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Enayet U, Tarafdar

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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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ON THE EXISTENCE OF SOLUTION OF THE EQUATION $L(x) = N(x)$ AND A GENERALIZED COINCIDENCE DEGREE THEORY I.

E. TARAFDAR

Abstract: Coincidence degree theory provides a method for proving the existence of solution of the equation $L(x) = N(x)$ where $L:\mathrm{dom}\, L \subset X \to Z$ is a linear Fredholm mapping of index equal to zero and N is a (completely continuous) mapping which is defined on the closure of a bounded open subset of X and takes values from Z, X and Z being Banach spaces over the reals. In this paper we have developed a method for proving the existence of the solution of the equation $L(x) = N(x)$ where L is a generalized Fredholm mapping (i.e., L is linear and kernel of L and image of L are complemented subspaces of X and Z respectively) having the additional property that kernel of L and cokernel of L are linearly homeomorphic and N is the same as above.

Key words and phrases: Coincidence degree, Leray-Schauder degree, admissible generalized Fredholm mapping.

Classification: Primary 47H15, 47A50
Secondary 47H10, 47A55

Introduction. Let X and Z be Banach spaces over the reals. Then the operator equation $L(x) = N(x)$ where $L:\mathrm{dom}\, L \subset X \to Z$ is a linear mapping and $N:\mathrm{dom}\, N \subset X \to Z$ is a (possibly nonlinear) mapping represents a wide variety of problems including nonlinear ordinary, partial and functional differential equations. When L^{-1} exists, then this equation reduces to $x = L^{-1}N(x)$ which is included in the class of Hammerstein operators and is under the scope of fixed point theory, or so to speak the Leray-Schauder degree theory. Extensive literature for this case can be had from the survey works of Dolph and Minty [3] and of Ehrmann [4]. When L is a noninvertible Fredholm mapping and is of finite index ≥ 0 and N is completely continuous, Mawhin [9] has developed a theory called the coincidence degree for the pair (L,N), which serves as a tool for proving the existence of solutions of the equation $L(x) = N(x)$. Hetzer [6] has extended the concept of coincidence degree to the situation when L is as above and N is a set-contraction mapping, and applied to the problem concerning neutral functional differential equations (see [7]). For application of coincidence degree to nonlinear differential equation we refer to Gaines and Mawhin [5].

It turns out that the coincidence degree of the pair (L,N) is zero if the index of $L>0$. Thus the coincidence degree plays an important role only when index of L is zero. In this paper we have dealt with the situation when L is a generalized Fredholm mapping, i.e., L is linear and ker L (kernel of L) and coker L (cokernel of L) are complemented subspaces of X and Z respectively with the additional conditions that ker L and coker L are linearly homeomorphic and ker L posseses a property of the type that it (ker L) has a Schauder basis when dimker $L = \infty$. Thus we allow ker L and coker L to be of infinite dimensions.

With the results of this paper, together with the condition (S) or (S)$'$ (see Section 3) we have built up in our next paper [10] a generalized coincidence degree for the pair (L,N).

1. <u>Notations and algebraic preliminaries</u>. Let X and Z be two vector spaces over the same scalar field and $L:\operatorname{dom} L \subset X \longrightarrow Z$ be a linear mapping where dom L stands for the domain of L. Ker $L = L^{-1}(0)$ and Im L denote respectively the kernel and image of L. An operator $P:X \longrightarrow X$ is said to be an algebraic projection if P is linear and idempotent, i.e. $P^2 = P$. Let $P:X \longrightarrow X$ and $Q:Z \longrightarrow Z$ be two algebraic projections. Then the pair (P,Q) is said to be an exact pair with respect to L or simply an exact pair if the sequence $X \xrightarrow{P} \operatorname{dom} L \xrightarrow{L} Z \xrightarrow{Q} Z$ is exact, i.e. $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q$. L_P will denote the restriction of L to $\ker P \cap \operatorname{dom} L$. Clearly L_P is an algebraic isomorphism. Let $K_P = L_P^{-1}$. Then $K_P:\operatorname{Im} L \longrightarrow \operatorname{dom} L \cap \ker P$ is a linear mapping. For an exact pair (P,Q) of algebraic projections we have the following:

(1.1) Obviously $PK_P = 0$ .....

