Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 22 (1981), No. 1, 87--111

Persistent URL: http://dml.cz/dmlcz/106055

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

22,1 (1981)

THE AXIOM OF REFLECTION

A. SOCHOR , P. VOPĚNKA

Abstract: A new very strong axiom of the alternative set theory is formulated. This axiom makes possible performance of further considerations and constructions. In the article some of its applications are shown; in particular, we construct a new type of standard extensions.

Key words: Alternative set theory, endomorphic universe, standard extension, ultrapower, enlargements.

Classification: Primary 02K10, 02K99
Secondary 02H20, 02H13

The alternative set theory can be considered as an intuitive theory similarly as the original Cantor's set theory was comprehended. In the book [V] the alternative set theory is described so that some basic principles are introduced as axioms from which all statements are derived. This approach makes it possible to formalize the alternative set theory and to construct a corresponding formal axiomatic theory (see [S1]). Such a formalization cannot of course describe the alternative set theory in the whole complexity above all, from the following two reasons.

At first in the alternative set theory intuitively taken we assume that for every property there is a class of all sets which fulfil this property. Formalizing this assumption, we have to formalize even the notion of property - this means that we are forced to restrict properties to properties which can be expressed in one formal language. Usually, we restrict ourselves to the usual language of set theory - to the language with the predicate $\in$ and one sort of variables only (cf. [Sl]). This approach is convenient for the usual set-theoretical considerations; on the other hand, in this article we shall show some sort of considerations which cannot be formalized in the language in question. Any formal theory is deprived of course from properties of such kind.

Secondly, it is advantageous to formulate new axioms which are not provable from the original till yet postulated axioms. We know that we are able to develop a great deal of mathematics in the axiomatic system of the alternative set theory (see [V]) but the development of mathematics in the alternative set theory is apparently not finished and therefore it is not possible to guarantee that this axiomatic system is strong enough. Furthermore, accepting the intuitive approach to the alternative set theory, it is natural to look for further axioms since it is our real endeavour to discover as many as possible true assertions independently whether we are able to prove them as theorems or if we formulate them as new axioms if they are not provable.

In this paper we are going to formulate the axiom of reflection which is consistent and independent to the alternative set theory as it was formulated in [V] or in [Sl] and so that this axiom agrees with the alternative set theory intuitively taken - in other words, which is intuitively true. This axiom asserts the existence of systems of classes (called reflecting systems) which are on one side codable ("small") and which on the second side have the "same" property as the system of all classes.

There are ultrafilters on reflecting systems which possess no countable classes. Using the axiom of reflection we are able to create ultrapowers using even such ultrafilters. Even these ultrapowers are isomorphic to the universal class and hence it is natural to combine both these constructions and this led us to the notion of ultralimit. In the second section we investigate properties of this notion.

In [S-V1] we dealt with a standard extension defined for all elements of one reflecting system - the system of all subclasses of the endomorphic universe in question. In the third section of the article we are generalizing the notion of standard extension also for other reflecting systems. Some properties of such a generalized notion are similar as in the original case; on the other hand, e.g., it is not clear whether for every reflecting system there is at most one standard extension. Furthermore, we can construct standard extensions, which corresponds to "enlargements" investigated in nonstandard methods. At the end of the paper we are going to show that the generalized notion agrees with the original one on a very large class (which is even an endomorphic universe).

We use the results and notions of [V] and [S-V1].

§ 1. Reflecting systems. We claimed that there are properties of classes which cannot be expressed in the usual language of set theory, i.e. using the predicate $\in$ and one sort of variables only. We are going to show an example of such a property, following the Tarski's idea of satisfaction.

Let us recall that every formula of the language FL_V is a set. Further if $\varphi(Z)$ is a formula of the language FL_V and if X is an arbitrary class then we use quite freely the symbol $\varphi(X)$ and we say that the class X fulfils the property φ. However, this needs a little explanation. Of course, if φ is a metamathematical (concrete, expressible in the natural non-formalized language) property, then we understand the meaning of the term $\varphi(X)$ as usual, but we claim that this term is meaningful for every property $\varphi \in FL_V$. In the alternative set theory intuitively taken (cf.[V]), we assumed that for every (even intuitive) property ψ the class $\{x, \psi(x)\}$ exists. Hence there is even the class of all formalizations of metamathematical formulas. Such a class must contain the class FL_V (since the class FN is the smallest transitive proper subclass of N) and therefore in the alternative set theory intuitively taken we are always justified to give a meaning to the term $\varphi(X)$ for every class X and for every $\varphi \in FL_V$. Moreover, every formula of the language FL_V is a formalization of a metamathematical formula in this case and we are going to neglect the difference between this metamathematical formula and its formalization. Some notes about the difference arising by formal approach can be found in [S1].

Let Stsf(z,Z) be a property of pairs of a set and class defined for every formula of the language FL_V and every class X by the equivalence $\mathrm{Stsf}(\varphi,X) \equiv \varphi(X)$. We shall show that the property Stsf cannot be expressed in the usual language of set theory. To get a contradiction let us suppose that ψ is a formula of the language FL_V such that for every $\varphi \in FL_V$

and every class X we have $\psi(\varphi, X) \equiv \mathrm{Stsf}(\varphi, X) \equiv \varphi(X)$. Put $\overline{\psi}(z) \equiv \neg\, \psi(z,z)$. Then for every $\varphi \in FL_V$ we have $\overline{\psi}(\varphi) \equiv$ $\equiv \neg\, \mathrm{Stsf}(\varphi, \varphi) \equiv \mathrm{Stsf}(\neg\, \varphi, \varphi) \equiv \neg\, \varphi(\varphi)$. In particular, $\overline{\psi}(\overline{\psi}) \equiv \neg\, \overline{\psi}(\overline{\psi})$ which is a contradiction.

Thus it is meaningful to deal with languages having more predicates since in such languages we can express more properties. This leads to the following definition.

Let $P_1(Z_1,\ldots,Z_{k_1}),\ldots,P_m(Z_1,\ldots,Z_{k_m})$ be predicates. We define analogically the formulas of the language $L_C(P_1,\ldots,P_m)$ as the formulas of the language L_C were defined in § 5 ch. II [V] but with the following complements:

(a) to the paragraph (1) of the definition of the alphabet we give more the requirement that $\langle 11,0\rangle,\ldots,\langle 10{+}m,0\rangle$ are (code) the signs $P_1,\ldots,P_m$ respectively.

(b) In the definition of formulas we add to the rule (1) that the words $P_1(\Gamma_1,\ldots,\Gamma_{k_1}),\ldots,P_m(\Gamma_1,\ldots,\Gamma_{k_m})$ are atomic formulas under the assumption that Γ_i are variables or constants for the elements of C.

The class of all formulas constructed in the way described above is denoted by $L_C(P_1,\ldots,P_m)$. The language $FL_C(P_1,\ldots$ $\ldots,P_m)$ is quite analogically defined from the language $L_C(P_1,\ldots,P_m)$ as the language FL_C is defined from the language L_C.

