle \in G \equiv \alpha = H"\{\beta\}(x)$. It is easy that G is a function, $G: A \rightarrow {}^{\xi}\xi$, and $G \in \mathfrak{M}$. We shall prove that, for each $x \in A$, $G(x)$ is a non-increasing function. We have $\beta \leq \delta \rightarrow S(\alpha,\beta) \supseteq S(\alpha,\delta)$. Thus, by using the facts from 3.0.2, we deduce that $H"\{\beta\}(x) \leq H"\{\delta\}(x)$. We conclude from this and by using the definition of G that G(x) is a non-increasing function. It remains to prove that

1) $x \in Q \equiv (\exists \delta \in \xi - FN)(\forall \beta \in \delta) G(x)(\beta) \in FN$ and

2) $x \in B \equiv G(x) = \{0\} \times \xi$.

1) We have $x \in Q \equiv (\exists \delta \in \xi - FN)(\exists m)(x \in S(m,\delta)) \equiv (\exists \delta \in \xi - FN) H"\{\delta\}(x) \in FN \equiv (\exists \delta \in \xi - FN)(G(x)(\delta) \in FN) \equiv (\exists \delta \in \xi - FN)(\forall \beta \leq \delta)((G(x)(\beta) \in FN)$.

2) We have: $x \in Q \equiv (\forall \beta \in \xi)(G(x)(\beta) = 0) \equiv G(x) = \{0\} \times \xi$.

Thus, G is a required valuation.

3.0.4. <u>Theorem</u>. Let $\mathcal{T}$ be a limit $\pi\sigma^{\mathfrak{M}}$-triad. Then there exists a valuation H of $\mathcal{T}$ in $\hat{\mathcal{T}}_{\pi\sigma}$, $H \in \mathfrak{M}$.

Proof of this theorem is analogical to the previous one and we omit it.

3.0.5. The analogous results as the ones formulated in the previous theorems can be proved also for monotonic valuations. We shall present the more simple of them.

<u>Theorem</u>. Let $\mathcal{T}$ be a closed, $\sigma\pi^{\mathfrak{M}}$-triad. [1] Then there is a monotonic valuation H of $\mathcal{T}$ in $\mathcal{T}_{\sigma\pi}$, $H \in \mathfrak{M}$.

Proof. Let $\vartheta \in N$ be such that $2\xi \leq \vartheta$. Let $R \in \mathfrak{M}$ be a $\sigma\pi$-matrix in A over B of the type ϑ such that 1) $\mathcal{H}_{\sigma\pi}(R) = Q$, 2) $[\![F,E,\lhd]\!] (R(\alpha,\beta)R(\alpha+1\ \beta))$ holds for each $\alpha+1$, $\beta \in \vartheta$. (The existence of R follows from 2.2.1.) Let M be a matrix such that $M(0,\beta) = B$, $M(\xi-1,\beta) = A$ holds for each $\beta \in \xi$ and $M(\alpha,\beta) = \trianglelefteq''R(2\alpha,\beta) \cap E''R(2\alpha,\beta))$ holds for each α, $\beta \in \xi$. We deduce from the lemma in 1.1.0 that $F''M^2(\alpha,\beta) \subseteq M(\alpha+1,\beta)$ $(\alpha+1, \beta \in \xi)$, $E''M(\alpha,\beta) \subseteq M(\alpha,\beta)$, $\lhd'' M(\alpha,\beta) \subseteq M(\alpha,\beta)$ $(\alpha, \beta \in \xi)$ and we have for each α, $\beta + 2 \in \xi$: $R(2\alpha,\beta) \subseteq M(\alpha,\beta+1) \subseteq R(2\alpha, 2(\beta+1))$. We deduce from this that $\mathcal{H}_{\sigma\pi}(M) = Q$. We put $P(\alpha,\beta) = \trianglelefteq''(R(\alpha,\beta) \cap \cap E''R(\alpha,\beta))$. We deduce from the lemma in 1.1.0 that $F_3''M^3(\alpha,\beta) = F_3''P^3(2\alpha,\beta) \subseteq F''P^2(2\alpha+1,\beta) \subseteq P(2(\alpha+1,\beta) = = M(\alpha+1,\beta)$. Thus, we have the following: M is a matrix in A over B, $M \in \mathfrak{M}$, and (a) each $M(\cdot,\beta)$ is a σ-string in $\mathcal{A}$ over B (where $M(\cdot,\beta) = \{\langle x,\alpha\rangle;\ x \in M(\alpha,\beta)\}$),

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1) Let $\mathcal{A} \in \mathfrak{M}$ be a commutative e-structure so that $\mathcal{A}$ has a u-expansion in $\mathfrak{M}$, and let $\mathcal{T}$ be over $\mathcal{A}$.

(b) $\trianglelefteq''M(\alpha,\beta)\subseteq M(\alpha,\beta)$ holds for each α, $\beta\in\xi$, (c) M is a $\sigma\pi$-matrix.

We put $\langle\langle y,x\rangle,\beta\rangle\in H\equiv\Phi(y,x,a,M(\cdot,\beta))$. We can prove quite analogously as in the proof of the theorem 3.0.1 and by using the arguments from the proof of the theorem in 1.1.0 that the mapping $G:A\rightarrow{}^{\xi}\xi$ such that $\langle\langle\alpha,\beta\rangle,x\rangle\in G\equiv\alpha=H''\{\beta\}(x)$ is the required monotonic valuation.

References

[M1] J. MLČEK: Approximations of $\sigma^{\mathfrak{M}}$-classes and $\pi^{\mathfrak{M}}$-classes, Comment. Math. Univ. Carolinae 20(1979), 669-679.

[M2] J. MLČEK: Valuations of structures, Comment. Math. Univ. Carolinae 20(1979), 681-695.

[V] P. VOPĚNKA: Mathematics in the alternative set theory, Teubner-Texte, Leipzig, 1979.

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(Oblatum 10.6. 1980)