(1.2) For each $y \in \operatorname{Im} L$, $LK_P(y) = L(I-P)K_P(y) = L_P(I-P)K_P(y) = y$ .....

where I is the identity operator on X. For each $x \in \operatorname{dom} L$

(1.3) $K_PL(x) = K_PL(I-P)(x) = K_PL_P(I-P)(x) = (I-P)(x)$.....

Coker L denotes the quotient space $Z/\operatorname{Im} L$ and $\pi:Z \longrightarrow \operatorname{coker} L$ the canonical surjection. We can easily see that

(1.4) $Q(z) = 0 \Longleftrightarrow z \in \operatorname{Im} L \Longleftrightarrow \pi(z) = 0$ .....

We will also use the well-known fact that since $\operatorname{Im} L = \ker Q$, the restriction $\hat{\pi}$ of π to Im Q is an algebraic isomorphism.

We should point out that the same symbol I will be used to denote the identity operator on X as well as on Z. We believe that this will create no confusion to the reader and will be clearly understood from the context.

Equivalence of solutions and fixed points

For proof of the following result we refer to Mawhin [9].

Proposition 1.1. Let X, Z be two vector spaces and Ω be a subset of X. Let $L:\operatorname{dom} L \subset X \to Z$ be a linear mapping and $N: \Omega \to Z$ be a mapping which is not necessarily linear. Further suppose that there exists a mapping $\psi :\operatorname{coker} L \to \ker L$ such that $\psi^{-1}(0) = \{0\}$. Then x is a solution of the operator equation $L(x) = N(x)$ if and only if x is a fixed point of the mapping $M:\Omega \to X$ is defined by $M(x) = P(x) + \psi\pi N(x) + K_P(I-Q)N(x)$, $x \in \Omega$ for every exact pair (P,Q) of projections. Clearly $M(\Omega) \subset \operatorname{dom} L$.

2. Admissible generalized Fredholm mapping and approximations. Unless otherwise stated, throughout the rest of the paper X and Z will denote Banach spaces over the field of reals.

Definition 2.1. A closed subspace M of X is said to be complemented if there exists a continuous projection (i.e., continuous linear and idempotent mapping) of X onto M.

If M and N are closed subspaces of X such that $M \cap N = \{0\}$ and $X = M+N$, then we write $X = M \oplus N$.

Definition 2.2. A linear mapping $L:\operatorname{dom} L \subset X \to Z$ is said to be generalized Fredholm mapping if ker L and Im L are complemented.

Remark 2.1. The class of bounded generalized Fredholm mappings had been studied by Caradus [1] and [2].

Definition 2.3. A linear mapping $L:\operatorname{dom} L \subset X \to Z$ is said to be an admissible generalized Fredholm mapping if L satisfies the following:

(i) L is a generalized Fredholm mapping in the sense of definition 2.2.

(ii) There is a topological isomorphism (i.e., linear homeomorphism) ψ of coker L = Z/Im L onto ker L.

(iii) There exists an increasing (not necessarily strictly, see remark 2.2 (3)) sequence $\{X_n\}$ of finite dimensional subspaces of ker L and a sequence $\{P_n\}$ of continuous linear projections P_n:ker $L \to X_n$ with the properties that

(a) $\lim_{n\to\infty} P_n(x) = x$ for each $x \in$ ker L;

and (b) if $P_j(x) = 0$ with $x \in$ ker L and for some positive integer j, then

$P_m(x) = 0$ for all positive integers $m < j$.

From now on by a positive integer n we shall mean n to be the dimension of X_n.

Remark 2.2.

(1) The condition (iii) implies that ker L is separable and hence by condition (ii) coker L is also separable.

(2) It is important to note that if ker L has Schauder basis, then the condition (iii) holds, i.e., there do exist sequences $\{X_n\}$ and $\{P_n\}$ satisfying (a) and (b) of condition (iii). For let $\{x_n\}$ be a Schauder basis for ker L. Let $\{x_n^*\}$, $x_n^* \in (\text{ker } L)^*$ be the system orthogonal to $\{x_n\}$, i.e. $x_i^*(x_j) = \delta_{ij}$. Let for each positive integer n, X_n be the linear span of $\{x_1, x_2, \dots, x_n\}$. Define P_n:ker $L \to X_n$ by $P_n(x) = \sum_{i=1}^{n} x_i^*(x)x_i$. Then it is trivial to check that $\{P_n\}$ is a sequence of continuous linear projections with the properties (a) and (b) of

condition (iii).