Let us remind that we are able to transfer the above described consideration of the property Stsf quite mechanically for languages with predicates $P_1,\ldots,P_m$ and hence even in this case there are properties of classes which cannot be expressed in the language in question. Hence the alternative set theory intuitively taken cannot be formalized fully in any language with finitely many predicates. We say that a class $\mathcal{R}$ is a reflecting system (of classes) in the language $FL_C(P_1,\ldots$ $\ldots,P_m)$ (in symbols $\mathrm{Refl}(\mathcal{R},P_1,\ldots,P_m)$ iff the following two conditions hold:

(a) for every formula $\varphi(Z,Z_1,\ldots,Z_k)$ of the language $FL_C(P_1,\ldots,P_m)$ and every $X_1,\ldots,X_k \in \mathcal{R}$ we have

$$(\exists X)\,\varphi(X,X_1,\ldots,X_k) \longrightarrow (\exists X \in \mathcal{R})\,\varphi(X,X_1,\ldots,X_k)$$

(b) if $\{X_n; n \in FN\} \subseteq \mathcal{R}$ then there is a coding pair $\langle K,S\rangle \in \mathcal{R}$ which codes $\{X_n; n \in FN\}$.

A class $\mathcal{R}$ satisfying the condition (a) only is called a simply reflecting system in the language $FL_C(P_1,\ldots,P_m)$, in symbols $\mathrm{SRefl}(\mathcal{R},P_1,\ldots,P_m)$. $\mathcal{R}$ is called a reflecting system (simply reflecting system respectively) iff it is a reflecting system (simply reflecting system respectively) in the language FL_V.

Let us emphasize that we did not express the predicate $\mathrm{Refl}(Z)$ in the language FL_V.

It is evident that if $\mathcal{R}$ is a reflecting system in the language $FL_{C_1}(P_1,\ldots,P_{m+1})$ and if $C_2 \subseteq C_1$, then $\mathcal{R}$ is a reflecting system in the language $FL_{C_2}(P_1,\ldots,P_m)$, too.

<u>Theorem</u>. If $\mathcal{R}$ is a simply reflecting system in the language $FL_C(P_1,\ldots,P_m)$, then $C \subseteq \mathcal{R}$.

Proof. Let a set $x \in C$ be given and let $\varphi(z)$ be the formula $z = x$. Then $\varphi \in FL_C$ and we have obviously $(\exists y)\,\varphi(y)$. Therefore we get $(\exists y \in \mathcal{R})\,\varphi(y)$ by the definition of a simply reflecting system.

If φ is a formula of the language $FL_C(P_1,\ldots,P_m)$ and if $\mathcal{R}$ is a system of classes, then $\varphi^{(\mathcal{R})}$ d[illegible]he formula resulting from φ by restriction of all quantifiers binding class variables to the elements of $\mathcal{R}$ (quantifiers binding set variables are let without change).

By induction according to the complexity of the formulas we are able to demonstrate the following statement.

Theorem. A system $\mathcal{R}$ is a simply reflecting system in the language $FL_C(P_1,\dots,P_m)$ iff for every formula $\varphi(Z_1,\dots$ $\dots,Z_k)$ of the language $FL_C(P_1,\dots,P_m)$ and every $X_1,\dots,X_k \in$ $\in \mathcal{R}$ we have

$$\varphi(X_1,\dots,X_k) \equiv \varphi^{(\mathcal{R})}(X_1,\dots,X_k).$$

In particular, if $\mathcal{R}$ is a simply reflecting system,then for every axiom φ of the alternative set theory it is $\varphi^{(\mathcal{R})}$ and hence $\mathcal{R}$ is closed under all usual set-theoretical operations.

In [V] it is proved that the system of all classes (of extended universe) is not codable. We can interprete this result that we are not able to sight the system of all classes as the whole or, in other words, that this system is not finished (neither potentially finished). In fact,we proved that this system is not codable by the Cantor's diagonal method - this means that assuming that all classes of extended universe are available (are in a list), then we can construct a new class of extended universe which is not in our list.

On the other hand, it is very convenient to suppose that we are able to make a list of all possible properties of classes - in other words, to make a list of classes such that for every property if there is a class fulfilling it, then there is even a class in our system fulfilling the property in question. Such an approach makes it possible for us - from some aspects - to sight the system of all classes of extended universe as the whole (as finished). Using our definition of a reflecting system, such a consideration leads us to the acceptance of the following axiom.

Axiom of reflection. For every class X there is a codable reflecting system $\mathcal{R}$ with $X \in \mathcal{R}$.

We can prove that the alternative set theory with the axiom of reflection is consistent (relatively to ZF, say). On the other hand, this axiom is also independent (cf. [Sl]).

We have proved that there are properties which cannot be expressed in the usual language of set theory. Thus it is natural to consider more complex languages. In this case we are led to accept the following general principle.

Principle of reflection. For any predicates $P_1,\dots,P_m$ and every class X there is a codable reflecting system $\mathcal{R}$ in the language $FL_V(P_1,\dots,P_m)$ with $X \in \mathcal{R}$.

Assuming the principle of reflection for predicates $P_1,\dots,P_m$, we can demonstrate the following two results.

Theorem. If $\mathcal{S}$ is a codable system of classes, then there is a codable reflecting system $\mathcal{R}$ in the language $FL_V(P_1,\dots,P_m)$ with $\mathcal{S} \subseteq \mathcal{R}$.

Proof. Let a coding pair $\langle K,S \rangle$ code the system $\mathcal{S}$ and let $\mathcal{R}$ be a codable reflecting system in the language $FL_V(P_1,\dots,P_m)$ with $\langle K,S \rangle \in \mathcal{R}$. Then for every x we have $x \in \mathcal{R}$ and hence even $S''\{x\} \in \mathcal{R}$.

Theorem. Let $\{\varphi_\alpha(Z); \alpha \in \Omega\}$ be a sequence of formulas of the language $FL_V(P_1,\dots,P_m)$ so that for every $\alpha \in \Omega$

the formula $(\exists X)\ \varphi_\alpha(X)$ holds. Then there is a codable class $\{X_\alpha, \alpha \in \Omega\}$ such that for every $\alpha \in \Omega$ we have $\varphi_\alpha(X_\alpha)$.

Proof. Let $\mathcal{R}$ be a codable reflecting system in the language $FL_V(P_1,\dots,P_m)$ and let a coding pair $\langle \Omega, S \rangle$ code $\mathcal{R}$. For every $\alpha \in \Omega$ let $F(\alpha)$ be the smallest ordinal number such that the formula $\varphi_\alpha(S''\{F(\alpha)\})$ holds. For every $\alpha \in \Omega$ the class of all $\beta \in \Omega$ for which it is $\varphi_\alpha(S''\{\beta\})$ is non-empty. Hence $\mathrm{dom}(F) = \Omega$. Therefore it is sufficient to put $X_\alpha = S''\{F(\alpha)\}$.

In particular, we have proved the following form of the axiom of choice:

$$(\forall n \in FN)(\exists X)\ \varphi(n,X) \rightarrow (\exists X)(\forall n \in FN)\ \varphi(n, X''\{n\})$$

under the assumption that φ is a formula of the language $FL_V(P_1,\dots,P_m)$.

§ 2. Ultralimits. Let us remind that every reflecting system is a ring of classes (cf. § 4 ch. II [V]) and hence there are ultrafilters on it. In this whole section, let $\mathcal{R}$ and $\mathcal{M}$ denote a reflecting system and an ultrafilter on $\mathcal{R}$ respectively.

