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ON THE EXISTENCE OF SOLUTION OF THE EQUATION L(x) = N(x) AND A GENERALIZED COINCIDENCE DEGREE THEORY, II

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Abstract: In this paper we have built up a coincidence degree theory for a pair (L,N) where $L:\mathrm{dom}L\subset X\longrightarrow Z$ is an admissible generalized linear Fredholm mapping and N is a mapping which is defined on the closure of a bounded open subset of X and takes values in Z, X and Z being Banach spaces over the reals. This coincidence degree is a generalization of the coincidence degree due to Mawhin. It has been shown that this generalized coincidence degree possesses most of the properties of a degree.

Key words and phrases: Leray-Schauder degree, admissible generalized Fredholm mapping, generalized coincidence degree, k-set contraction.

Classification: Primary 47H15, 47A50
Secondary 47H10, 47A55

Introduction. Let X and Z be real Banach spaces. Mawhin [9] has developed a coincidence degree theory for the pair (L,N) where $L:\mathrm{dom}L\subseteq X\longrightarrow Z$ is a linear Fredholm mapping of index equal to zero and $N:\mathrm{Cl}\,\Omega\longrightarrow Z$ is a mapping where $\mathrm{Cl}\,\Omega$ is the closure of a bounded open subset of X. This degree serves as a tool for proving the existence of solution of the equation $L(x) = N(x)$.

The purpose of this paper is to build up with the help of the results obtained in [12] a coincidence degree for the couple (L,N) a coincidence degree where L is an admissible generalized Fredholm mapping and N is as above. It is shown that this generalized coincidence degree has most of the properties of Leray-Schauder degree.

1. <u>A generalized coincidence degree for the pair</u> (L,N). For definitions of an admissible generalized Fredholm mapping L and an associated scheme $\Gamma(X_n, P_n, P, \psi)$ we refer to section 2 of [12]. The symbols K_P M, π, $K_{P_n P}$, L_n, M_n, ψ_n, π_n which will be used throughout this paper have the same meanings as explained in section 2 of [12] while the definitions of conditions (S) and (S)$'$ have been given in Section 3 of [12].

In this section we will define the generalized coincidence degree for the pair (L,N).

We will do this in two different cases - the general case when Ω is an open bounded subset of a Banach space X and the special case when Ω is an open bounded subset of a reflexive Banach space X with the property that $Cl\Omega = \omega\text{-}Cl\Omega$. For this we will have to consider for each case a separate set of assumptions.

Let us now consider the first set of assumptions:

(a) X is a Banach space and Ω is a bounded open subset of X.

(b) $L: \mathrm{dom} L \subset X \longrightarrow Z$ is an admissible generalized Fredholm mapping and $N: Cl\Omega \longrightarrow Z$ is a mapping. $\Gamma = (X_n, P_n, P, 0, \psi)$ is an associated scheme for L.

(c) For each positive integer n, $\pi_n N$ is continuous and $\pi_n(Cl\Omega)$ is bounded.

(d) For each positive integer n, $K_{P_n P}(I-Q_n' Q)N$ is completely continuous (i.e. $K_{P_n P}(I-Q_n' Q)N$ is continuous and $K_{P_n P}(I-Q_n' Q)$ $N(Cl\Omega)$ are relatively compact in X).

(e) N is continuous.

(f) Either L is continuous, or K_P is continuous.

(g) The condition (S) holds for the triple (L,N,Ω).

(h) $0 \notin (L-N)(\partial\Omega \cap \operatorname{dom}L)$.

The second set of assumptions is as follows:

(a)′ X is a reflexive Banach space and Ω is a bounded open subset of X with the property that $\mathrm{Cl}\Omega = \omega\text{-}\mathrm{Cl}\Omega$ (in particular is an open bounded convex subset).

(b)′ Same as (b).

(c)′ Same as (c).

(d)′ Same as (d).

(e)′ N is weakly continuous.

(f)′ Either L is weakly continuous or K_P is weakly continuous.

(g)′ The condition (S)′ holds for the triple (L,N,Ω).

(h)′ Same as (h).

Remark 1.1. (i) Clearly the condition that ′N is continuous and $N(\mathrm{Cl}\Omega)$ is bounded′ implies the condition (c).

(ii) We can also show that if K_P and N are both continuous, then $K_{P_nP}(I-Q_n' Q)N$ is continuous for each positive integer n.

Lemma 1.1. If the triple (L,N,Ω) satisfies either (a) to (d) or (a)′ to (d)′, then for each positive integer n, $M_n:\mathrm{Cl}\Omega \to$ $\to X$ is completely continuous.

Proof. By assumption (c) (or (c)′) $\psi_n \pi_n N$ is continuous and $\psi_n \pi_n N(\mathrm{Cl}\Omega)$ being a bounded subset of a finite dimensional subspace is relatively compact. P_nP is linear and continuous and $P_nP(\mathrm{Cl}\Omega)$ is a bounded subset of a finite dimensional subspace ($\mathrm{Cl}\Omega$ being bounded by assumption). Hence P_nP is completely continuous. By assumption (d) or (d)$'$ $K_{P_n P}(I-Q_n' Q)N$ is completely continuous. Thus it follows that M_n is completely continuous and the lemma is proved.

If for a positive integer m, $0 \notin (I-M_m)(\partial\Omega)$ then by virtue of the lemma 1.1, the Leray-Schauder degree of $I-M_n$ on Ω is well-defined. We will denote by $d(I-M_m, \Omega, 0)$ the Leray-Schauder degree of $I-M_m$ on Ω over 0.

We are now in a position to define a generalized coincidence degree of the pair (L,N). To do this we will employ a device introduced by Browder and Petryshyn [1] and [2] (see also Fitzpatrick [3]) in a context entirely different from that of ours. In the following definition Z' will denote the set of all integers (positive, negative and zero) together with $+\infty$ and $-\infty$.