(3) If $L:\mathrm{dom}\, L \subset X \to Z$ is a Fredholm operator of index zero, i.e. $\dim\ker L = \dim\operatorname{coker} L < \infty$, where dim means dimension, then L is clearly an admissible generalized Fredholm mapping. This is because we can take $X_n = \ker L$ for each positive integer n.

(4) If $L:\mathrm{dom}\, L \subset X \to Z$ is a linear mapping such that ker L and Im L are closed subspaces and if ker L and coker L are infinite dimensional separable Hilbert spaces, then L is an admissible generalized Fredholm mapping. Since all infinite dimensional separable Hilbert spaces are isomorphic to l^2, ψ of condition (ii) exists. The condition (iii) holds as ker L has an orthonormal basis.

We will now develop an approximation technique for approximating an admissible generalized Fredholm mapping by Fredholm mappings of index zero.

Let $L:\mathrm{dom}\, L \subset X \to Z$ be an admissible generalized Fredholm mapping. Let (P,Q) be an exact pair, of continuous projections with respect to L, which exists by condition (i) of definition 2.3. Let $\{X_n\}$ and $\{P_n\}$ be a pair of sequences obtained from condition (iii) of definition 2.3 and ψ is a topological isomorphism obtained from condition (ii) of definition 2.3. The system $\Gamma = (X_n, P_n, P, Q, \psi)$ is said to be an associated scheme for L.

For each positive integer n we define the mapping

$L_n:\mathrm{dom}\, L \subset X \to Z$ by setting

$$L_n(x) = L(x) + \phi\psi^{-1}(P-P_nP)(x), \quad x \in \mathrm{dom}\, L,$$

where $\phi = \tilde{\pi}^{-1}:\operatorname{coker} L \to \operatorname{Im} Q$ is the (natural) topological isomorphism, $\tilde{\pi}'$ being the restriction to Im Q of the natural surjection $\pi : Z \longrightarrow$ coker L. We note that since Im L is closed and Z is a Banach space, coker L is a Banach space with the usual quotient topology. Also π is continuous and $\tilde{\pi}$ is a topological isomorphism.'

We first prove the following:

(A) For each positive integer n, L_n is a Fredholm mapping of index zero with dimker L_n = dimcoker L_n = dim X_n.

To prove this let n be a fixed but arbitrary positive integer. We first consider the following direct sum representations:

(2.1) $\qquad \ker L = U_n \oplus X_n$ .....

where $U_n = \ker P_n$.

Since ψ is a topological isomorphism, it is easy to see that $Q_n = \psi^{-1} P_n \psi : \text{coker } L \longrightarrow \psi^{-1}(X_n)$ is a continuous projection and

$$\text{coker } L = \psi^{-1}(U_n) \oplus \psi^{-1}(X_n) = \psi^{-1}(U_n) \oplus Z_n,$$

where $Z_n = \psi^{-1}(X_n)$, is the corresponding direct sum representation.

Again since Φ is a topological isomorphism we can as before see that $Q_n' = \Phi Q_n \Phi^{-1} : \text{Im } Q \longrightarrow \Phi(Z_n)$ is a continuous projection and

(2.2) $\qquad \text{Im } Q = \Phi(\psi^{-1}(U_n)) \oplus \Phi(Z_n)$ .....

is the corresponding direct sum representation.

Now we consider the following direct sum representations: Using (2.1) and (2.2) we have respectively

(2.3) $\qquad X = \ker P \oplus \ker L = \ker P \oplus U_n \oplus X_n$ .....

and $Z = \ker Q \oplus \text{Im } Q = \ker Q \oplus \Phi(\psi^{-1}(U_n)) \oplus \Phi(Z_n) =$

(2.4) $= \operatorname{Im} L \oplus \Phi(\psi^{-1}(U_n)) \oplus \Phi(Z_n)$ .....

We can easily see that the following mappings

(2.5) $P_n P : X \longrightarrow X_n$, ...

(2.6) $P - P_n P : X \longrightarrow U_n$, ...

(2.7) $I - P_n P : X \longrightarrow \ker P \oplus U_n$, ...

(2.8) $Q_n' Q : Z \longrightarrow \Phi(Z_n)$, ...

(2.9) $Q - Q_n' Q : Z \longrightarrow \Phi(\psi^{-1}(U_n))$, ...

(2.10) and $I - Q_n' Q : Z \longrightarrow \operatorname{Im} L \oplus \Phi(\psi^{-1}(U_n))$, ...

are all continuous projections.