A mapping $\mathcal{U}\ell$ with $\mathrm{dom}(\mathcal{U}\ell) = \{G \in \mathcal{R}\ ;\ \mathrm{dom}(G) = V\}$ & & $\mathrm{rng}(\mathcal{U}\ell) \subseteq V$ is called an ultralimit on $\mathcal{R}$ according to $\mathcal{M}$ iff for every $G_1,\dots,G_k \in \mathrm{dom}(\mathcal{U}\ell)$ and for every set-formula $\varphi(z_1,\dots,z_k)$ of the language FL we have

$$\varphi(\mathcal{U}\ell(G_1),\dots,\mathcal{U}\ell(G_k)) \equiv \{x\,;\ \varphi(G_1(x),\dots,G_k(x))\} \in \mathcal{M}.$$

Theorem. If $\mathcal{U}\ell$ is an ultralimit on $\mathcal{R}$ according to $\mathcal{M}$, then a mapping $\mathcal{F}$ with $\mathrm{dom}(\mathcal{F}) = \mathrm{dom}(\mathcal{U}\ell)$ & $\mathrm{rng}(\mathcal{F}) \subseteq V$

is an ultralimit of $\mathcal{R}$ according to $\mathcal{M}$ iff there is a similarity H such that $\mathrm{dom}(H) \supseteq \mathrm{rng}(\mathcal{U}\ell)$ and such that for every $G \in \mathrm{dom}(\mathcal{U}\ell)$ we have $\mathcal{F}(G) = H(\mathcal{U}\ell(G))$.

Proof. If $\mathcal{F}$ is an ultralimit on $\mathcal{R}$ according to $\mathcal{M}$, then for every $y \in \mathrm{rng}(\mathcal{U}\ell)$ we define $H(y) = \mathcal{F}(G)$ where G is a function with $\mathcal{U}\ell(G) = y$. Evidently, if $G_1, G_2 \in \mathrm{dom}(\mathcal{U}\ell)$ and $\mathcal{U}\ell(G_1) = \mathcal{U}\ell(G_2)$, then $\{x; G_1(x) = G_2(x)\} \in \mathcal{M}$ and hence $\mathcal{F}(G_1) = \mathcal{F}(G_2)$ according to the definition of ultralimit. Further, if $\varphi(z_1,\dots,z_k)$ is a set-formula of the language FL and if $G_1,\dots,G_k \in \mathrm{dom}(\mathcal{U}\ell)$ and $\mathcal{U}\ell(G_1) = y_1, \dots, \mathcal{U}\ell(G_k) = y_k$, then the equivalences $\varphi(H(y_1),\dots,H(y_k)) \equiv \varphi(\mathcal{F}(G_1),\dots \dots, \mathcal{F}(G_k)) \equiv \{x; \varphi(G_1(x),\dots,G_k(x))\} \in \mathcal{M} \equiv \varphi(\mathcal{U}\ell(G_1),\dots \dots, \mathcal{U}\ell(G_k)) \equiv \varphi(y_1,\dots,y_k)$ hold. Thus we have proved that H is a similarity.

On the other hand, if H is a similarity with $\mathrm{dom}(H) \supseteq \mathrm{rng}(\mathcal{U}\ell)$, then for every set-formula $\varphi(z_1,\dots,z_k)$ of the language FL and for every $G_1,\dots,G_k \in \mathrm{dom}(\mathcal{U}\ell)$ we have $\varphi(H(\mathcal{U}\ell(G_1)),\dots,H(\mathcal{U}\ell(G_k))) \equiv \varphi(\mathcal{U}\ell(G_1),\dots,\mathcal{U}\ell(G_k)) \equiv \{x; \varphi(G_1(x),\dots,G_k(x))\} \in \mathcal{M}$ and therefore the composition of H and $\mathcal{U}\ell$ is an ultralimit on $\mathcal{R}$ according to $\mathcal{M}$.

Let K_y denote the function G such that for every $x \in V$ we have $G(x) = y$.

<u>Theorem</u>. Let $\mathcal{U}\ell$ be an ultralimit on $\mathcal{R}$ according to $\mathcal{M}$. Put $F(y) = \mathcal{U}\ell(K_y)$ for every $y \in V$. Then F is an endomorphism.

Proof. Evidently, the equality $\mathrm{dom}(F) = V$ holds. If $\varphi(z_1,\dots,z_k)$ is a set-formula of the language FL, then for every $y_1,\dots,y_k$ we have $\varphi(F(y_1),\dots,F(y_k)) \equiv \varphi(\mathcal{U}\ell(K_{y_1}),\dots$

$\ldots, \mathcal{U}\ell(K_{y_k})) \equiv \{x; \varphi(K_{y_1}(x),\ldots,K_{y_k}(x))\} \in \mathfrak{M} \equiv$
$\equiv \varphi(y_1,\ldots,y_k)$, since the class $\{x; \varphi(y_1,\ldots,y_k)\}$ is either O or V.

Theorem. If $\mathcal{U}\ell$ is an ultralimit on $\mathcal{R}$ according to $\mathfrak{M}$, then rng($\mathcal{U}\ell$) is an endomorphic universe.

Proof. We are going to verify the condition (3) of the first theorem of [S-V 1].

(a) Let $\varphi(z,z_1,\ldots,z_k)$ be a set-formula of the language FL and let $G_1,\ldots,G_k$ be elements of dom($\mathcal{U}\ell$) so that the formula $(\exists y)\varphi(y, \mathcal{U}\ell(G_1),\ldots, \mathcal{U}\ell(G_k))$ holds. Thus we have $\{x;(\exists y)\varphi(y,G_1(x),\ldots,G_k(x))\} \in \mathfrak{M}$ and thence by the axiom of choice there is a function G with $\{x;(\exists y)\varphi(y,G_1(x),\ldots$ $\ldots,G_k(x))\} = \{x;\varphi(G(x),G_1(x),\ldots,G_k(x))\}$ & $\mathrm{dom}(G) = V$. Therefore according to the definition of a reflecting system we can suppose moreover that $G \in \mathcal{R}$. Hence $\mathcal{U}\ell(G) \in$ rng($\mathcal{U}\ell$) and we have $\varphi(\mathcal{U}\ell(G), \mathcal{U}\ell(G_1),\ldots, \mathcal{U}\ell(G_k))$.

(b) Let $\{\mathcal{U}\ell(G_n); n \in FN\}$ be a countable function (which is a subclass of rng($\mathcal{U}\ell$)). Thus for every $n \in FN$, the class $X_n = \{x;\mathrm{Fnc}(\{G_0(x),\ldots,G_n(x)\})\} \in \mathfrak{M}$. Let $G \in \mathcal{R}$ be a function such that for every $x \in (X_n - X_{n+1})$ we have $G(x) = \{G_0(x),\ldots,G_n(x)\}$ and for every $x \in \bigcap\{X_n;n \in FN\}$ the set $G(x)$ is a function with $\{G_n(x);n \in FN\} \subseteq G(x)$. Then evidently it is $\mathcal{U}\ell(G) \supseteq \{\mathcal{U}\ell(G_n);n \in FN\}$ and $\mathrm{Fnc}(\mathcal{U}\ell(G))$.