Definition 1.1. If the triple (L,N,Ω) satisfies either (a) to (h) or (a)$'$ to (h)$'$, then we define $d((L,N),\Omega)$ a generalized coincidence degree of the pair (L,N) on Ω with respect to Γ as follows:

$d((L,N),\Omega) = \{t \in Z'$; there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that for each m, $d(I-M_m, \Omega, 0)$ is well-defined and $d(I-M_m, \Omega, 0) \to t\}$.

Thus $d((L,N),\Omega)$ is a subset of Z'. Hence as in Browder and Petryshyn [1] our degree $d((L,N),\Omega)$ depends on Γ.

Lemma 1.2. If the triple (L,N,Ω) satisfies either (a) to (h) or (a)$'$ to (h)$'$, then $d(L,N,\Omega)$ is well-defined and, in particular, a nonempty subset of Z'.

Proof. Noting that conditions (i)$'$ and (ii)$'$ in Corollary 3.3 (the conditions (i) and (ii) in Corollary 3.5) of

[12] are respectively the same as in the conditions (e) and (f) (the conditions (e)′ and (f)′) we obtain by Corollary 3.3 (Corollary 3.5) of [12] an integer $m_0 \geq 1$ such that $0 \notin (I-M_m)(\partial\Omega)$ for all $m \geq m_0$. Hence in view of lemma 1.1 $d(I-M_m,\Omega,0)$ is well-defined for all $m \geq m_0$. Now if $\{d(I-M_m,\Omega,0) : m \geq m_0\}$ is bounded then it is clear that there is an infinite sequence $\{m_j\}$ with $m_j \geq m_0$ for each j and $m_j \to \infty$ such that $d(I-M_m,\Omega) = t$, a finite integer for all j. Hence $t \in d((L,N),\Omega)$. On the other hand, if $\{d(I-M_m,\Omega,0) : m \geq m_0\}$ is unbounded, then there exists an infinite sequence $\{n_j\}$ of positive integers with $n_j \geq m_0$ for all j and $n_j \to \infty$ such that $d(I-M_n,\Omega,0) \to +\infty$ or $-\infty$. Hence in this case $+\infty$ or $-\infty \in d((L,N),\Omega)$. Thus in either case $d((L,N),\Omega)$ is nonempty. This completes the proof of the lemma.

We now prove the following properties of the coincidence degree.

Theorem 1.1 (Existence theorem). If the triple (L,N,Ω) satisfies either (a) to (h) (or (a)′ to (h)′) and $d((L,N),\Omega) \neq \{0\}$, then there exists an element $x_0 \in \Omega$ such that x_0 is a solution of the equation $L(x) = N(x)$.

Proof. Since $d((L,N),\Omega) \neq \{0\}$, there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that $d(I-M_m,\Omega,0) \to t \neq 0$ where $t \in d((L,N),\Omega)$. Clearly we can find an infinite subsequence $\{m_j\}$ of $\{m\}$ such that $d(I-M_{m_j},\Omega,0) \neq 0$ for each j. Hence for each j, by properties of Leray-Schauder degree there exists $x_{m_j} \in \Omega$ such that $x_{m_j} = M_{m_j}(x_{m_j})$. Hence by proposition 3.3 (respectively proposition 3.2) of [12] there is a point $x_0 \in \mathrm{Cl}\,\Omega$ such that x_0 is a solution of $L(x) = N(x)$. However, by (h) (respectively by (h)′) x_0 cannot belong to $\partial\Omega$. Hence $x_0 \in \Omega$. Thus the theorem is proved.

Theorem 1.2. (Borsuk theorem.) If the triple (L,N,Ω) satisfies (a) to (h) (or (a)$'$ to (h)$'$), Ω is symmetric with respect to the origin and contains it and N is an odd mapping, i.e. $N(-x) = -N(x)$ for each $x \in \mathrm{Cl}\Omega$, then $d((L,N),\Omega) \neq \{0\}$ and is odd in the sense that if the integer $t \in d((L,N),\Omega)$ is finite, then t must be an odd integer. Thus, in particular, there is a solution $x_0 \in \Omega$ of the equation $L(x) = N(x)$.

Proof. $d((L,N),\Omega)$ is well-defined, i.e., nonempty. Let $t \in d((L,N),\Omega)$. Then there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that $d(I-M_m,\Omega,0)$ is well-defined for each m and $d(I-M_m,\Omega,0) \to t$. If $t \neq +\infty$ or $-\infty$, then it follows that there is a subsequence $\{m_j\}$ of $\{m\}$ such that $d(I-M_{m_j},\Omega,0) = t$ for each j. However, since all the operators (other than N) involved in the definition of M_n are linear and N is odd, M_n is an odd mapping for each positive integer n. Hence by the corresponding property of Leray-Schauder degree $d(I-M_{m_j},\Omega,0)$ is nonzero and odd. Thus t is an odd integer. Thus t is an odd integer, $+\infty$ or $-\infty$. Hence the theorem is proved.

Theorem 1.3. (Additivity theorem.) (α) If the triple (L,N,Ω) satisfies (a) to (h) and if Ω_1 and Ω_2 are open with $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \phi$, then $d((L,N),\Omega) \subset d((L,N),\Omega_1) + d((L,N),\Omega_2)$.

(β) If the triple (L,N,Ω) satisfies (a)$'$ to (h)$'$ and if Ω_1, Ω_2 are open with $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \phi$, and $\mathrm{Cl}\Omega_i = \omega - \mathrm{Cl}\Omega_i$, $i=1,2$, then $d((L,N),\Omega) \subset d((L,N),\Omega_1) + d((L,N),\Omega_2)$.

The equality both in (α) and (β) holds if either of the right hand side is a singleton integer. In the above for two subsets A and B of Z' we define $A + B = \{t = t_1 + t_2 : t_1 \in A$ and $t_2 \in B\}$ and we use the convention that $+\infty + (-\infty) = Z'$.