We can easily prove that $\ker L_n = X_n$.
We can see without difficulty that $\operatorname{Im} L_n = \operatorname{Im} L \oplus \Phi(\psi^{-1}(U_n))$. Therefore, it is immediate from (2.4) that $\operatorname{coker} L_n \simeq \Phi(Z_n)$. Since $\dim X_n = \dim \Phi(Z_n)$, we conclude that L_n is a Fredholm mapping of index zero with $\dim\ker L_n = \dim\operatorname{coker} L_n = \dim X_n$.

(B) $\{\Phi(Z_n)\}$ is an increasing sequence of finite dimensional subspaces of Im Q. Also if $Q_j'(z) = 0$ with $z \in \operatorname{Im} Q$ for some positive integer j, then $Q_m'(z) = 0$ for all $m < j$, where Q_n' is defined as in (A).

The first part follows from the fact that $\{X_n\}$ is an increasing sequence and that ψ^{-1} and Φ are isomorphisms. To see the second part let $Q_j'(z) = 0$ for some j. Then $\Phi Q_j \Phi^{-1}(z) = 0$ which implies that $Q_j \Phi^{-1}(z) = 0$ as Φ is an isomorphism. Hence $\psi^{-1} P_j \psi \Phi^{-1}(z) = 0$. Thus $P_j \psi \Phi^{-1}(z) = 0$ as ψ^{-1} is an isomorphism. Now by condition (iii)(b) of definition 2.3, $P_m \psi \Phi^{-1}(z) = 0$ for all $m < j$. Hence $Q_m'(z) =$
$= \Phi(\psi^{-1} P_m \psi)\Phi^{-1}(z) = 0$ for all $m < j$.

(C) $\lim_{n\to\infty} Q'_n(z) = z$ for each $z \in \operatorname{Im} Q$. Let $z \in \operatorname{Im} Q$. Then $\psi\Phi^{-1}(z) \in \ker L$. Hence by condition (iii)(a) of definition 2.3, $\lim_{n\to\infty} P_n \psi\Phi^{-1}(z) = \psi\Phi^{-1}(z)$. Therefore by using continuity of ψ^{-1}, $\lim_{n\to\infty} Q_n\Phi^{-1}(z) = \lim_{n\to\infty} \psi^{-1}P_n \psi\Phi^{-1}(z) = $
$= \psi^{-1}(\lim_{n\to\infty} P_n\psi\Phi^{-1}(z)) = \Phi^{-1}(z)$. Finally using the continuity of Φ, $\lim_{n\to\infty} Q'_n(z) = \lim_{n\to\infty} \Phi Q_n \Phi^{-1}(z) =$
$= \Phi(\lim_{n\to\infty} Q_n\Phi^{-1}(z)) = \Phi\Phi^{-1}(z) = z.$

(D) For each $x \in \operatorname{dom} L$, $\lim_{n\to\infty} L_n(x) = L(x)$. This follows from the condition (iii)(a) of definition 2.3.

In the sequel we will use L_n and Q'_n defined in this section without further reference.

3. <u>Approximate equations.</u> Our aim is to obtain a method for proving the existence of the solution of the operator equation

$$(3.1) \qquad L(x) = N(x) \quad \ldots\ldots$$

where $L:\operatorname{dom} L \subset X \longrightarrow Z$ is an admissible generalized Fredholm mapping and $N:\operatorname{Cl}\Omega \longrightarrow Z$ is a mapping not necessarily linear, Ω being a bounded open subset of X. Let $\Gamma = (X_n, P_n, P, Q, \psi)$ be an associated scheme for L. We will now consider the approximate equations

$$(3.2) \qquad L_n(x) = N(x) \quad \ldots\ldots$$

for each positive integer n.

Let us also consider the mapping $M:\operatorname{Cl}\Omega \longrightarrow X$ defined by

$$(3.3) \qquad M(x) = P(x) + \psi\pi N(x) + K_P(I-Q)N(x),\ x \in \operatorname{Cl}\Omega \quad \ldots\ldots$$

where K_P is the inverse of the restriction to $\ker P \cap \operatorname{dom} L$ of L and π is the natural surjection of Z onto $Z/\operatorname{Im} L = \operatorname{coker} L$;

and for each positive integer n, the mapping $M_n: Cl\Omega \longrightarrow X$ defined by

$$(3.4)\quad M_n(x) = P_nP(x) + \psi_n \pi_n N(x) + K_{P_nP}(I-Q_n'Q)N(x), \quad x \in Cl\Omega \; .....$$

where K_{P_nP} is the inverse of the restriction to $\ker P_nP \cap \operatorname{dom} L$ of L_n, π_n is the natural surjection of Z onto $Z/(\operatorname{Im} L \oplus \oplus \Phi\psi^{-1}(U_n)) = \operatorname{coker} L_n$, and ψ_n is an isomorphism of coker L_n onto ker L_n (ψ_n always exists).