Theorem. If $\mathcal{R}$ is a codable reflecting system, then there is an ultralimit on $\mathcal{R}$ according to $\mathfrak{M}$.

Proof. Let a coding pair $\langle K,S\rangle$ extensionally code $\mathcal{R}$. If $R = S \upharpoonright \{x \in K;\mathrm{Fnc}(S''\{x\})$ & $\mathrm{dom}(S''\{x\}) = V\}$, then by the proof of the last but one theorem of [V] there is an endomorphism

F_1 and a set r such that $F_1"R \subseteq r$ and such tha for every set-formula $\varphi(z)$ of the language FL we have $(\forall x)((x \subseteq R \ \& \ \& \ Fin(x)) \longrightarrow \varphi(x)) \longrightarrow \varphi(r)$. For every $G \in \mathcal{R}$ with $dom(G) = V$ there is exactly one $\tilde{x} \in K$ so that $G = R"\{\tilde{x}\}$. Hence putting $\tilde{G} = r"\{F_1(\tilde{x})\}$, we see that $\tilde{G}$ is a set and moreover from the trivial fact $(\forall x \subseteq R)(Fin(x) \longrightarrow (\forall y)Fnc(x"\{y\}))$ and from the above stated property we conclude that $\tilde{G}$ is a function.

Let M denote the class of all sets of the form $\{x; \varphi(\tilde{G}_1(x),\dots,\tilde{G}_k(x)) \ \& \ x \in dom(\tilde{G}_1) \ \& \dots \& \ x \in dom(\tilde{G}_k)\}$ where $dom(G_1) = \dots = dom(G_k) = V$, $G_1,\dots,G_k \in \mathcal{R}$ and where $\varphi(z_1,\dots,z_k)$ is a set-formula of the language FL for which the statement $\{x; \varphi(G_1(x),\dots,G_k(x))\} \in \mathfrak{M}$ holds. Evidently $x_1, x_2 \in M \longrightarrow x_1 \cap x_2 \in M$. For every set-formula $\varphi(z_1,\dots,z_k)$ of the language FL and for every $G_1,\dots,G_k \in \mathcal{R}$ with $dom(G_1) = \dots = dom(G_k) = V$ and for every $\tilde{x}_1,\dots,\tilde{x}_k \in K$ with $G_1 = R"\{\tilde{x}_1\} \ \& \dots \& \ G_k = R"\{\tilde{x}_k\}$ we have $0 \neq F_1"\{x; \varphi(G_1(x),\dots$ $\dots,G_k(x))\} = F_1"\{x; \varphi((R"\{\tilde{x}_1\})(x),\dots,(R"\{\tilde{x}_k\})(x))\} =$ $= \{x; \varphi(((F_1"R)"\{F_1(\tilde{x}_1)\})(x),\dots,((F_1"R)"\{F_1(\tilde{x}_k)\})(x)) \ \& \ x \in$ $\in dom((F_1"R)"\{F_1(\tilde{x}_1)\}) \ \& \dots \& \ x \in dom((F_1"R)"\{F_1(\tilde{x}_k)\})\} \subseteq$ $\subseteq \{x; \varphi((r"\{F_1(\tilde{x}_1)\})(x),\dots,(r"\{F_1(\tilde{x}_k)\})(x)) \ \& \dots \& \ x \in$ $\in dom(r"\{F_1(\tilde{x}_1)\}) \ \& \ x \in dom(r"\{F_1(\tilde{x}_k)\})\} = \{x; \varphi(G_1(x),\dots$ $\dots,\tilde{G}_k(x)) \ \& \ x \in dom(\tilde{G}_1) \ \& \dots \& \ x \in dom(\tilde{G}_k)\}$. We proved just now that every element of M is nonempty. Thus we can choose an ultrafilter $\mathfrak{M}_2$ on the ring of all set-theoretically definable classes so that $M \subseteq \mathfrak{M}_2$.

According to § 2 ch. V [V] there is an endomorphism F_2 and a set d such that F_2, $\mathfrak{M}_2$, d are coherent. If $G \in \mathcal{R} \ \& \ \& \ dom(G) = V$, then $dom(G) \in \mathfrak{M}$ and hence $dom(\tilde{G}) \in M$ from which the statement $d \in dom(\tilde{G})$ follows by the definition of coherentness. For every $G \in \mathcal{R}$ with $\mathrm{dom}(G) = V$ we put $\mathcal{Ul}(G) = \tilde{G}(d)$. If $\varphi(z_1,\dots,z_k)$ is a set-formula of the language FL and if $G_1,\dots,G_k \in \mathrm{dom}(\mathcal{Ul})$, then $\varphi(\mathcal{Ul}(G_1),\dots,\mathcal{Ul}(G_k)) \equiv \varphi(\tilde{G}_1(d),\dots,\tilde{G}_k(d)) \equiv \{x;\ \varphi(\tilde{G}_1(x),\dots,\tilde{G}_k(x))\ \&\ x \in \mathrm{dom}(\tilde{G}_1)\ \&\ \dots\ \&\ x \in \mathrm{dom}(\tilde{G}_k)\} \in \mathcal{M}_2 \equiv \{x;\ \varphi(\tilde{G}_1(x),\dots,\tilde{G}_k(x))\ \&\ x \in \mathrm{dom}(\tilde{G}_1)\ \&\ \dots\ \&\ x \in \mathrm{dom}(\tilde{G}_k)\} \in M \equiv \{x;\ \varphi(G_1(x),\dots,G_k(x))\} \in \mathcal{M}$. We have proved that the mapping $\mathcal{Ul}$ is an ultralimit on $\mathcal{R}$ according to $\mathcal{M}$.

An ultralimit $\mathcal{Ul}$ on $\mathcal{R}$ according to $\mathcal{M}$ is called total iff $\mathrm{rng}(\mathcal{Ul}) = V$.

Combining the results of this section, we obtain the following result.

Theorem. If $\mathcal{R}$ is a codable reflecting system, then there is a total ultralimit on $\mathcal{R}$ according to $\mathcal{M}$.

Theorem. Let $\mathcal{Ul}$ be a total ultralimit on $\mathcal{R}$ according to $\mathcal{M}$. Put $\overline{X} = \{\mathcal{Ul}(G);\ G \in \mathrm{dom}(\mathcal{Ul})\ \&\ \{x; G(x) \in X\} \in \mathcal{M}\}$. Then for every normal formula $\varphi(z_1,\dots,z_k,Z_1,\dots,Z_m)$, for every $G_1,\dots,G_k \in \mathrm{dom}(\mathcal{Ul})$ and for every $X_1,\dots,X_m \in \mathcal{R}$ we have

$$\varphi(\mathcal{Ul}(G_1),\dots,\mathcal{Ul}(G_k),\overline{X}_1,\dots,\overline{X}_m) \equiv \{x;\ \varphi(G_1(x),\dots,G_k(x),X_1,\dots,X_m)\} \in \mathcal{M}.$$

Proof. According to the definition of ultralimit it is sufficient to deal only with the atomic formulas of the form $z_i \in Z_j$, but in this case the statement follows immediately from the definition of $\overline{X}$. The induction steps for negation and conjunction are obvious. Let us consider the induction step of equivalences using the fact that $\mathcal{Ul}$ is a total ultralimit for the first equivalence and the fact that $\mathcal{R}$ is a reflecting system for the last one:

$(\exists y)\ \varphi(y, \mathcal{U\ell}(G_1),\ldots, \mathcal{U\ell}(G_k),\overline{X}_1,\ldots,\overline{X}_m) \equiv (\exists G \in \mathrm{dom}(\mathcal{U\ell}))$
$\varphi(\mathcal{U\ell}(G), \mathcal{U\ell}(G_1),\ldots, \mathcal{U\ell}(G_k),\overline{X}_1,\ldots,\overline{X}_m) \equiv (\exists G \in \mathrm{dom}(\mathcal{U\ell}))\{x;$
$\varphi(G(x),G_1(x),\ldots,G_k(x),X_1,\ldots,X_m)\} \in \mathfrak{M} \equiv \{x;(\exists y)\ \varphi(y,$
$G_1(x),\ldots,G_k(x),X_1,\ldots,X_m)\} \in \mathfrak{M}$.