Proof. Since $\Omega_1 \cap \Omega_2 = \phi$, it follows that $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$. Hence by virtue of (h) and (h)$'$ in either of the cases (α) and (β), $0 \notin (L-N)(\partial\Omega_i \cap \mathrm{dom} L)$, i=1,2. In case ($\alpha$) it follows easily from (g) that (L,N,Ω_i), i=1,2 satisfy the condition (S). We now prove that in case (β), (L,N,Ω_i), i=1,2 satisfy (S)$'$. Let for each n, $x_n \in \partial\Omega_1$ be a fixed point of M_n restricted to $CL\Omega_1$ and $x_n \rightharpoonup x_0$. Then since $CL\Omega_1 = \omega\text{-}Cl\Omega_1$, $x_0 \in Cl\Omega_1$. Also by (g)$'$ $x_0 \in \partial\Omega$. Now if $x_0 \notin \partial\Omega_1$, then x_0 would belong to Ω_1 and $x_0 \in \partial\Omega_2$. But this would then imply that $\Omega_1 \cap \Omega_2 \neq \phi$. Hence we conclude that $x_0 \in \partial\Omega_1$. Thus (L,N,Ω_1) satisfies (S)$'$. Similarly (L,N,Ω_2) satisfies (S)$'$. Thus in either of the cases (α) and (β), $d((L,N),\Omega_i)$, i=1,2 are well-defined. The rest of the argument covers (α) and (β) simultaneously.

Let $t \in d((L,N)\Omega$. Then there exists an infinite sequence $\{n\}$ of positive integers with $n \to \infty$ such that $d(I-M_n,\Omega,0)$ is well-defined for each n and $d(I-M_n,\Omega,0) \to t$. By additivity property of Leray-Schauder degree we have

$$d(I-M_n,\Omega_1,0) + d(I-M_n,\Omega_2,0) \to t.$$

The rest of the proof is exactly the same as in the corresponding case of Browder and Petryshyn ([1],p. 223). However, we will include the proof for completeness. Now if $d(I-M_n,\Omega_1,0) \not\to \pm\infty$, then we can find a subsequence $\{n_j\}$ of $\{n\}$ such that $d(I-M_{n_j},\Omega_1,0) = t_1$ for each j. Hence $d(I-M_{n_j},\Omega_2,0) = t-t_1 = t_2$

for each j. Thus $t_i \in d(L,N), \Omega_1)$, $i=1,2$ and $t = t_1 + t_2 \in$ $\in d((L,N), \Omega_1) + d((L,N), \Omega_2)$. Next, let $d(I-M_n, \Omega_1, 0) \longrightarrow$ $\longrightarrow \pm \infty$. If t is finite, then we must have $d(I-M_n, \Omega_2, 0) \rightarrow$ $\longrightarrow \mp \infty$, so that $t = +\infty + (-\infty) \in d((L,N), \Omega_1) +$ $+ d((L,N), \Omega_2)$. If $t = \pm\infty$, then we have $t = \pm\infty - t_2$ for all $t_2 \in d((L,N), \Omega_2)$ by convention and thus $t \in d((L,N), \Omega_1) +$ $+ d((L,N), \Omega_2)$. The only possibility that remains is $t = \mp\infty$, in which case $d((L,N), \Omega_2, 0) \longrightarrow \mp\infty$ and $t \in d((L,N), \Omega_1) +$ $+ d((L,N), \Omega_2)$.

To complete the proof, let for the sake of definiteness $d((L,N), \Omega_1) = \{t_1\}$ where t_1 is an integer. Then we can assume the existence of an integer $n_o \geq 1$ such that $d(I-M_n, \Omega_1, 0) =$ $= t_1$ for all $n \geq n_o$.

Therefore $d(I-M_n, \Omega, 0) - d(I-M_n, \Omega_2, 0) \longrightarrow t_1$.

From this it follows that the equality holds whenever either of the right hand side is a singleton. Thus the theorem is proved.

One of the important properties of degree theory is the homotopy invariance property. In order to prove this we will need to prove few lemmas.

Let L be an admissible generalized Fredholm mapping and $\widetilde{N}: Cl\,\Omega \times [0,1] \longrightarrow Z$ be a mapping, where Ω is an open bounded subset of X. We define the mappings: $\widetilde{M}: Cl\,\Omega \times [0,1] \longrightarrow X$ by

$$M(x,t) = P(x) + \psi\pi\, \widetilde{N}(x,t) + K_p(I-Q)\widetilde{N}(x,t),$$

and for each positive integer n, $\widetilde{M}_n: Cl\,\Omega \times [0,1] \longrightarrow X$ by

$$M_n(x,t) = P_n P(x) + \psi_n \pi_n \widetilde{N}(x,t) + K_{P_n P}(I-Q_n' Q)\widetilde{N}(x,t).$$

For each fixed $t \in [0,1]$ we denote by $\widetilde{M}_t$ and $M_{n,t}$ respectively the mappings $\widetilde{M}_t(x) = \widetilde{M}(x,t):\mathrm{Cl}\,\Omega \longrightarrow X$ and $\widetilde{M}_{n,t}(x) = \widetilde{M}_n(x,t):$ $:\mathrm{Cl}\,\Omega \longrightarrow X$. Also for fixed $t \in [0,1]$ let us denote the mapping $\widetilde{N}(x,t):\mathrm{Cl}\,\Omega \longrightarrow Z$ by $\widetilde{N}_t$.

Lemma 1.3. Let X, Ω and L be as in (a)$'$ and (b)$'$ and let $\widetilde{N}:\mathrm{Cl}\,\Omega \times [0,1] \longrightarrow Z$ be a mapping such that

(0) whenever $x_n \rightharpoonup x \in \mathrm{Cl}\,\Omega$, $x_n \in \mathrm{Cl}\,\Omega$ and $t_n \longrightarrow t \in [0,1]$, $t_n \in [0,1]$

we have $\widetilde{N}(x_n,t_n) \rightharpoonup \widetilde{N}(x,t)$;

(00) Either L is weakly continuous or K_P is weakly continuous.

