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REMARK ON COMPLETELY BAIRE-ADDITIVE FAMILIES IN ANALYTIC SPACES

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Abstract: Completely Baire-additive families in $\varkappa$-analytic spaces are investigated. A characterization of point-countable completely Baire-additive families in ω-analytic spaces is proved. The results and methods follow that of [H], [P], and [F-H_2].

Key words: $\varkappa$-analytic space, Baire set, completely additive family, σ-discretely decomposable family, Suslin set.

Classification: Primary 54C50, 54H05
Secondary 54C60, 54C65

The aim of this remark is to notify that the result on completely Baire-additive families from [P, Prop. 1] proved for complete metric spaces also holds for $\varkappa$-analytic spaces introduced in [F-H_1]. Essentially it means that it holds in the product of a complete metric space by a compact space. The method combines Pol's proof and Hansell's original procedure [H, Th. 2] with Frolík's result [F, Th. 1]. A similar procedure was used in [F-H_2] to extend a characterization of point-finite completely Suslin-additive families from complete metric spaces [K-P] to analytic spaces.

The result is used for a characterization of point-countable completely Baire-additive families in ω-analytic spaces

(Corollary 2).

1. Preliminaries. The topological space X is regular and Hausdorff if we do not say more.

A Suslin set in X is a set of the form $\bigcup_{\sigma \in N^N} \bigcap_{n \in N} F_{\sigma|n}$ where $N = \{1,2,\dots\}$, $\sigma | n$ stands for $\sigma_1, \sigma_2, \dots, \sigma_n$ and $F_{\sigma|n}$ are closed in X.

If S is Suslin in X then there is a "Suslin stratification" of S; it means that there are sets $(S)_{\sigma|n}$ for $\sigma \in N^N$, $n \in N$, such that

$$S = \bigcup_{\sigma \in N^N} (S)_{\sigma|n}$$

$$(S)_{\sigma|n+1} \subset (S)_{\sigma|n} \text{ for } n \in N \text{ and}$$

$$S = \bigcup_{\sigma \in N^N} \bigcap_{n \in N} \overline{(S)}_{\sigma|n}.$$

We suppose that some such stratification is fixedly chosen for any Suslin set in the corresponding space, and the notation analogical to the above one ($(S)_{\sigma|n}$) will be used for it without other comments.

Baire sets are the elements of the smallest σ-algebra of subsets of a topological (uniform) space that is closed under unions of topologically (uniformly) discrete unions and contains zero sets of continuous functions. Any Baire set is Suslin.

The family $\mathcal{F}$ of subsets of a topological (uniform) space X is said to be completely Suslin (Baire)-additive if $\bigcup \mathcal{G}$ is Suslin (Baire) in X for any $\mathcal{G} \subset \mathcal{F}$.

The indexed family $\mathcal{F} = \{F(A) \mid A \in \mathcal{A}\}$ is said to be σ-dd or σ-discretely decomposable if there are sets $F_n(A)$ for

$A \in \mathcal{A}$ and $n \in N$ such that $f(A) = \bigcup_{n \in N} F_n(A)$ and the families $\{F_n(A) | A \in \mathcal{A}\}$ are discrete in the topology (uniformity) of X.

The topological (uniform) space X is called $\varkappa$-analytic if $\varkappa \geq \omega$ is a cardinal number and if there is an upper semi-continuous compact-valued (further usco-compact) correspondence $f: M \longrightarrow X$ with $f(M) = X$ such that M is a complete metric space of weight $\leq \varkappa$, and f is σ-dd-preserving, i.e. f takes the families with σ-discrete decomposition to systems with the same property.

The fundamental properties of analytic spaces and Baire sets can be found in [F-H$_1$]. Especially any Baire set is Suslin and Suslin subsets of $\varkappa$-analytic spaces are $\varkappa$-analytic.

2. Results

Theorem. Let $f: M \longrightarrow X$ be an usco-compact correspondence of the complete metric space M onto the topological space X. Let $\mathcal{A}$ be a completely Baire-additive family in X. Then the family

$$f^{-1}\mathcal{A}^* = \{f^{-1}(A) | A \in \mathcal{A}^*\}$$

(here $\mathcal{A}^* = \{A^* = A \setminus \bigcup \{B \in \mathcal{A} | B \neq A\} | A \in \mathcal{A}\}$)

is σ-discretely decomposable.

The proof of Theorem is left to sections 3 - 5.

According to the definition of $\varkappa$-analytic spaces we can immediately derive the following assertion.

Corollary 1. Let X be a $\varkappa$-analytic topological or uniform space and let $\mathcal{A}$ be a completely Baire-additive family. Then the family $\mathcal{A}^*$ is σ-dd in the topology or uniformity, respectively.