At the end let us note that all results of this section except the third and fifth theorem hold for simple reflecting systems, too.

§ 3. Standard extensions of reflecting systems. Let A denote an endomorphic universe in the whole of this section.

A system $\mathcal{R}$ of subclasses of A is called a reflecting system on A iff the formula $\mathrm{Refl}^A(\mathcal{R})$ holds. Similarly we define simple reflecting systems and corresponding notions in more complex languages.

Let us mention that $\mathcal{R}$ is a reflecting system iff it is a reflecting system on V. Moreover, let us note that if F is an endomorphism with rng(F) = A, then the following conditions are equivalent (this follows trivially from the second theorem of § 1 ch. V [V]):

(a) $\mathcal{R}$ is a reflecting system on A

(b) there is a reflecting system $\mathcal{S}$ such that $\mathcal{R} =$
$= \{F''X; X \in \mathcal{S}\}$

(c) $\{F^{-1}{}''X; X \in \mathcal{R}\}$ is a reflecting system .

Up to the end of this section we shall suppose that $A \neq V$ and that $\mathcal{R}$ is a reflecting system on A. Let us realize that $A \in \mathcal{R}$ and that $(\forall X \in \mathcal{R})X \subseteq A$.

An operation Ex defined for all elements of $\mathcal{R}$ is called a standard extension of $\mathcal{R}$ iff for arbitrary n.. ..al formula $\varphi(Z_1,\dots,Z_k)$ of the language FL_A and for arbitrary $X_1,\dots,X_k \in \mathcal{R}$ we have

$$\varphi^A(X_1,\dots,X_k) \equiv \varphi(\mathrm{Ex}(X_1),\dots,\mathrm{Ex}(X_k)) .$$

Repeating the proof of the first theorem of § 2 [S-V 1], we get the following result.

Theorem. An operation Ex defined for all elements of $\mathcal{R}$ is a standard extension of $\mathcal{R}$ iff for arbitrary normal formula $\varphi(z_1,\dots,z_k,Z_1,\dots,Z_m)$ of the language FL_A and for arbitrary $X_1,\dots,X_m \in \mathcal{R}$ we have

$$\mathrm{Ex}(\{\langle x_1,\dots,x_k\rangle \in A; \varphi^A(x_1,\dots,x_k,X_1,\dots,X_m)\}) =$$
$$= \{\langle x_1,\dots,x_k\rangle; \varphi(x_1,\dots,x_k,\mathrm{Ex}(X_1),\dots,\mathrm{Ex}(X_m))\} .$$

Let us fix a standard extension Ex of $\mathcal{R}$ up to the end of this section. Using the definition of standard extension of $\mathcal{R}$ and the previous theorem,we see again that all formulas of the list on pp. 617 and 618 of [S-V 1] hold. Moreover, repeating considerations of the second proof of the section in question, we obtain the following statement.

Theorem. If $\varphi(z,Z_1,\dots,Z_k)$ is a normal formula of the language FL_A and if $X_1,\dots,X_k \in \mathcal{R}$, then the equivalence

$$(\exists x)\varphi(x,\mathrm{Ex}(X_1),\dots,\mathrm{Ex}(X_k)) \equiv (\exists x \in A)\varphi(x,\mathrm{Ex}(X_1),\dots,\mathrm{Ex}(X_k))$$

holds.

Theorem. $\mathrm{Ex}(FN) \neq FN$.

Proof. Let $\stackrel{\circ}{=}$ be the equivalence defined in § 1 ch. V [V] and let F be an endomorphism with $\mathrm{rng}(F) = A$. Then $\{F^{-1}"X; X \in \mathcal{R}\}$ is a reflecting system and hence there is a class $B \in \mathcal{R}$ so that $F^{-1}"B$ is a selector of this equivalence. Further, using the statement $(\forall x)(\exists y \in A)x \stackrel{\circ}{=} y$ and the fact that F is an endomorphism, we get that even B is a selector of the investigated equivalence. Furthermore $(\forall x,y \in \in Ex(B))(x \neq y \rightarrow (\exists \varphi \in Ex(FL))(\neg \varphi(x) \equiv \varphi(y)))$. To obtain a contradiction, let us suppose that Ex(*FN*) = *FN*. In this case we have Ex(FL) = *FL*, since the class *FL* is defined by a normal formula from the class FN. Therefore $Ex(B) \supseteq B$ is a selector of the equivalence $\doteq$, too. Thus we have proved *B* = = Ex(B). On the other hand, there is a one-one mapping *G* of A onto B and the statement $B \neq Ex(B)$ is a consequence of the formula Ex(dom(G)) = Ex(A) = $V \neq A$ = dom(G).

<u>Theorem</u>. The class Ex(X) is fully revealed for every $X \in \mathcal{R}$.

Proof. It is sufficient to prove that Ex(X) is revealed according to the first theorem of this section and according to the assumption that $\mathcal{R}$ is a reflecting system. Let us assume that Ex(X) is not revealed for a class $X \in \mathcal{R}$. Then there is a countable class $Y \subseteq Ex(X)$ such that $(\forall u)(Y \subseteq u \rightarrow \rightarrow \neg u \subseteq Ex(X))$. By the prolongation axiom there is a set f so that f"FN = Y. Using our assumption we can conclude that $(\forall \alpha \notin FN)(\exists \beta \leq \alpha)f(\beta) \notin Ex(X)$. On the other hand, $(\forall n \in FN)f"n \subseteq Ex(X)$ and thence we obtain the statement $(\forall \alpha)(f"\alpha \subseteq Ex(X) \rightarrow f(\alpha) \in Ex(X))$. Moreover, by the last theorem $Ex(FN) \neq FN$ and therefore we get $(\exists \gamma \in Ex(FN))f(\gamma) \notin \notin Ex(X)$. Thus we have proved the formula $(\exists f)((\forall \alpha)(f"\alpha \subseteq \subseteq Ex(X) \rightarrow f(\alpha) \in Ex(X)) \& (\exists \gamma \in Ex(FN))f(\gamma) \notin Ex(X))$. By the definition of standard extension of $\mathcal{R}$ we get $(\exists f)((\forall \alpha)(f"\alpha \subseteq X \rightarrow f(\alpha) \in X) \& (\exists \gamma \in FN)f(\gamma) \notin X)$ which is a contradiction.