If there exists an infinite sequence $\{m\}$ of positive integers with $m \longrightarrow \infty$ such that x_m is a fixed point of $\widetilde{M}_{m,t_m}$ for each m and $t_m \longrightarrow t$, then there exists a point $x_0 \in \mathrm{Cl}\,\Omega$ such that x_0 is a fixed point of $\widetilde{M}_t$.

Proof. Since X is reflexive, we can find a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ such that $x_{m_j} \rightharpoonup x_0 \in \mathrm{Cl}\,\Omega$. By (0) $\widetilde{N}(x_{m_j},t_{m_j}) \rightharpoonup$ $\rightharpoonup \widetilde{N}(x_0,t)$. By proposition 3.1 of [12] it follows that $L_{m_j}(x_{m_j}) = \widetilde{N}_{t_{m_j}}(x_{m_j}) = \widetilde{N}(x_{m_j},t_{m_j})$. Thus $\widetilde{N}(x_{m_j},t_{m_j}) \in \mathrm{Im}\,L_{m_j} =$ $= \mathrm{Im}L \oplus \phi\psi^{-1}(U_{m_j})$ for each j. From this it follows that $\pi_m\widetilde{N}(x_{m_j},t_{m_j}) = 0$ and $Q'_{m_j}\,Q\,\widetilde{N}(x_{m_j},t_{m_j}) = 0$ for each j. Now proceeding exactly as in the proof of lemma 3.2 of [12] we can prove that $N(x_0,t) \in \mathrm{Im}L$.

Let for each j, $\widetilde{N}(x_{m_j},t_{m_j}) = r_{m_j} + \omega_{m_j}$, where $r_{m_j} \in \mathrm{Im}L$ and $\omega_{m_j} \in \phi\psi^{-1}(U_{m_j})$. As in the proof of proposition 3.2 of 12 , $(I-Q)\,N(x_{m_j},t_{m_j}) = r_{m_j}$ for each j. Hence $r_{m_j} \rightharpoonup \widetilde{N}(x_0,t)$ by (1.4) as $\widetilde{N}(x_0,t) \in \mathrm{Im}L$. Thus $\omega_{m_j} \rightharpoonup 0$. Now proceeding exactly as in the proof of proposition 3.2 of [12] we can prove that x_o is a fixed point $\widetilde{M}_t$.

Lemma 1.4. Let X, Ω and L be as in (a) and (b) and let $\widetilde{N}: \mathrm{Cl}\,\Omega \times [0,1] \to Z$ be a mapping such that

(O)′ Whenever $x_n \to x$ $\mathrm{Cl}\,\Omega$, $x_n \in \mathrm{Cl}\,\Omega$ and $t_n \to t \in$ $\in [0,1]$, $t_n \in [0,1]$ we have $\widetilde{N}(x_n, t_n) \to \widetilde{N}(x,t)$;

(OO)′ Either L is continuous or K_P is continuous.

If there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that x_m is a fixed point of $\widetilde{M}_{m,t_m}$ for each m and $x_m \to x_o$ and $t_m \to t$, then x_o is a fixed point of $\widetilde{M}_t$.

Proof. The proof is exactly the same as that of lemma 1.3 except that we replace everywhere the weak convergence $\rightharpoonup$ by convergence $\to$ and use the continuity in place of weak continuity.

Let us now consider the following two sets of assumptions:

(i) X and Ω are as in (a).

(ii) L is as in (b) and $\widetilde{N}: \mathrm{Cl}\,\Omega \times [0,1] \to Z$ is a mapping.

(iii) For each positive integer n, $\sigma_n \widetilde{N}$ is continuous and $\sigma \widetilde{N}(\mathrm{Cl}\,\Omega \times [0,1])$ is bounded.

(iv) For each positive integer n, $K_{P_n P}(I - Q_n' Q)\, \widetilde{N}$ is completely continuous.

(v) Same as (O)′ in lemma 1.4.

(vi) Same as (f).

(vii) If there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that for each m, x_m is a fixed point of $\widetilde{M}_{m,t_m}$ and $t_m \to t$, then $x_m \to x_o \in \mathrm{Cl}\,\Omega$.

(viii) $(L(x) - \widetilde{N}(x,t)) \neq 0$ for any $x \in \partial\Omega$ and any $t \in [0,1]$, and (i)′ x and Ω are as in (a)′

(ii)′ Same as (ii) above.

(iii)′ Same as (iii) above.

(iv)′ Same as (iv) above.

(v)′ Same as (0) in lemma 1.3.

(vi)′ Same as (f)′.

(vii)′ If there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that for each m, $x_m \in \partial\Omega$ is a fixed point of M_{m,t_m} and $x_m \rightharpoonup x_0$ and $t_m \to t$, then $x_0 \in \partial\Omega$.

(viii)′ Same as (viii) above.

Theorem 1.4. (Homotopy invariance theorem.) If the triple $(L,\widetilde{N},\Omega)$ satisfies (i) to (viii), or (i)′ to (viii)′, then $d((L,N_t),\Omega)$ is independent of $t \in [0,1]$.