The following proposition follows from the proposition 1.1.

Proposition 3.1. x is a solution of the equation 3.1 if and only if x is a fixed point of M. Also for positive integer n, x_n is a solution of the equation 3.2 if and only if x_n is a fixed point of M_n.

To see the second part we need only to observe that $(P_nP, Q_n'Q)$ is an exact pair of projections with respect to L_n.

Corollary 3.1. If there exists $x \in X$ such that x is a fixed point of each of M_{n_j} where $\{n_j\}$ is an infinite sequence of positive integers with $n_j \longrightarrow \infty$, then x is a solution of the equation 3.1.

Proof. By proposition 3.1 we have

$$L_{n_j}(x) = L(x) + \Phi\psi^{-1}(P-P_{n_j}P)(x).$$

Now the corollary follows from (D) of Section 2.

Lemma 3.1. If $\{u_n\}$ is a sequence of points in ker L such that $P_nu_n = 0$ for all n and $u_n \rightharpoonup u_o$ ($\rightharpoonup$ denotes weak convergence), then $u_o = 0$.

Proof. With fixed n_o it follows from (iii)(b) of definition 2.3 that $P_{n_o}(u_n) = 0$ for all $n \geq n_o$. From the weak continuity of P_{n_o} and the fact that $u_n \rightharpoonup u_o$, we have $P_{n_o}(u_n) \rightharpoonup P_{n_o}(u_o)$. Thus $P_{n_o}(u_o) = 0$. As n_o is arbitrary, the lemma follows from (iii)(a) of definition 2.3.

Lemma 3.2. If $\{m\}$ is an infinite sequence of positive integers with $m \to \infty$ such that x_m is a fixed point of M_m for each m and $N(x_m) \rightharpoonup y$, then $\pi_m N(x_m) = 0$ for each m and $y \in \operatorname{Im} L$.

Proof. By proposition 3.1, $L_m(x_m) = N(x_m)$ for each m. Thus for each m, $N(x_m) \in \operatorname{Im} L_m = \operatorname{Im} L \oplus \Phi\psi^{-1}(U_m)$ (see the proof of (A) in Section 2). From this it follows that $\pi_m N(x_m) = 0$ and $Q_m' QN(x_m) = 0$ for each m (the last equality follows from the fact that $QN(x_m) \oplus \Phi\psi^{-1}(U_m)$). Let $m = m_o$ be fixed but arbitrary. Then since $Q_m' QN(x_m) = 0$ for all m, it follows from (B) of Section 2 that $Q_{m_o}' QN(x_m) = 0$ for all $m \geq m_o$. But since $N(x_m) \rightharpoonup y$ and both Q and Q_{m_o}' are weakly continuous (we will no longer repeat the argument that a continuous linear mapping is weakly continuous), $Q_{m_o}' QN(x_m) \rightharpoonup Q_{m_o}' Q(y)$. Thus $Q_{m_o}' Q(y) = 0$. Now since m_o is arbitrary, $Q_m' Q(y) = 0$ for all m. Hence by (C) of Section 2 $\lim_{m\to\infty} Q_m' Q(y) = Q(y) = 0$. Therefore $y \in \operatorname{Im} L$ by (1.4).

Proposition 3.2. Let X be reflexive and Ω be an open bounded subset of X such that $\mathrm{Cl}\,\Omega = \omega\text{-}\mathrm{Cl}\,\Omega$ where $\omega\text{-}\mathrm{Cl}\,\Omega$ is the weak closure of Ω. Let $L:\operatorname{dom} L \subset X \to Z$ be an admissible generalized Fredholm mapping and $N:\mathrm{Cl}\,\Omega \to Z$ be a mapping. Further assume that

(i) N is weakly continuous;

(ii) either L is weakly continuous or K_P is weakly continuous, where K_P is as defined before.

If there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that x_m is a fixed point of M_m for each m, then there is a fixed point x_o of M, i.e. there is a solution x_o of the equation 3.1.