Corollary 2. Let X be an ω-analytic topological space and let $\mathcal{A}$ be a point-countable completely Baire-additive family. Then $\mathcal{A}$ is countably refinable, i.e. there is a countable family $\mathcal{C}$, such that $\mathcal{C} \subset \mathcal{A}$, and $\bigcup \mathcal{C} = \bigcup \mathcal{A}$.

Remarks. Corollary 2 extends a part of a result of R. Pol from [P], where the analogical result is proved for complete metric spaces of weight less or equal to $\aleph_1$.

It follows that for any $\mathcal{B} \subset \mathcal{A}$ in Corollary 2 the family $\mathcal{B}$ is countably refinable. If a family $\mathcal{A}$ consists of Baire sets, and $\mathcal{B}$ is countably refinable for $\mathcal{B} \subset \mathcal{A}$ then $\mathcal{A}$ is completely Baire-additive, so that Corollary 2 gives a characterization of completely Baire-additive families among point-countable families of Baire sets.

Let us remark that in the one-point compactification K of an uncountable discrete space D there is a completely Suslin-additive family $\mathcal{A}$ such that $\mathcal{A}^*$ is uncountable. Put e.g. $\mathcal{A} = \{\{x,d\} \mid d \in D, x \in K \setminus D\}$.

Proof of Corollary 2. Suppose that $\mathcal{A}$ is not countably refinable. Let the points $x_\beta \in X$ and the sets $A_\beta \in \mathcal{A}$ be chosen for $\beta < \alpha < \aleph_1$. The family $\mathcal{A}_\alpha = \{A \in \mathcal{A} \mid x_\beta \in A$ for some $\beta < \alpha\}$ is countable. Therefore $(\mathcal{A} \setminus \mathcal{A}_\alpha) \cup \{A_\beta \mid \beta < \alpha\}$ is not countably refinable, and there is a set $A_\alpha \in \mathcal{A}$ such that we can choose an $x_\alpha \in A_\alpha \setminus \bigcup\{A_\beta \mid \beta < \alpha\}$. We construct in this way $A_\alpha \in \mathcal{A}$ for $\alpha < \aleph_1$ such that $x_\alpha \in A_\alpha \setminus \bigcup\{A_\beta \mid \beta \neq \alpha, \beta < \aleph_1\}$. This contradicts Corollary 1.

Corollary 2 can be used for an assertion concerning separability of the range of a measurable correspondence. Notice that the following corollary enables us to use the selection theorem from [K-RN] for such correspondences:

Corollary 3. Let F be a Baire-measurable separable- and closed-valued correspondence from the ω-analytic space X to a complete metric space M. Then there is a separable subspace S of M such that $F^{-1}(S)$ $(\equiv\{x \in X \mid F(x) \cap S \neq \emptyset\})$ = DF $(\equiv\{x \in X \mid F(x) \neq \emptyset\})$.

Proof. Let $\mathcal{C}_n$ be a σ-discrete closed cover of M by sets with diameters less than $1/n$. Then $F^{-1}\mathcal{C}_n = \{F^{-1}(C) \mid C \in \in \mathcal{C}_n\}$ is a completely Baire-additive point-countable family which covers the ω-analytic space $DF \subset X$. According to Corollary 2 the family $F^{-1}\mathcal{C}_1$ has a countable refinement, i.e. there is a countable family $\mathcal{S}_1 \subset \mathcal{C}_1$ such that $F^{-1}\mathcal{S}_1$ covers DF. For any $S_1 \in \mathcal{S}_1$ consider the restriction $F_{S_1}: :F^{-1}(S_1) \longrightarrow S_1$ and construct $\mathcal{S}_2^{S_1}$ from $\mathcal{C}_2^{S_1} = \mathcal{C}_2 \cap S_1$ similarly as $\mathcal{S}_1$ was constructed from $\mathcal{C}_1$. By induction we construct families $\mathcal{S}_n$, $\mathcal{S}_{n+1}^{S_n}$ and $\mathcal{C}_{n+1}^{S_n}$ for $S_n \in \mathcal{S}_n$ such that

(i) $\mathcal{S}_{n+1} = \bigcup\{\mathcal{S}_{n+1}^{S_n} \mid S_n \in \mathcal{S}_n\}$

(ii) $\mathcal{C}_{n+1}^{S_n} = \mathcal{C}_{n+1} \cap S_n$

(iii) $\mathcal{S}_{n+1}^{S_n} \subset \mathcal{C}_{n+1}^{S_n}$ and

(iv) $F^{-1}\mathcal{S}_{n+1}^{S_n}$ covers $F^{-1}(S_n)$.