Proof. Under hypotheses it is clear that $d((L,\widetilde{N}_t),\Omega)$ is well-defined for each fixed $t \in [0,1]$. We first prove that there exists an integer $n_0 \geq 1$ such that $0 \notin (I-\widetilde{M}_{n,t})(\partial\Omega)$ for all integers $n \geq n_0$ and for all $t \in [0,1]$. If possible, let us suppose that this is not true. Then we can find a sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ and a sequence $\{t_{n_j}\}$, $t_{n_j} \in [0,1]$ with $t_{n_j} \to t$ (passing to a subsequence if necessary) such that $0 \in (I-M_{n_j,t_{n_j}})(\partial\Omega)$ for all j. From this we obtain a sequence $\{x_{n_j}\}$, $x_{n_j} \in \mathrm{Cl}\Omega$ such that x_{n_j} is a fixed point of $\widetilde{M}_{n_j,t_{n_j}}$ for each j. Now in case of (i)′ to (viii)′, it follows that $x_{n_j} \rightharpoonup x_0$ $\mathrm{Cl}\Omega$ (if necessary passing to a subsequence) as $\mathrm{Cl}\Omega$ is bounded and X is reflexive. Hence by lemma 1.3 x_0 is a fixed point of $\widetilde{M}_t$ and by (vii)′ $x_0 \in \partial\Omega$. In case of (i) to to (viii), it follows from the condition (vii) and lemma 1.4 that there is a fixed point x_0 of $\widetilde{M}_t$ with $x_0 \in \partial\Omega$. Thus in either case $(I-\widetilde{M}_t)(x_0) = (L(x_0) - \widetilde{N}(x_0,t) = 0$, $x_0 \in \partial\Omega$ which is a contradiction. Hence we have an integer $n_0 \geq 1$ such that $0 \notin (I-M_{n,t})(\partial\Omega)$ for all integers $n \geq n_0$ and all $t \in [0,1]$. Now from the homotopy invariance property of Leray-Schauder degree we obtain that $d(I-M_{n,t},\Omega,0)$ is independent of t for all $n \geq$ $\geq n_0$. It is thus clear from this and from the definition of $d((L,\widetilde{N}_t),\Omega)$ that $d((L,\widetilde{N}_t),\Omega)$ is independent of t.

Corollary 1.1. $d((L,N),\Omega)$ depends on L,Ω,Γ and the restriction of N to $\partial\Omega$.

Proof. See [4], p. 22.

Theorem 1.5 (Excision property).

(1) If the triple (L,N,Ω) satisfies (a) to (h) and if $\Omega_0 \subset \Omega$ is an open subset such that $(L-N)^{-1}(0) \subset \Omega_0$, or

(2) If the triple (L,N,Ω) satisfies (a)′ to (h)′ and if $\Omega_0 \subset \Omega$ is an open subset such that $\mathrm{Cl}\,\Omega_0 = \omega - \mathrm{Cl}\,\Omega_0$ and $(L-N)^{-1}(0) \subset$ $\subset \Omega_0$, then

$$d((L,N),\Omega) = d((L,N),\Omega_0).$$

Proof. The proof is omitted.

2. Extension of continuation theorem. Mawhin [9] (see also Gaines and Mawhin [4]) has extended the well-known Leray-Schauder continuation theorem in the frame of coincidence degree. In this section we will extend this theorem in the setting of our generalized coincidence degree.

To this end let $L:\mathrm{dom}L \subset X \to Z$ be an admissible generalized Fredholm mapping and $N^*:\mathrm{Cl}\,\Omega \times [0,1] \to Z$ be a mapping and let the triple (L,N^*,Ω) satisfy either the conditions (i) to (vi) or the conditions (i)' to (vi)' of section 1. We also write $N^*(\cdot,1) = N_1^*(\cdot) = N$. For $y \in \text{Im}L$, let us consider the family of equations

$$L(x) = t\, N^*(x,t) + y,\ t \in [0,1], \quad \ldots.. \tag{2.1}$$

and for each positive integer n, the family of equations

$$L_n(x) = t\, N^*(x,t) + y,\ t \in [0,1]. \quad \ldots.. \tag{2.2}$$

We first prove the following lemma.

Lemma 2.1. Let n be a positive integer. Then for each $t \in [0,1]$, the set of solutions of the equation (2.2) coincides with the set of solutions of the equation

$$L_n(x) = Q_n' Q\, N^*(x,t) + t(I-Q_n' Q)\, N^*(x,t) + y, \ \ldots \tag{2.3}$$

and for $t = 0$, each solution of the equation (2.3) is also a solution of the equation (2.2).

Proof. Let $t \neq 0$. First we suppose that x is a solution of the equation (2.2). Then since $L_n(x) \in \text{Im}L_n = \text{Im}L \oplus \phi\psi^{-1}(U_n)$ we have by (1.4) of [12] $Q_n' QL_n(x) = 0$. Hence $Q_n' QN^*(x,t) = 0$. It now follows that x is a solution of the equation (2.3). Next, let x be a solution of the equation (2.3). Then by the same reason given above $Q_n' QL_n(x) = 0$. Thus the equation (2.3) reduces to the equation (2.2), i.e. x is a solution of the equation (2.2).

Finally if $t = 0$ and x is a solution of the equation (2.3), then by (1.4) of [12] as before $Q_n' Q\, N^*(x,0) = 0$. Hence the equation (2.3) reduces to $L_n(x) = y$. Thus x is a solution of the equation (2.2).

Remark 2.1. It is trivial to see that if we use $y_n \in \text{Im}L_n$

in the equation (2.2), the lemma (2.1) still holds.

Theorem 2.1. (Continuation theorem.) Let the triple (L,N^*,Ω) be as above. Further assume that

(1) $L(x) \neq t\, N^*(x,t) + y$ for every $x \in \partial\Omega \cap \mathrm{dom}L$ and every $t \in (0,1)$;

(2) There exists a positive integer n_o such that

$\pi_{n_o} N^*(x,0) \neq 0$ for every $x \in L^{-1}\{y\} \cap \partial\Omega$;

(3) there exists a positive integer m_o such that

$d(\pi_m N^*(\cdot,0)/L_m^{-1}\{y\}, \Omega \cap L_m^{-1}\{y\}, 0) \neq 0$ for all positive integers $m \geq m_o$, where the last number is the Brouwer degree at $0 \in \mathrm{coker}\, L_m$ of the continuous mapping $\pi_m N^*(\cdot,0)$ from the affine finite dimensional topological space $L^{-1}\{y\} = L_m^{-1}\{y\}$ (see remark below) into $\mathrm{coker}\, L_m$ (for details see Gaines and Mawhin ([4], p. 27);

(4) The triple $(L,\overline{N},\Omega)$ satisfies the condition (vii) (respectively (vii)$'$) of section 1, where $\overline{N}: \mathrm{Cl}\Omega \times [0,1] \to Z$ is defined by $\overline{N}(x,t) = t\, N^*(x,t) + y$, $(x,t) \in \mathrm{Cl}\Omega \times [0,1]$.