Remark 3.1. It is well known that every convex subset Ω has the property that $Cl\Omega = \omega\text{-}Cl\Omega$.

Proof of Proposition 3.2. Since $\{x_m\}$ is a bounded sequence in the reflexive Banach space X, there is a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ such that $x_{m_j} \rightharpoonup x_o \in Cl\Omega = \omega\text{-}Cl\Omega$. As N is weakly continuous $N(x_{m_j}) \rightharpoonup N(x_o)$. Hence by lemma 3.2 we have

(3.5) $N(x_o) \in \operatorname{Im} L$ and $\pi_{m_j} N(x_{m_j}) = 0$ for each j. .....

The last equality is equivalent to

$$N(x_{m_j}) \in \operatorname{Im} L \oplus \Phi\psi^{-1}(U_{m_j}) \text{ for each } j.$$

Let for each j, $N(x_{m_j}) = \gamma_{m_j} + \omega_{m_j}$ where $\gamma_{m_j} \in \operatorname{Im} L$ and $\omega_{m_j} \in \Phi\psi^{-1}(U_{m_j})$, [i.e., for each j, $(I-Q)N(x_{m_j}) = \gamma_{m_j}$, $(Q-Q'_{m_j}Q)N(x_{m_j}) = \omega_{m_j}$ and $Q'_{m_j}QN(x_{m_j}) = 0$]. Since $(I-Q)$ is weakly continuous, $\gamma_{m_j} = (I-Q)N(x_{m_j}) \rightharpoonup (I-Q)N(x_o) = N(x_o)$ by (1.4) as $N(x_o) \in \operatorname{Im} L$ by (3.5). Thus it follows that $\omega_{m_j} \rightharpoonup 0$.

By definition of $K_{P_{m_j}} P$, we can write for each j,

(3.6) $K_{P_{m_j}} PN(x_{m_j}) = u_{m_j} + v_{m_j}$ .....

where $u_{m_j} \in \ker P \cap \operatorname{dom} L$ and $v_{m_j} \in U_{m_j}$. [This is possible as

$K_{P_{m_j}P}N(x_{m_j}) \in \ker P_{m_j}P \cap \operatorname{dom} L$ and $\ker P_{m_j}P = \ker P \oplus U_{m_j}$. Thus $u_{m_j} + v_{m_j} \in \operatorname{dom} L$, $u_{m_j} \in \ker P$ and $v_{m_j} \in U_{m_j} \subset \operatorname{dom} L$. Hence $u_{m_j} \in \operatorname{dom} L$]. Again since by proposition 3.1, $L_{m_j}(x_{m_j}) = N(x_{m_j})$, we have $N(x_{m_j}) \in \operatorname{Im} L_{m_j}$ and hence by (1.2) $L_{m_j}K_{P_{m_j}P}N(x_{m_j}) = N(x_{m_j})$. Thus for each j,

$$\begin{aligned} \gamma_{m_j} + \omega_{m_j} &= N(x_{m_j}) = L_{m_j}(u_{m_j} + v_{m_j}) \\ (3.7) \qquad &= L(u_{m_j} + v_{m_j}) + \Phi\psi^{-1}(P-P_{m_j}P)(u_{m_j} + v_{m_j}) \\ &= L(u_{m_j}) + \Phi\psi^{-1}(v_{m_j}) \quad \ldots\ldots \end{aligned}$$

as $v_{m_j} \in \ker L$ and $u_{m_j} \in \ker P$.

But since both of γ_{m_j} and $L(u_{m_j})$ are in Im L and both of ω_{m_j} and $\Phi\psi^{-1}(v_{m_j})$ are in $\Phi\psi^{-1}(U_{m_j})$, it follows from (3.7) and the fact that $\operatorname{Im} L \cap \Phi\psi^{-1}(U_{m_j}) = \{0\}$ that for each j,

$$(3.8) \quad L(u_{m_j}) = \gamma_{m_j} \text{ and } \Phi\psi^{-1}(v_{m_j}) = \omega_{m_j}, \text{ i.e. } v_{m_j} = \psi\Phi^{-1}(\omega_{m_j}) \quad \ldots\ldots$$

Now as ψ and Φ^{-1} are topological isomorphism and the sequence $\omega_{m_j} \to 0$, it follows from the last part of (3.8) that $v_{m_j} \to 0$.