Theorem. If $X \in \mathcal{R}$, then $X = Ex(X)$ iff X is finite.

Proof. If X is finite, then X is a set which is an element of A. Hence the equality $X = Ex(X)$ is obvious in this case. On the other hand, if X is an infinite class, then there is a one-one mapping $f \in A$ such that $f''FN \subseteq X$. Thus $Ex(f''FN) = f''Ex(FN) \subseteq Ex(X)$. If we would have $Ex(X) = X$, then the formula $f''Ex(FN) \subseteq A$ would hold and therefore we would obtain $Ex(FN) \subseteq A$ which contradicts $Ex(FN) \cap A = FN \neq Ex(FN)$.

Repeating the proof of the fourth theorem before the last one of § 2 [S-V 1] we get the following statement.

Theorem. If $u \subseteq A$, then u is a finite set.

Theorem. For every $X \in \mathcal{R}$ we have $Ex(P(X)) \subseteq P(Ex(X))$.

Proof. Using the last theorem and the definition of standard extension of $\mathcal{R}$, we obtain $Ex(P(X)) = Ex(\{u; (\forall x)(x \in u \rightarrow x \in X)\}) = Ex(\{u \in A; (\forall x)(x \in u \rightarrow x \in X)\}) \subseteq Ex(\{u \in A; (\forall x \in A)(x \in u \rightarrow x \in X)\}) = \{u; (\forall x)(x \in u \rightarrow x \in Ex(X)\} = P(Ex(X))$.

We are going to use the following lemma in the proof of the next theorem, however, let us realize that it is meaningful itself since there are classes (eg. Ω) with $(\forall u)(u \cap X \preceq FN)$ and thence there are elements of $\mathcal{R}$ fulfilling the condition of the lemma in question.

Lemma. If $X \in \mathcal{R}$ and if the formula $(\forall u \in A) X \cap u \preceq FN$ holds, then $Ex(P(X)) = P(Ex(X))$.

Proof. According to the proof of the last theorem it is sufficient to show that under our condition it is $\{u \in A; (\forall x)(x \in u \rightarrow x \in X)\} = \{u \in A, (\forall x \in A)(x \in u \rightarrow x \in X)\}$, i.e. $(\forall u \in A)(u \cap A \subseteq X \rightarrow u \subseteq X)$. If $u \cap A \subseteq X$, then $u \cap A \subseteq u \cap X$ and hence the class $u \cap A$ is at most countable. Using the eighth theorem of § 4 ch. I [V] relativised to the endomorphic universe A, we conclude that u is finite. Thus $u \subseteq A$ and therefore the required implication is trivial.

The next theorem will be later used for construction of particular standard extensions.

<u>Theorem</u>. The following conditions are equivalent:

(a) $(\exists v)(A \subseteq v \in Ex(P(A)))$

(b) $(\exists X \in \mathcal{R})(\exists v)(\neg FN \preceq X \,\&\, X \subseteq v \in Ex(P(X)))$

(c) $(\forall X \in \mathcal{R})(\exists v)(X \subseteq v \in Ex(P(X)))$

(d) $(\forall X \in \mathcal{R})(\exists v)(X \subseteq v \in P(Ex(X)))$.

Proof. The implication (a) $\longrightarrow$ (b) is obvious because there is no countable endomorphic universe, the statement (c) $\longrightarrow$ (d) is a trivial consequence of the last theorem. The implication (c) $\longrightarrow$ (a) is evident and moreover the result (d) $\longrightarrow$ (b) follows from the last lemma and from the fact that there is an uncountable $X \in \mathcal{R}$ with $(\forall u \in A)(X \cap u \preceq FN)$.

Therefore it remains to prove the implication (b) $\longrightarrow$ (c) only. If $X \subseteq A$ is a countable class, then our claim obviously follows from the fact that $X \subseteq f''\alpha \in Ex(P(X))$ for every $\alpha \in Ex(FN) - FN$ and every $f \in A$ with $f''FN = X$. Thence we can suppose that there are Y, u with $Y \subseteq u \in Ex(P(X)) \,\&\, \neg(Y \preceq FN) \,\&\, Y \in \mathcal{R}$ and that there is $G \in \mathcal{R}$ which is a one-one mapping of Y onto X. There are only finite subsets of Y by the last but one theorem and therefore for every $x \in P(Y)$, the class $G''x$ is a set which is an element of P(X). According to the definition of standard extension of $\mathcal{R}$, the class $v = Ex(G)''u$ is a set which is an element of Ex(P(X)), furthermore we have $X = Ex(G)''Y \subseteq Ex(G)''u$ and hence we have proved $X \subseteq v \in Ex(P(X))$.

*

For every d we put $A_{Ex}[d] = \{(Ex(G))(d); G \in \mathcal{R} \& d \in Ex(dom(G))\}$.

In [S-V 1] we investigated classes of the form A[d] where $d \in \bigcup A$. If there is a countable class $X \subseteq A$ with $d \in Ex(X)$, then $A[d] = A_{Ex}[d]$ since for every $G \in \mathcal{R}$ with $d \in Ex(dom(G))$ we have $(Ex(G))(d) = (Ex(G \upharpoonright X))(d)$ and moreover there is $g \in A$ such that $g \upharpoonright X = G \upharpoonright X$. In the mentioned paper we further dealt with the standard extension of the reflecting system containing all subclasses of A. In that case we showed that $V = \bigcup \{Ex(X); X \subseteq A \& Count(X)\}$ and therefore, introducing classes of the form $A_{Ex}[d]$, we should have got nothing new in that case.

Let F be an endomorphism with rng(F) = A. If we put $\mathcal{R}_1 = \{F^{-1}"X; X \in \mathcal{R}\}$ and $\mathfrak{M}_1 = \{F^{-1}"X; X \in \mathcal{R} \& d \in Ex(X)\}$, then $\mathcal{R}_1$ is a reflecting system and $\mathfrak{M}_1$ is an ultrafilter on $\mathcal{R}_1$. If $G \in \mathcal{R}_1$ and dom(G) = V, then $F"G \in \mathcal{R}$ and dom(F"G) = A and thence $d \in dom(Ex(F"G))$. Thus it is meaningful to define $\mathfrak{Ul}(G) = (Ex(F"G))(d)$ for all such G. The mapping $\mathfrak{Ul}$ is an ultralimit on $\mathcal{R}_1$ according to $\mathfrak{M}_1$ because for $G_1, \dots, G_k \in \mathcal{R}_1$ with $dom(G_1) = \dots = dom(G_k) = V$ and every set-formula $\varphi(Z_1, \dots, Z_k)$ of the language FL we have

$$\varphi(\mathfrak{Ul}(G_1), \dots, \mathfrak{Ul}(G_k)) \equiv \varphi((Ex(G"G_1))(d), \dots, (Ex(F"G_k))(d)) \equiv$$
$$\equiv d \in \{x; \varphi((Ex(F"G_1))(x), \dots, (Ex(F"G_k))(x))\} \equiv$$
$$\equiv d \in Ex(\{x \in A; \varphi((F"G_1)(x), \dots, (F"G_k)(x))\}) \equiv$$
$$\equiv d \in Ex(F"\{x; \varphi(G_1(x), \dots, G_k(x)\}) \equiv$$
$$\equiv F^{-1}"F"\{x; \varphi(G_1(x), \dots, G_k(x))\} \in \mathfrak{M}_1 \equiv \{x; \varphi(G_1(x), \dots$$
$$\dots, G_k(x))\} \in \mathfrak{M}_1.$$

<u>Theorem</u>. For every d the class $A_{Ex}[d]$ is an endomorphic universe.

