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LUZIN SETS ARE ADDITIVE

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Abstract: The assertion from the title is proved in the case of ω- Luzin spaces. The proof of the general (non-separable) case is also indicated.

Key words: Luzin space, Baire set, correspondence: upper semi-continuous, compact-valued, σ-dd preserving.

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The following theorem solves a problem which has been open for some time.

Theorem. If L_1 and L_2 are Luzin subspaces of a completely regular space X then so is $L_1 \cup L_2$.

Corollary. If L_1 and L_2 are ω-Luzin subspaces of a completely regular space X then so is $L_1 \cup L_2$.

In fact we shall prove just Corollary because the proof of Theorem is analogous, one just uses the corresponding results in "non-separable" theory developed in [F-H_1] and [F-H_2]; see also the concluding remark which points out an additional trick needed in the general case. For convenience

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of the reader we shall recall all definitions and properties needed in the proof.

In what follows all topological spaces are assumed to be completely regular (this includes Hausdorff).

Before recalling the definitions, note that ω-analytic introduced in [F-H_1] means exactly analytic in the sense of [F_1] or [R_1], i.e. K-analytic in the sense of G. Choquet [Ch] and V. Šnejder [Š]. Also note that the term ω-Luzin introduced in [F-H_1] means exactly Borelian in the sense of [F_1], i.e. descriptive Borel in the sense of [R_2], i.e. K-Luzin in analogy to K-analytic.

A space X is called ω-analytic if there exists an upper semi-continuous compact-valued correspondence f (abb. usco-compact correspondence) from a separable completely metrizable space S onto X; the correspondence f is called an ω-analytic parametrization of X. If f can be chosen disjoint (i.e. $s,t \in S$, $s \neq t$ implies $f[s] \cap f[t] = \emptyset$) then X is called ω-Luzin, and f is called an ω-Luzin parametrization of X.

Note that a correspondence $f: S \longrightarrow X$ is a triple such that f is a subset of $S \times X$; the image of $T \subset S$ under f is the set

$$f[T] = \{x \mid \langle t,x \rangle \in f \text{ for some } t\},$$

and if $T = \{t\}$ is a singleton we write simply $f[t]$ instead of $f[\{t\}]$. The images of singletons are called the "values" of f.

Our proof of Corollary is based on playing with correspondences. For basic properties of correspondences we refer to [F_2], or [F_3], [F_1].

Recall that if $f_i: S_i \longrightarrow X$ are usco-compact correspondences with X Hausdorff, then so are the correspondences

$$\bigwedge\{f_i\}:\prod\{S_i\}\longrightarrow X$$

$$\bigvee\{f_i\}:\sum\{S_i\}\longrightarrow X$$

where

$$\bigwedge\{f_i\}[\{s_i\}] = \bigcap f_i[s_i],$$

$$\bigvee\{f_i\}[j,s] = f_j[s].$$

It is trivial that $\bigwedge\{f_i\}$ is disjoint if all f_i are disjoint, and $\bigvee\{f_i\}$ is disjoint if all f_i are disjoint and the family of the ranges of f_i is disjoint.

It follows that both ω-analytic and ω-Luzin subspaces of a given space are closed under countable intersections, ω-analytic subspaces are closed under countable unions, and ω-Luzin spaces are closed under countable disjoint unions. (In fact we can show that ω-analytic subspaces are closed under the Suslin operation, and ω-Luzin subspaces are closed under the disjoint Suslin operation.) It was shown in [F_1, Remark Th. 10] that a σ-compact space does not need to be ω-Luzin (hence, in general, ω-Luzin subspaces are not closed under countable unions). The main trick in the proof of Corollary is the following

Lemma 1. Let L be an ω-Luzin subspace of an ω-Luzin space X. If $h:S\longrightarrow L$ is an ω-Luzin parametrization of L, then there exists an ω-Luzin parametrization $k:T\longrightarrow X$ of X such that the family $\{L\cap k[t] \mid t\in T\}$ refines the family $\{h[s] \mid s\in \in S\}$.

For the proof of Lemma 1 we need to recall several facts about ω-Baire sets (called Baire sets usually). If X is a space then $Ba_\omega(X)$ is the smallest σ-algebra making all continuous real-valued functions measurable (or equivalently, containing the zero sets of continuous functions); the elements are called ω-Baire sets.

Fact 1. If A_1 and A_2 are ω-analytic subsets of X, $A_1 \cap A_2 = \emptyset$, then $A_1 \subset B \subset X - A_2$ for some ω-Baire set B (this is the first separation principle, see [F_1, Th. 5] or [F_3, Th. 5.3]).

Fact 2. If X is ω-Luzin, B is a ω-Baire set in X, then B is ω-Luzin ([F_1, Th. 12], [F_3, Th. 7.7]).