Next, since x_{m_j} is a fixed point of M_{m_j} for each j, by using (3.4),(3.6) and the fact that $\psi_{m_j}\pi_{m_j}N(x_{m_j}) = 0$ which is a consequence of (3.5), we can write, for each j

$$(3.9) \quad x_{m_j} = P_{m_j}P(x_{m_j}) + u_{m_j} + v_{m_j}. \quad \ldots\ldots$$

Hence

$$(3.10) \qquad P(x_{m_j}) = P_{m_j}P(x_{m_j}) + v_{m_j} \quad \ldots..$$

as $P_{m_j}P(x_{m_j})$ and v_{m_j} are in ker L = Im P and $u_{m_j} \in$ ker P. Now letting $j \to \infty$ in (3.10) we obtain

$$(3.11) \qquad P_{m_j}P(x_{m_j}) \rightharpoonup P(x_o) \quad \ldots..$$

as we know that $x_{m_j} \rightharpoonup x_o$ and $v_{m_j} \rightharpoonup 0$. Let us first consider the case when L is weakly continuous. Since x_{m_j} is a fixed point of M_{m_j} for each j, by proposition 3.1 we have that for each j, $L_{m_j}(x_{m_j}) = N(x_{m_j})$, i.e.

$$(3.12) \qquad L(x_{m_j}) + \Phi\psi^{-1}(P-P_{m_j}P)(x_{m_j}) = N(x_{m_j}). \quad \ldots..$$

Noting that Φ, ψ^{-1}, L are all weakly continuous and that N is weakly continuous by hypothesis, letting $j \to \infty$ in (3.12) and using (3.11) and the fact that $N(x_{m_j}) \rightharpoonup N(x_o)$ we obtain that $L(x_o) = N(x_o)$. This proves the conclusion of the proposition when L is weakly continuous.

Finally, we consider the case when K_P is weakly continuous. We have already obtained the following:

$$(3.13) \qquad \text{and} \quad \begin{cases} v_{m_j} \rightharpoonup 0 \\ \omega_{m_j} \rightharpoonup 0 \\ \gamma_{m_j} \rightharpoonup N(x_o) \end{cases} \quad \ldots..$$

Again, since from (3.8) for each j, $Lu_{m_j} = \gamma_{m_j}$ and $u_{m_j} \in$ ker $P \cap$ dom L, it follows readily that $K_P\gamma_{m_j} = u_{m_j}$. From this, together with (3.13) and the fact that K_P is weakly continuous, we obtain that $u_{m_j} \rightharpoonup K_PN(x_o)$. Hence letting $j \to \infty$

in (3.9) and using (3.11) and the fact that $v_{m_j} \rightharpoonup 0$ and the above limit we obtain that $x_0 = P(x_0) + K_P N(x_0) = P(x_0) + \psi\pi N(x_0) + K_P N(x_0)$ as in view of (3.5) and (1.4) we have $\pi N(x_0) = 0$ (and hence $\psi\pi N(x_0) = 0$). Thus $x_0 = M(x_0)$. This completes the proof of our proposition (the last part follows from the proposition 3.1).

<u>Proposition 3.3.</u> Let Ω be an open bounded subset of X (not necessarily reflexive). Let $L: \mathrm{dom}\, L \subset X \to Z$ be an admissible generalized Fredholm mapping and $N: \mathrm{Cl}\Omega \to Z$ be a mapping. Further assume that

(i)′ N is continuous;

(ii)′ either L is continuous or K_P is continuous.

If there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that x_m is a fixed point of M_m for each m and $x_m \to x_0$. Then x_0 is a fixed point of M and is, therefore, a solution of the equation 3.1.

Proof. The proof is exactly the same as that of proposition 3.2 if we replace everywhere the weak convergence $\rightharpoonup$ by convergence $\to$ and use the continuity in place of weak continuity.

To define the coincidence degree in our next paper [10] we will need the following assumptions which are given in the definition below:

<u>Definition 3.1.</u> The triple (L,N,Ω) is said to satisfy the condition (S) if the following condition holds:
If $\{m\}$ is an infinite sequence of positive integers with $m \to \infty$ such that x_m is a fixed point of M_m for each m, then there exists a subsequence $\{x_{m_i}\}$ of $\{x_m\}$ such that $x_{m_i} \to x_0$

for some $x_0 \in Cl\Omega$.