Proof. We have $A_{Ex}[d] = \{(Ex(G))(d); G \in \mathcal{R}$ & $d \in$ $\in Ex(dom(G))\} = \{(Ex(G)(d); G \in \mathcal{R}$ & $dom(G) = A\} = \{(Ex(F"G))(d);$ $G \in \mathcal{R}_1$ & $dom(G) = V\} = rng(\mathcal{U})$. Hence it suffices to use the third theorem of the last section.

<u>Theorem</u>. If $A_{Ex}[c] = A_{Ex}[d]$, then there is a one-one mapping $G \in \mathcal{R}$ such that $(Ex(G))(d) = c$.

Proof. There are H and $\tilde{H}$ which are elements of $\mathcal{R}$ and such that $(Ex(H))(d) = c$ and $(Ex(\tilde{H}))(c) = d$. Evidently $d \in$ $\in Ex(\{x; \tilde{H}(H(x)) = x\})$ and we put $G = H \upharpoonright \{x; \tilde{H}(H(x)) = x\}$ and we get $G(d) = c$. We have to prove that G is a one-one-mapping. Let $x, y \in dom(G)$ and $x \neq y$. If we would have $G(x) = G(y)$, then the statement $H(x) = H(y)$ would hold and hence we would obtain $\tilde{H}(H(x)) = \tilde{H}(H(y))$. Further from the assumption $x, y \in dom(G)$ we would get $x = y$ which is a contradiction.

We say that a set d is generic (w.r.t. Ex) iff $A_{Ex}[d] = V$.

If c, d are generic, then there is a one-one mapping $G \in \mathcal{R}$ so that $G(d) = c$.

For every $d \in V - A$ the class $\{X \in \mathcal{R}; d \in Ex(X)\}$ is an ultrafilter on $\mathcal{R}$.

We say that d realizes an ultrafilter $\mathcal{M}$ on $\mathcal{R}$ iff $d \in$ $\in Ex(X) \equiv X \in \mathcal{M}$ for every $X \in \mathcal{R}$. An ultrafilter is realized if there is a set which realizes it.

Let us mention that an ultrafilter $\mathcal{M}$ on $\mathcal{R}$ is realized iff $\bigcap \{Ex(X); X \in \mathcal{M}\} \neq 0$.

<u>Theorem</u>. Let $\mathcal{M}$ be an ultrafilter on a reflecting system $\mathcal{S}$. Then there is an endomorphism F such that there is a standard extension Ex on $F"\mathcal{S}$ and moreover there is a set d which is generic and realizes $F"\mathcal{M}$.

Proof. Let $\mathcal{U}\ell$ be a total ultralimit of $\mathcal{S}$ according to $\mathfrak{M}$. Put $F(x) = \mathcal{U}\ell(K_x)$ for every set x, $A = rng(\mathcal{U}\ell)$ and $Ex(F''X) = \overline{X}$ for every $X \in \mathcal{S}$. Then F is an endomorphism by the second theorem of the last section. Furthermore, using the last theorem of that section and the second theorem of § 1 ch. V [V], we can prove that the equivalence $\varphi^A(F(x_1),\ldots,F(x_k), F''X_1,\ldots,F''X_m) \equiv \varphi(x_1,\ldots,x_k,X_1,\ldots,X_m) \equiv \{x; \varphi(x_1,\ldots,x_k, X_1,\ldots,X_m)\} \in \mathfrak{M} \equiv \varphi(\mathcal{U}\ell(K_{x_1}),\ldots,\mathcal{U}\ell(K_{x_k}),\overline{X}_1,\ldots,\overline{X}_m) \equiv$
$\equiv \varphi(F(x_1),\ldots,F(x_k),Ex(F''X_1),\ldots,Ex(F''X_m))$ holds for every $X_1,\ldots,X_m \in \mathcal{S}$ and every normal formula $\varphi(z_1,\ldots,z_k,Z_1,\ldots$ $\ldots,Z_m)$ of the language FL. Hence we have proved that Ex is a standard extension of $F''\mathcal{S}$.

Let $d = \mathcal{U}\ell(Id)$ where Id is the identity. It is $d \in$ $\in Ex(F''X) \equiv d \in \overline{X} \equiv \{x; x \in X\} \in \mathfrak{M} \equiv X \in \mathfrak{M}$ for every $X \in \mathcal{S}$. We proved just now that d realizes $F''\mathfrak{M}$ and therefore it remains only to show that d is generic. If y is given, then there is $G \in \mathcal{S}$ with $dom(G) = V$ & $\mathcal{U}\ell(G) = y$ because $\mathcal{U}\ell$ is a total ultralimit. We have $\{x; G(x) = G(Id(x))\} = V \in \mathfrak{M}$ and hence by the last theorem of the second section we get $\mathcal{U}\ell(G) = \overline{G}(\mathcal{U}\ell(Id))$, therefore at the end we obtain y = = $(Ex(F''G))(d)$. Thus d is generic and we are done.

We have of course proved a little stronger result. Let us formulate it explicitly.

Let $\mathfrak{M}$ be an ultrafilter on a reflecting system $\mathcal{S}$ and let $\mathcal{U}\ell$ be a total ultralimit of $\mathcal{S}$ according to $\mathfrak{M}$. Put $F(x) = \mathcal{U}\ell(K_x)$ for every set x. Then F is an endomorphism and the operation Ex defined by $Ex(F''X) = \overline{X}$ is a standard extension of $F''\mathcal{S}$. Furthermore , $\mathcal{U}\ell(Id)$ is a generic set such that $(\forall X \in \mathfrak{M})\ \mathcal{U}(\mathrm{Id}) \in \mathrm{Ex}(F''X)$.

<u>Theorem</u>. For every reflecting system $\mathcal{S}$ there is an endomorphism F such that there is a standard extension Ex of $F''\mathcal{S}$ with

$$(\forall X \in \mathcal{S})(\exists u) F''X \subseteq u \subseteq \mathrm{Ex}(F''X).$$

Proof. By the last theorem of the first part of this section we have to construct an endomorphism F such that there is a standard extension of $F''\mathcal{S}$ with $\{v; A \subseteq v \in \mathrm{Ex}(P(A))\} \neq 0$ where $A = \mathrm{rng}(F)$. Let us realize that $\{v; A \subseteq v \in \mathrm{Ex}(P(A))\} = \bigcap \{\{u; x \in u \in \mathrm{Ex}(P(A))\}; x \in A\} = \bigcap \{\mathrm{Ex}(\{u \in A; x \in u \subseteq A\}); x \in A\}$.