Fact 3. If $L \subset M$, L is ω-Luzin and M is metrizable then L is a Baire set in M ([F_1, Th. 16], [F_3, Prop. 7.10]).

Lemma 2. If $\mathcal{B}$ is a countable collection of ω-Baire sets in an ω-Luzin space X, then there exists an ω-Luzin parametrization $k:T \longrightarrow X$ such that for each B in $\mathcal{B}$ and $t \in T$ we have either $k[t] \subset B$ or $k[t] \subset X - B$.

Proof. For each B in $\mathcal{B}$ let f_B be an ω-Luzin parametrization of B, and let g_B be an ω-Luzin parametrization of L - - B. Put $k_B = f_B \vee g_B$,

$$k = \bigwedge \{ k_B \mid B \in \mathcal{B} \} .$$

Proof of Lemma 1: Let $\mathcal{U}$ be a countable base for the topology of S. For each U in $\mathcal{U}$ the sets h[U] and h[S - U] are analytic and mutually disjoint in X, and hence by Fact 1 we can choose an ω-Baire set B_U in X such that

$$h[U] \subset B_U \subset X \setminus h[S \setminus U].$$

By Lemma 2 there exists an ω-Luzin parametrization $k:T \longrightarrow X$ such that for each $U \in \mathcal{U}$ and each $t \in T$ we have either $k[t] \subset$ $\subset B_U$ or $k[t] \subset X \setminus B_U$. Obviously k has the required property.

Proof of Corollary. We may and shall assume that X = $= L_1 \cup L_2$. For i = 1,2 let f_i^o be any ω-Luzin parametrization of L_i. Since $f_1^o \wedge f_o^o$ is an ω-Luzin parametrization of $L = L_1 \cap L_2$, by Lemma 1 there exists an ω-Luzin parametrization f_1^1 of L_i, i = 1,2, such that

$$\{f_1^1[s] \cap L\} \text{ refines } \{(f_1^o \wedge f_2^o)[t]\} ,$$

and by induction there exist ω-Luzin parametrizations f_i^n of L_i, $n \in \omega$, i = 1,2, such that

the traces on L of the values of f_i^{n+1} refine the values of $f_1^n \wedge f_2^n$. Put

$$f_i = \bigwedge \{f_i^n \mid n \in \omega\} .$$

Thus f_i is an ω-Luzin parametrization of L_i, i = 1,2, and it is easy to check that

(*) if $L \cap f_1[s_1] \cap f_2[s_2] \neq \emptyset$ then $L \cap f_1[s_1] = L \cap f_2[s_2]$,

that means, the traces on L of the values of f_1 coincide with those of the values of f_2.

Denote by S_i the domain of f_i, i = 1,2, and consider the set

$$C = \{\langle s_1, s_2 \rangle \mid f_1[s_1] \cap f_2[s_2] \neq \emptyset\} .$$

Since f is usco, C is a closed set in $S_1 \times S_2$, hence C is a separable complete metric space, and by (*) the projections $\pi_i : S_1 \to S_2$ restricted to C are one-to-one. Put (i = 1,2)

$$T_i = \pi_i[C] .$$

Since $\pi_i : C \to T_i$ is continuous and 1-1, T_i is an ω-Luzin set in S_i, and since S_i is metrizable, T_i is a ω-Baire set in S_i, and so is the complement $T_i' = S_i \setminus T_i$. Hence T_i' is an ω-Luzin space, and hence $f_i[T_i']$ is an ω-Luzin space. So it remains to show that

(**) $\qquad L \setminus (f_1[T_1'] \cup f_2[T_2'])$

is ω-Luzin, because the sets $f_1[T_1']$ and $f_2[T_2']$ are disjoint.

Define $g: C \longrightarrow X$ by

$$g[\langle s_1, s_2 \rangle] = f_1[s_1] \cup f_2[s_2] .$$

By ($*$) g is a disjoint correspondence, and one easily checks that g is usco-compact. Hence $g[C]$ is ω-Luzin, and it follows from the disjointness of f_1 and f_2, and from the definition of T_1' and T_2', that $g[C]$ is the set ($**$).

Remark. In the general case of Luzin spaces one should check that

$$\pi_i : C \longrightarrow S_i$$

are σ-dd-preserving to prove that T_i are Baire sets and that g is σ-dd-preserving. To this end it is convenient to prove the following more general result:

Lemma 3. Let $f_i : S_i \longrightarrow X$, $i = 1,2$, be usco-compact and σ-dd-preserving. Assume that S_i are metric. Put

$$C = \{\langle s_1, s_2 \rangle \mid f_1[s_1] \cap f_2[s_2] \neq \emptyset\} .$$

Then the projections $S_1 \times S_2$ restricted to C are σ-dd-preserving.

R e f e r e n c e s

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