The triple (L,N,Ω) is said to satisfy the condition $(S)'$ if the following condition on $\partial\Omega$ holds:
If $\{m\}$ is an infinite sequence of positive integers with $m \to \infty$ such that if for each m, $x_m \in \partial\Omega$ and is a fixed point of M_m and $x_m \to x_0$, then $x_0 \in \partial\Omega$.

<u>Remark 3.2.</u> Clearly the condition (S) implies the condition $(S)'$.

We will now make some remarks on condition (S) and $(S)'$.

(1) For each positive integer n, let $S_n = \{x \in Cl\Omega : x = M_n(x)\}$. If $\bigcup_m S_n$ is relatively compact, then clearly the condition (S) holds.

(2) If $[\mathrm{dom}\ L \cap (\bigcup_m S_n)]$ is relatively compact, then the condition (S) holds. This is because $S_n \subset Cl\Omega \cap \mathrm{dom}\ L$ for each n.

(3) If $(\partial\Omega \cap \mathrm{dom}\ L \cap (\bigcup_m S_n))$ is weakly closed, then it is clear that the condition $(S)'$ holds. The condition $(S)'$ obviously holds if $\partial\Omega$ is weakly closed. However, $\partial\Omega$ is not, in general, weakly closed. For example, let X be an infinite dimensional uniformly convex Banach space and $\Omega = \{x \in X : \|x\| < 1\}$. Then $\partial\Omega$ is not weakly closed (e.g., see Kelley and Namioka ([8], p. 161-162)).

(4) The condition S obviously holds if the sequence $\{M_n\}$ is collectively completely continuous, that is $\bigcup_m M_n(Cl\Omega)$ is relatively compact and M_n is continuous for each n.

<u>Corollary 3.2.</u> Let X, Ω, L and N be as in proposition 3.3 satisfying the conditions $(i)'$ and $(ii)'$ of proposition 3.3 and also let (S) hold. If there exists an infinite se-

quence $\{m\}$ of positive integers with $m \to \infty$ such that for each m, x_m is a fixed point of M_m and $x_m \in \partial\Omega$, then there exists a fixed point x_0 of M such that $x_0 \in \partial\Omega$.

Proof. By condition (S) there exists a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ such that $x_{m_j} \to x_0 \in Cl\Omega$. Now by proposition 3.3, x_0 is a fixed point of M. Also since $x_{m_j} \in \partial\Omega$ for each j and $\partial\Omega$ is closed, $x_0 \in \partial\Omega$. This completes the proof.

Corollary 3.3. Let X, Ω, L and N be as in proposition 3.3 satisfying (i)$'$ and (ii)$'$. Also let (S) hold. If $0 \notin (L-N)(\partial\Omega \cap \text{dom } L)$, then there exists an integer $m_0 \geq 1$ such that $0 \notin (L_m-N)(\partial\Omega \cap \text{dom } L)$ for all positive integers $m \geq m_0$, or equivalently $0 \notin (I-M_m)(\partial\Omega)$ for all $m \geq m_0$.

Proof. This follows from proposition 3.1 and corollary 3.2.

Corollary 3.4. Let X, Ω, L and N be as in proposition 3.2 satisfying the conditions (i) and (ii) of proposition 3.2, and also let (S)$'$ hold.

If there exists an infinite sequence $\{m\}$ of positive integers with $m \to \infty$ such that for each m, x_m is a fixed point of M_m and $x_m \in \partial\Omega$, then there exists a fixed point x_0 of M with $x_0 \in \partial\Omega$.

Proof. Following the proof of the proposition 3.2 we obtain a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ such that $x_{m_j} \to x_0$ and x_0 is a fixed point of M. Since $x_{m_j} \in \partial\Omega$, by condition (S)$'$ $x_0 \in \partial\Omega$. This completes the proof.

Corollary 3.5. Let X, Ω, L and N be as in proposition 3.2 satisfying the conditions (i) and (ii) of proposition 3.2. Also let (S)$'$ hold. If $0 \notin (L-N)(\partial\Omega \cap \text{dom } L)$, then there exists an integer $m_0 \geq 1$ such that $0 \notin (L_m - N)(\partial\Omega \cap \mathrm{dom}\, L)$ for all positive integers $m \geq m_0$, or equivalently $0 \notin (I-M_m)(\partial\Omega)$ for all $m \geq m_0$.

Proof. The corollary follows from corollary 3.4.

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Department of Mathematics
University of Queensland
St. Lucia, Queensland
Australia, 4067.

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