It suffices to put $S = \bigcap_{n \in N} \bigcup \mathcal{S}_n$ for example.

Remark. It cannot be proved that F(X) is separable in ZFC. Assume that $\aleph_1 = 2^{\aleph_0}$. Let $\{x_\alpha \mid \alpha < \aleph_1\} = [0,1]$, and put $F(x_\alpha) = \{x_\beta \mid \beta \leq \alpha\}$. Then F is a correspondence from Corollary 3 if X = [0,1] with its usual topology (uniformity), and $M = [0,1]$ is endowed with the discrete metric. However $F(X) = M$ is not separable.

3. Auxiliary assertions. We suppose that the assumptions on f, M and X of Theorem are satisfied.

Lemma 1. Let $\mathcal{F}$ be a family of subsets of X and let $f^{-1}\mathcal{F}$ be not σ-dd. Then there are families $\mathcal{F}_1, \mathcal{F}_2$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ and $f^{-1}\mathcal{F}_i$ is not σ-dd for $i = 1,2$.

Proof. The family $f^{-1}\mathcal{F} = \mathcal{D}$ can be divided into two subfamilies $\mathcal{D}_1, \mathcal{D}_2$ such that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{D}$ and $\mathcal{D}_i$ is not σ-dd for $i = 1,2$ (see [K-P]). Put

$$\mathcal{F}_i = \{F \in \mathcal{F} \mid f^{-1}(F) \in \mathcal{D}_i\}.$$

Lemma 2. Let $\overline{f(W_n)} \cap \overline{A_n} \neq \emptyset$ and $A_{n+1} \subset A_n$ for $n = 1,2,\dots$. Let $\{w\} = \bigcap \overline{W_n}$ and diam W_n converge to zero. Then

$$\bigcap_{n \in N} \overline{A_n} \neq \emptyset \text{ and } \bigcap_{n \in N} \overline{f(W_n)} \subset f(w).$$

Proof. Let $f(w) \cap \overline{A_n} = \emptyset$ for some $n_1 \in N$. Then there is an $n_2 \in N$ such that $\overline{f(W_{n_2})} \cap \overline{A_{n_1}} = \emptyset$. Therefore $f(w) \cap \overline{A_n}$ is a decreasing sequence of non-empty compact sets and has the non-empty intersection. Thus the non-emptiness of $\bigcap_{n \in N} \overline{A_n}$ is proved.

Let $x \in (\bigcap \overline{f(W_n)}) \setminus f(w)$. There is an open set $G \supset f(w)$ such that $x \notin \overline{G}$. However there is an $n_3 \in N$ such that $f(W_{n_3}) \subset G$. This is a contradiction. Hence $\bigcap \overline{f(W_n)} \subset f(w)$.

4. Proof of Theorem. Suppose that Theorem does not hold. The following objects with properties (1) - (5) can be constructed in the k-th step of induction for any $i \in I = D^N$ where

$D = \{1,2\}$ and $i|k$ stands for $i_1,\ldots,i_k$ for $k \in N$ ("$i|0$," must be ignored): sets $\mathcal{A}_{i|k}, U_{i|k}, F_{i|k} = \overline{X_{i|k}}$, and natural numbers $n_1^{i|k}, n_2^{i|k-1}, \ldots, n_k^{i|1}$:

(1) $\mathcal{A}_{i|k-1,1} \cup \mathcal{A}_{i|k-1,2} = \mathcal{A}_{i|k-1}$

(2) $\mathcal{A}_{i|k-1,1} \cap \mathcal{A}_{i|k-1,2} = \emptyset$

(3) $X_{i|k} = X_{i|k-1} \cap (\mathcal{L}\mathcal{A}_{i|1})_{n^{i|1}|k} \cap \ldots \cap (\mathcal{L}\mathcal{A}_{i|k})_{n_1^{i|k}} \cap$

$\cap f(U_{i|k})$

(We use the notation $\mathcal{L}\mathcal{F} = \bigcup\mathcal{F} \setminus \bigcup(\mathcal{A}\setminus\mathcal{F})$ for any $\mathcal{F} \subset \mathcal{A}$.)

(4) The diameter of $U_{i|k}$ is less than $1/k$.

(5) $f^{-1}(X_{i|k} \cap \bigcup \mathcal{A}^*_{i|k}) \cap U_{i|k}$ is not σ-dd, where $\mathcal{A}^*_{i|k} =$
$= \mathcal{A}_{i|k} \cap \bigcup \mathcal{A}^*$.