Let $\mathfrak{M}$ be an ultrafilter on $\mathcal{S}$ such that for every finite u, the class $\{v; u \subseteq v \,\&\, \mathrm{Fin}(v)\}$ is an element of $\mathfrak{M}$ (such an ultrafilter exists since for every finite u_1, u_2 we have $\{v; u_1 \subseteq v \,\&\, \mathrm{Fin}(v)\} \cap \{v; u_2 \subseteq v \,\&\, \mathrm{Fin}(v)\} \supseteq \{v; (u_1 \cup u_2) \subseteq v \,\&\, \mathrm{Fin}(v)\} \neq 0$). Let F be an endomorphism such that there is a standard extension Ex of $F''\mathcal{S}$ and such that $F''\mathfrak{M}$ is realized. Let us put $\mathrm{rng}(F) = A$. Then $\bigcap \{\mathrm{Ex}(\{u \in A; x \in u \subseteq A\}); x \in A\} \supseteq \bigcap \{\mathrm{Ex}(\{u \in A; x \in u \subseteq A \,\&\, \mathrm{Fin}(u)\}); x \in A\} \supseteq \bigcap \{\mathrm{Ex}(F''X); X \in \mathfrak{M}\} \neq 0$. This finishes the proof.

*

We define $A^{\mathrm{Ex}} = \bigcup \{\mathrm{Ex}(X); X \subseteq A \,\&\, \mathrm{Count}(X)\}$.

Let us realize that this definition has sense since for every countable $X \subseteq A$ we have $X \in \mathcal{R}$ and hence Ex(X) is defined.

<u>Theorem</u>. A^{Ex} is a revealed endomorphic universe.

Proof. If $\{x_n; n \in FN\} \subseteq A^{\mathrm{Ex}}$, then there is a sequence $\{X_n; n \in FN\}$ so that for every $n \in FN$ we have $x_n \in \mathrm{Ex}(X_n) \,\&\, X_n \subseteq A \,\&\, \mathrm{Count}(X_n)$. Put $X = \bigcup \{X_n; n \in FN\}$. Evidently X is a countable subclass of A and further for every $n \in FN$ it is $x_n \in$

$\in Ex(X_n) \subseteq Ex(X) \subseteq A^{Ex}$. We have proved that A^{Ex} is revealed since Ex(X) is revealed.

Let $\varphi(z, z_1, \ldots, z_k)$ be a set-formula of the language FL and let $x_1, \ldots, x_k \in A^{Ex}$ be given. Let us choose a countable class $X \subseteq A$ so that $\langle x_1, \ldots, x_k \rangle \in Ex(X)$ and at the end let us suppose that the formula $(\exists y)\varphi(y, x_1, \ldots, x_k)$ holds. Put $Y = \{\langle y_1, \ldots, y_k \rangle \in X;\ (\exists y \in A)\varphi(y, y_1, \ldots, y_k)\}$. Evidently, $\langle x_1, \ldots, x_k \rangle \in Ex(Y)$ by the definition of standard extension of $\mathcal{R}$. Further we can choose a countable class $Z \subseteq A$ such that $(\forall y_1, \ldots, y_k)(\exists z \in Z)(\langle y_1, \ldots, y_k \rangle \in Y \rightarrow \varphi(z, y_1, \ldots, y_k))$.

Therefore we have

$(\forall y_1, \ldots, y_k)(\exists z \in Ex(Z))(\langle y_1, \ldots, y_k \rangle \in Ex(Y) \rightarrow \varphi(z, y_1, \ldots, y_k))$.

Substituting the constants $x_1, \ldots, x_k$ instead of the variables $y_1, \ldots, y_k$ respectively, we obtain $(\exists z \in Ex(Z))\varphi(z, x_1, \ldots, x_k)$ and thus even $(\exists z \in A^{Ex})\varphi(z, x_1, \ldots, x_k)$. Thence to finish the proof it suffices to use a theorem of § 1 [S-V 1].

Theorem. If $A^{Ex} \neq V$ then $Sms(A^{Ex})$, i.e. A^{Ex} is a semiset.

Proof. Let F be an endomorphism with rng(F) = A. There is a one-one mapping of A onto $F''\Omega$, moreover, there is a $c \in V - A^{Ex}$ and hence there is $d \in Ex(F''\Omega) - \bigcup\{Ex(X); X \subseteq F''\Omega \ \& \ Count(X)\} \subseteq Ex(F''\Omega) - \bigcup\{Ex((F''\Omega) \cap \alpha); \alpha \in F''\Omega\}$. Thus for every $\alpha \in F''\Omega$ we have $(F''\Omega) \cap \alpha \subseteq d$ and hence $F''\Omega \subseteq d$. If we deal with the operation $\overline{P}(z)$ defined in § 1 ch. II [V], then from the statement $V = \bigcup\{\overline{P}(\alpha); \alpha \in \Omega\}$ we can conclude that $A = \bigcup\{\overline{P}(\alpha), \alpha \in F''\Omega\} \subseteq \overline{P}(d)$. We proved just now Sms(A). For every countable $X \subseteq A$ there is $u \in A$ with $X \subseteq u$ and hence $Ex(X) \subseteq u \subseteq \bigcup A$. Thus $A^{Ex} \subseteq \bigcup A$ and therefore the statement $Sms(A^{Ex})$ is a trivial consequence of the formula $Sms(A)$.

Theorem. If X is a countable subclass of A, then $Ex(X) \cap \cap A^{Ex} = \bigcap \{u \cap A^{Ex}; u \in A \& X \subseteq u\}$.

Proof. If $X \subseteq u \in A$, then $X \cap u \subseteq A$ and moreover, $u \cap A \in \mathcal{R}$, since both X and u are elements of $\mathcal{R}$. Hence $Ex(X) \cap A^{Ex} \subseteq \subseteq Ex(u \cap A) \cap A^{Ex} = u \cap A^{Ex}$. To prove the converse implication let us assume that $y \in A^{Ex}$, i.e. that there is a countable class $Y \subseteq A$ such that $y \in Ex(Y)$ and that $y \in \bigcap \{u \cap A^{Ex}; u \in A \& X \subseteq u\}$. There are $u_1, u_2 \in A$ so that $u_1 \cap u_2 = 0 \& X \subseteq u_1 \& (Y - X) \subseteq u_2$. Evidently we have $y \in u_1$. Since $y \notin u_2$, the formula $y \notin Ex(Y - X)$ follows from the first part of the proof. However, this implies $y \in Ex(X)$.

Let us consider the operation $\mathcal{F}(X) = Ex(X) \cap A^{Ex}$ defined for every $X \in \mathcal{R}$. Since $A^{Ex} = \bigcup \{Ex(Y); Y \subseteq A \& Count(Y)\}$, we have $\mathcal{F}(X) = \bigcup \{Ex(X) \cap Ex(Y); Y \subseteq A \& Count(Y)\} = \bigcup \{Ex(X \cap Y); Y \subseteq A \& Count(Y)\} = \bigcup \{Ex(Y) \cap A^{Ex}; Y \subseteq X \& Count(Y)\} = \bigcup \{\mathcal{F}(Y), Y \subseteq \subseteq X \& Count(Y)\}$.

A^{Ex} is an endomorphic universe and hence we are able to apply the results of [S-V 1] considering A^{Ex} as the universal class. In this case there is a standard extension of the reflecting system on A consisting of all subclasses of A (cf. the mentioned article) and the operation $\mathcal{F}(X)$ agrees with this standard extension.

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(Oblatum 6.6. 1980)