Then for each $t \in [0,1]$ the equation (2.1) has at least one solution in Ω and the equation

$$L(x) = N(x) + y$$

has at least one solution in $\mathrm{Cl}\Omega$.

Before we attempt to prove this theorem it is important to note the following remark:

Remark 2.2.

(p) By definition of L_n, $\mathrm{dom}L_n = \mathrm{dom}L$ for each positive integer n. Also since $y \in \mathrm{Im}L$, it follows that for each positive integer n, $L_n^{-1}\{y\} = L^{-1}\{y\}$.

(q) (2) implies that for each positive integer $n \geq n_o$,

$\pi_n\, N^*(x,0) \neq 0$ for every $x \in L^{-1}\{y\} \cap \partial\Omega$ and that

$\pi\, N^*(x,0) \neq 0$ for every $x \in L^{-1}\{y\} \cap \partial\Omega$. This follows from B of section 2 of [12].

Proof of Theorem 2.1. If there exists $x \in \partial\Omega \cap \mathrm{dom}L$ such that $L(x) = N^*(x,1) + y = N(x) + y$, then the first part of our theorem holds. We suppose that

(2.4) $\qquad L(x) \neq N(x) + y$ for all $x \in \partial\Omega \cap \mathrm{dom}L$ .....

In view of assumption (1)

$L(x) \neq tN^*(x,t) + y$ for each $x \in \partial\Omega \cap \mathrm{dom}L$ and each $t \in (0,1)$.

On the other hand if $t = 0$, our equation (2.3) is equivalent to

$$QN^*(x,0) = 0, \quad L(x) = y$$

i.e.

(2.5) $\quad \pi\, N^*(x,0) = 0$ (by (1.4)), $x = L^{-1}\{y\}$ .....

Thus by assumption (2) vide (q) above, there is no solution of (2.5) in $\partial\Omega$. Hence $L(x) \neq t\, N^*(x,t) + y$ for each $x \in \partial\Omega \cap \mathrm{dom}L$ and each $t \in [0,1]$.

Let $\overline{N}(x,t) = t\, N\,(x,t) + y$. Then we can easily verify that all the conditions of Theorem 1.4 hold for the triple $(L,\overline{N},\Omega)$. Hence as in the proof of Theorem 1.4 we can find a positive integer n_1 such that $0 \notin (I-\overline{M}_{n,t})(\partial\Omega)$ for all $n \geq n_1$ and all $t \in$ $\in [0,1]$ where, as before,

$$\overline{M}_{n,t}(x) = P_nP(x) + \psi_n\pi_n\overline{N}_t(x) + K_{P_nP}(I-Q_n'\, Q)\,\overline{N}_t(x).$$

Thus by proposition 3.1 of [12] for each $n \geq n_1$

(2.6) $\qquad L_n(x) \neq \overline{N}(x,t) = \overline{N}_t(x)$ .....

for each $x \in \partial\Omega \cap \mathrm{dom}L$ and each $t \in [0,1]$. Let $m_1 = \max(n_0, m_0, 1$. Let $n = n$ be a fixed but arbitrary positive integer with $\overline{n} \geq m_1$

Let us consider the mapping $\hat{N}: Cl\,\Omega \times [0,1] \to Z$ defined by

$$\hat{N}(x,t) = Q'_{\bar{n}}\, Q\, N^*(x,t) + t(I-Q'_{\bar{n}}\, Q)\, N^*(x,t) + y.$$

Then by lemma 2.1 and (2.6) it follows that $L_{\bar{n}}(x) \neq \hat{N}(x,t)$ for each $x \in \partial\Omega \cap \mathrm{dom}L$ and each $t \in [0,1]$. Hence $0 \notin (I-\hat{M}_{\bar{n},t})(\partial\Omega)$ for each $t \in [0,1]$, where

$$\hat{M}_{\bar{n},t}(x) = P_{\bar{n}}P(x) + \psi_{\bar{n}}\, \pi_{\bar{n}}\, \hat{N}_t(x) + K_{P_{\bar{n}}P}(I-Q'_{\bar{n}}\, Q)\, \hat{N}_t(x)$$

which is completely continuous for each $t \in [0,1]$. Hence by the homotopy invariance of Leray-Schauder degree $d(I-\hat{M}_{n,t},\Omega,0)$ is independent of t and is equal to its value at $t = 0$.

Now $d(I-M_{\bar{n},0},\Omega,0) = d(I-P_{\bar{n}}P - \psi_{\bar{n}}\, \pi_{\bar{n}}\, N^*(\cdot,0) - K_{P_{\bar{n}}P}y, \Omega, 0)$

where

$$N_0^*(\cdot) = N^*(\cdot,0).$$

Proceeding exactly as in Gaines and Mawhin ([4],p. 28) we prove that $d(I-M_{\bar{n},0},\Omega,0) \neq 0$. If $\ker L_{\bar{n}} = \{0\}$, then by the same argument as given in Gaines and Mawhin ([4],p. 28) we have that $d(I-M_{\bar{n},0},\Omega,0) = 1$. We assume that $\mathrm{Ker}L_n \neq \{0\}$. Then by using the invariance property of Leray-Schauder degree, we have

(2.7) $d(I-\hat{M}_{\bar{n},0},\Omega,0) = d(I-P_{\bar{n}}P - \psi_{\bar{n}}\, \pi_{\bar{n}}\, N^*(\cdot,0) - K_{P_{\bar{n}}P}y, \Omega, 0)$