The first step of the construction can be done as follows:

Using Lemma 1 we find $\mathcal{A}^*_1, \mathcal{A}^*_2$ such that $\mathcal{A}^*_1 \cap \mathcal{A}^*_2 = \emptyset$, $\mathcal{A}^*_1 \cup \mathcal{A}^*_2 = \mathcal{A}^*$ and $f^{-1}\mathcal{A}^*_i$ is not σ-dd. Put $\mathcal{A}_i = \{A \in \mathcal{A} \mid A^* \in \mathcal{A}^*_i\}$ for $i = 1,2$. Now we can choose n_1^1 and n_1^2 such that $f^{-1}((\mathcal{L}\mathcal{A}_{i|1})_{n_1^{i|1}} \cap \mathcal{A}^*_{i|1})$ is not σ-dd. Since M is paracompact we can find U_1, U_2 such that (4) is satisfied and $f^{-1}((\mathcal{L}\mathcal{A}_{i|1})_{n_1^{i|1}} \cap \mathcal{A}^*_{i|1}) \cap U_{i|1}$ is not σ-dd for any $i \in I$. It is enough to put $X_{i|1} = X \cap (\mathcal{L}\mathcal{A}_{i|1})_{n_1^{i|1}} \cap f(U_{i|1})$ and all properties from (1) to (5) are satisfied for $k = 1$.

The induction continues analogically, and we will omit it.

We will finish the proof of Theorem by proofs of the following statements:

(a) $Y = \bigcup_{i \in I} \bigcap_{k \in N} F_{i|k}$ is ω-analytic

(b) $\bigcap_{k \in N} F_{i|k} \neq \emptyset$ for any $i \in I$

(c) $\bigcap_{k \in N} F_{i|k} \cap \bigcap_{k \in N} F_{j|k} = \emptyset$ whenever $i \neq j, i, j \in I$

(d) $\{\bigcap F_{i|k} | i \in I\}$ is completely Suslin-additive in Y.

These four assertions are in the contradiction with Lemma 2 from [F-H_2] which says that disjoint completely Suslin-additive families in ω-analytic spaces are countable. This lemma is an immediate corollary of [F, Th.1].

5. Proofs of (a) to (d).

(a) $Y = \bigcup_{i \in I} \bigcap_{k \in N} F_{i|k} \subset \bigcup_{i \in I} \bigcap_{k \in N} \overline{f(U_{i|k})}$.

The intersection of $\overline{U_{i|k}}$, k = 1,2,..., is non-empty with respect to the construction. Thus $Y \subset \bigcup_{i \in I} f(\bigcap_{k \in N} \overline{U_{i|k}})$ according to the second assertion of Lemma 2. Hencefore $Y \subset f(\bigcup_{i \in I} \bigcap_{k \in N} \overline{U_{i|k}})$ and this is a compact set because f is usco-compact and (4) holds.

Obviously Y is Suslin and thus it is ω-analytic.

(b) Since $X_{i|k} \subset f(U_{i|k})$ and it is non-empty we know that $F_{i|k} \cap \overline{f(U_{i|k})} \neq \emptyset$ and the first part of Lemma 2 guarantees that $\bigcap_{k \in N} F_{i|k} \neq \emptyset$.

(c) and (d). Let $i \neq j$ and $x_i \in \bigcap_{k \in N} F_{i|k}$, $x_j \in \bigcap_{k \in N} F_{j|k}$. (3) implies that $x_i \in \bigcap_{k \in N} \mathcal{L}\mathcal{A}_{i|k}$. Especially there is $A_i \in \mathcal{A}$ such that $x_i \in A_i$ and A_i has to be in $\mathcal{A}_{i|k}$ for k = 1,2,... . Similarly $x_j \in A_j$ with $A_j \in \bigcap_{k \in N} \mathcal{A}_{j|k}$ but $\bigcap_{k \in N} \mathcal{A}_{i|k} \cap \bigcap_{k \in N} \mathcal{A}_{j|k} = \emptyset$. Thus $x_i \neq x_j$ and (c) is proved.

We easily see that $\bigcup \{\bigcap_{k \in N} F_{i|k} | i \in J \subset I\} = \bigcup \{A \in \mathcal{A} | A \in \bigcap \mathcal{L}\mathcal{A}_{i|k}$ for some $i \in J\} \cap Y$, and thus (d) is verified, too.

The family $\{\bigcap_{k \in N} F_{i|k} \mid i \in I\}$ is even Baire-additive in Y.

6. Problems. We do not know the answers to the following natural questions concerning completely-additive families:

(a) Can Theorem be extended for completely Suslin-additive families in complete (separable) metric spaces?

(b) Can Corollary 2 be extended to $\varkappa$-analytic spaces with $\varkappa > \omega$?

(Consider σ-discretely refinable instead of countably refinable!)

(c) Can Corollary 2 be extended for Suslin-additive families?

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