$= d(I-P_{\bar{n}}P - \psi_{\bar{n}}\, \pi_{\bar{n}}\, N^*(\cdot + K_{P_{\bar{n}}P}y, 0), -K_{P_{\bar{n}}P}y + \Omega, 0)$

$= d((I-P_{\bar{n}}P - \psi_{\bar{n}}\, \pi_{\bar{n}}\, N^*(\cdot + K_{P_{\bar{n}}P}y,0))/\mathrm{Ker}L_{\bar{n}}, (-K_{P_{\bar{n}}P}y + \Omega) \cap$
$\cap \mathrm{Ker}L_{\bar{n}}, 0)$

$= d((-\psi_{\bar{n}}\, \pi_{\bar{n}}\, N^*(\cdot + K_{P_{\bar{n}}P}y,0)/\mathrm{Ker}L_n, (-K_{P_{\bar{n}}P}y + \Omega) \cap \mathrm{Ker}L_n, 0)$

$= \pm\, d(\pi_n\, N^*(\cdot + K_{P_{\bar{n}}P}y,0)/\mathrm{Ker}L_{\bar{n}}, (-K_{P_{\bar{n}}P}y + \Omega) \cap \mathrm{Ker}L_{\bar{n}}, 0)$

$= \pm\, d(\pi_{\bar{n}}\, N^*(\cdot,0)/L_{\bar{n}}^{-1}(y), \Omega \cap L_{\bar{n}}^{-1}(y), 0)$ .....

Where we have used the multiplication property of Brouwer degree and the fact that degree of a linear isomorphism is equal to ± 1. Hence by assumption (3) and (2.7) it follows that $d(I-\hat{M}_{\bar{n},0},\Omega,0)\neq 0$. Thus $d(I-\hat{M}_{\bar{n},t},\Omega)\neq 0$ for all $t\in[0,1]$ and all $n\geq m_1$.

Let $t_0\in[0,1]$ be fixed but arbitrary. Then since we have already proved that $d(I-\hat{M}_{n,t_0},\Omega,0)\neq 0$ for all $n\geq m_1$ it follows from the existence property of Leray-Schauder degree that for each $n\geq m_1$ there exists a fixed point x_n of $\hat{M}_{n,t_0}$. Hence by proposition 3.1 of [12] for each $n\geq m_1$ we have

$$L_n(x_n) = O_n' \, Q \, N^*(x_n,t_0) + t_0(I-Q_n' \, Q) \, N^*(x_n,t_0) + y.$$

Thus by lemma 2.1 for each $n\geq m_1$ $L_n(x_n) = t_0 \, N^*(x_n,t_0) + y$. Hence by condition (S) (respectively (S)$'$) of the triple $(L,t_0N^*_{t_0} + y,\Omega)$ derived from hypothesis (4) it follows that there is a point $x_0\in \mathrm{Cl}\,\Omega$ such that $L(x_0) = t_0N(x_0,t_0) + y$. But since we have already seen that x_0 cannot belong to $\partial\Omega$, it follows that $x_0\in\Omega$ and $L(x_0) = t_0N(x_0,t_0) + y$. This completes the proof.

When X is reflexive, we can state a continuation theorem in the following form which does not involve the concept of generalized coincidence degree.

Theorem 2.2. Let the triple (L,N^*,Ω) satisfy the conditions (i)$'$ to (vii)$'$ of section 1. Further assume that there exists an infinite sequence $\{m\}$ of positive integers with $m\to\infty$ such that for each m,

(1) $L_m(x)\neq t \, N^*(x,t) + y$ for every $x\in\partial\Omega\cap \mathrm{dom}L$ and every $t\in(0,1)$;

(2) $\pi_m \, N^*(x,0)\neq C$ for every $x\in L^{-1}\{y\}\cap\partial\Omega$;

(3) $d(\pi_m \, N^*(\cdot,0)/L_m^{-1}(y),\ \Omega\cap L_m^{-1}\{y\},0)\neq 0.$

Then for each $t \in [0,1]$ the equation (2.1) has at least one solution in $Cl\Omega$.

Proof. Clearly for each fixed m, the continuation theorem IV.I in Gaines and Mawhin ([4], p. 27) is applicable to the equation

$$L_m(x) = t\, N^*(x,t) + y \quad \ldots\ldots \tag{2.8}$$

Hence by this theorem for each $t \in [0,1]$ there is a solution $x_m \in Cl\Omega$ of the equation (2.8).

Let $t_o \in [0,1]$ be arbitrary. Then by what we have said before, there is a solution $x_m \in Cl\Omega$ of the equation $L_m(x) = t_o\, N^*(x,t_o) + y$ for each m. Since X is reflexive and $Cl\Omega$ is bounded, there is a subsequence $\{x_{m_i}\}$ of $\{x_m\}$ such that $x_{m_i} \rightharpoonup x_o \in Cl\Omega$.

Replacing N(x) by $t_o N(x,t_o) + y$ in proposition 3.2 of [12] we can easily see that x_o is a solution of the equation $L(x) = t_o N(x,t) + y$. Since t_o is arbitrary, the theorem is proved.

As in Mawhin [9] (also Gaines and Mawhin [4]) we can now deduce the following existence theorems, the proofs of which being in the same line as in [4] are omitted.

Theorem 2.3. Let the $(L,\tilde{N},\Omega)$ satisfy either (i) to (viii) or (i)′ to (viii)′ of section 1 and let Ω be symmetric with respect to the origin and containing it. Further suppose that $\tilde{N}(-x,0) = -\tilde{N}(x,0)$ for each $x \in Cl\Omega$. Then for each $t \in [0,1]$ the equation $L(x) = N(x,t)$ has at least one solution in Ω.

Theorem 2.4. (Krasnosel′skii Theorem [7].) Let the triple (L,N,Ω) satisfy (a) to (f) (respectively (a)′ to (f)′) of section 1. Let Ω be symmetric with respect to the origin and containing it. Further assume that

(1) the triple $(L,\hat{N},\Omega)$ satisfies the condition (vii) (respectively (vii)$'$) of section 1 where $\hat{N}:\mathrm{Cl}\Omega\times[0,1]\to Z$ is defined by

$$\hat{N}(x,t) = \frac{1}{1+t}\,(N(x) - t\,N(-x)),\ (x,t)\in \mathrm{Cl}\Omega\in[0,1];$$

(2) $(L-N)(x)\neq t(L-N)(-x)$ for each $x\in\partial\Omega\cap \mathrm{dom}L$ and each $t\in[0,1]$.

Then the equation $L(x) = N(x)$ has at least one solution in Ω.

<u>Theorem 2.5</u>. Let the triple $(L,\hat{N},\Omega)$ satisfy either (i) to (viii) or (i)$'$ to (viii)$'$ of section 1. Let for some $t_o\in$ $\in[0,1]$, $d((L,\hat{N}_{t_o}),\Omega)\neq\{0\}$. Then for each $t\in[0,1]$, the equation $L(x) = N(x,t)$ has a solution in Ω.

Proof. The proof follows from Theorem 1.4 and Theorem 1.1.

3. <u>A generalized coincidence degree for k-set contractive perturbations of admissible generalized Fredholm mappings</u>

The concepts of Kuratowski measure introduced by Kuratowski [8] and Hausdorff-ball measure introduced by Gol'denstein, Gohberg and Markus [5] are well known. For definition of k-set contraction with respect to these two measures and their properties we refer to Gaines and Mawhin [4].

As in section 1, if we consider the same two sets of assumptions with both (d) and (d)$'$ replaced by the following condition:

(d) for each positive integer n, there exists a non-negative real number $k_n<1$ such that $K_{P_n P}(I-Q_n' Q)N$ is a k_n-set contraction; then we easily see that with the triple (L,N,Ω) satisfying either (a),(b),(c) and $(\bar{d})$ or (a)′,(b)′,(c)′ and $(\bar{d})$, the mapping $M_n:Cl\Omega \longrightarrow X$ is a k_n-set contraction for each positive integer n and therefore the degree $\hat{d}(I-M_n,\Omega,0)$ is well-defined provided $0 \notin (I-M_n)(\partial\Omega)$. Here $\hat{d}(I-M_n,\Omega,0)$ is the degree obtained by Nussbaum [10] and [11] with respect to Kuratowski measure and by Vainikko and Sadovskii [13] with respect to Hausdorff-ball measure.

Thus if the triple (L,N,Ω) satisfies (a) to (h) or (a)′ to (h)′ with both (d) and (d)′ being replaced by $(\bar{d})$, we can define $d((L,N),\Omega)$, a generalized coincidence degree of the pair (L,N) on Ω with respect to Γ in the same way as in definition 1.1 by replacing $d(I-M_m,\Omega,0)$ by $\hat{d}(I-M_m,\Omega,0)$ and can show that this degree has the same properties as proved in Theorems 1.1 - 1.4 by using the corresponding properties of $\hat{d}(I-M_m,\Omega,0)$. This will then be the extension of Hetzer's result [10] to our situation.

<u>Remark 3.1</u>. Degree theory presented here is applicable at least to the equation L(x) = N(x) where L is a closed self-adjoint operation on a Hilbert space H, K_P is compact and N is a k-set contraction with $k<1$. Application to such case and the existence of periodic solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t^2} = N(x,t,u)$$

will appear elsewhere.

References

[1] BROWDER F.E. and PETRYSHYN W.V.: Approximation methods and the generalized topological degree for nonlinear

mappings in Banach spaces, J. Functional Anal. 39(1968), 217-245.

[2] BROWDER F.E. and PETRYSHYN W.V.: The topological degree and the Galerkin approximations for non-compact operator in Banach spaces, Bull. Amer. Math. Soc. 74 (1968), 641-646.

[3] FITZPATRICK P.M.: A generalized degree for uniform limits of A-proper mappings, J. Math. Anal. Appl. 35 (1971), 536-552.

[4] GAINES R.E. and MAWHIN J.L.: Coincidence degree and non-linear differential equations, Lecture Notes in Mathematics, No. 568 (Edited by Dold,A. and Eckmann, B.) Springer-Verlag (1977).

[5] GOLDENSTEIN L.S., GOHBERG I. Ts and MARKUS A.S.: Investigation of some properties of bounded linear operators in connection with their q-norm, Uch. Zap. Kishinev, Gos. Univ. 29(1957), 29-36.

[6] HETZER G.: Some remarks on ϕ_+ operators and on the coincidence degree for Fredholm equation with non-compact nonlinear perturbation, Ann. Soc. Sci. Brùxelles Ser. I 89(1975), 497-508.

[7] KRASNOSEL'SKII M.A.: Some problems of nonlinear analysis, Amer. Math. Soc. Transl. (2)10(1958), 345-409.

[8] KURATOWSKI C.: Sur les espaces complets, Fund. Math. 15 (1930), 301-309.

[9] MAWHIN J.: Equivalence Theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, J. Differential Equations 12 (1972), 610-636.

[10] NUSSBAUM R.D.: The fixed point index for local condensing maps, An. Mat. pura Appl. (4)89(1971), 217-258.

[11] NUSSBAUM R.D.: Degree theory for local condensing maps, J. Math. Anal. Appl. 37(1972), 741-766.

[12] TARAFDAR E.: On the existence of solution of the equation $L(x) = N(x)$ and a generalized coincidence degree theory I, Comment. Math. Univ. Carolinae 21(1980), 805-823.

[13] VAINIKKO G.M. and SADOVSKII B.N.: On the rotation of condensing vector fields (Russian), Probl. Matem., Analiza Slozhn. Sist. No. 2, Voronezh (1968), 84-88

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