Commentationes Mathematicae Universitatis Carolinae

Jan Čerych

A characterization of $C(X)$

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 2, 379--383

Persistent URL: http://dml.cz/dmlcz/106004

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21,2 (1980)

A CHARACTERIZATION OF C(X)

Jan ČERYCH

Abstract: If A is a function algebra, this note gives a sufficient condition in order that A contains any continuous function.

Key words: C(X), function algebra, pervasive algebra.

Classification: 46J10

Let A be a function algebra on compact Hausdorff topological space X, i.e., a closed subalgebra of the sup-norm algebra C(X) of all continuous complex-valued functions on X.

A is called a pervasive algebra provided it satisfies the following condition:

(1) Whenever F is a proper closed nonvoid subset of X and f in C(F), then, for an arbitrary $e > 0$, there exists a g in A such that

$$|g - f|_F < e.$$

(Here $|h|_H$ for $H \subset X$ and h in C(H) denotes the supremum of the function $|h(x)|$ on the set H.)

The notion "pervasivity" is due to Hoffman and Singer who were also first to investigate properties of such algebras. (See [1].)

C(X) is of course a pervasive algebra. More interesting are its proper pervasive subalgebras. The simplest of them is the classical disc algebra, the set of all uniform limits of polynomials on the unit circle in z-plane, and related algebras. Another way of constructing pervasive algebras was developed by Hoffman and Singer in [1]. A nice example of this type is due to Fuka. In his construction [2] the space X is a discontinuum of zero planar measure.

Our aim in this remark is to show that if the approximation of the type (1) is, moreover, a bounded one, then A contains all continuous complex-valued functions on X.

More specifically, we shall prove the following

<u>Theorem</u>. Let A be a function algebra on X. Suppose that for any closed nonvoid proper subset F of X and for any function f in C(F) there exists a positive constant k(F,f) having the following property:

Whenever e is a positive number, then there exists a g in A satisfying

$$|g - f|_F < e, \ |g| \leqq k(F,f).$$

Then A is equal to C(X).

<u>Proof</u>: Choose an f in C(X) and a positite number e. Our aim is to find an h in A such that

(2) $$|h - f| < e.$$

Without loss of generality we may suppose that $|f| \leqq 1$.

If X consists of one or two points, Theorem is trivial. Suppose now that X contains at least three distinct points. Then there exists a pair P, Q of closed disjoint subsets of

X which have nonvoid interiors, and whose union $P \cup Q$ does not cover all X. Setting

$$M = \overline{X - Q}, \quad N = \overline{X - P},$$

(where the bar denotes the closure in X) we have obtained a pair of proper closed nonvoid subsets of X.
A function j which is equal to 1 in P and to 0 in Q is a well-defined continuous function on $P \cup Q$.

Put

$$k = \max \{k(M,f), k(N,f), k(P \cup Q, j) + 1\}$$

and

$$\check{e} = \frac{e}{3k}.$$

Take, by the hypotheses, a triple of functions m,n,g in A satisfying the conditions

$$|m - f|_M < \check{e}, \quad |m| \leqq k(M,f),$$

$$|n - f|_N < \check{e}, \quad |n| \leqq k(N,f),$$

$$|g - 1|_P < \check{e}, \quad |g|_Q < \check{e}, \quad |g| \leqq k(P \cup Q, j).$$

Finally, setting $h = gm + (1 - g)n$, the h satisfies (2).
Really, it is obvious that

$$P \subset M, \; Q \subset N, \; X = P \cup Q \cup (M \cap N),$$

and

$$|h - f|_P \leqq |gm - f + gf - gf|_P + |1 - g|_P |n| \leqq$$

$$\leqq |m - f|_M |g| + |g - 1|_P |f| + |g - 1|_P |n| <$$

$$< \check{e}.k(P \cup Q, j) + \check{e} + \check{e}.k\,(N,f) \leqq e,$$

$$|h - f|_Q \leqq |(1 - g)n - f + (1 - g)f - (1 - g)f|_Q +$$

$$+ |g|_Q |m| \leqq |n - f|_N |1 - g| + |g|_Q |f| + |g|_Q |m| <$$

$$< \check{e}.[k(P\cup Q,j) + 1] + \check{e} + \check{e}.k(M,f) \leqq e,$$

$$|h - f|_{M\cap N} \leqq |gm - gf + (1 - g)n - (1 - g)f|_{M\cap N} \leqq$$

$$\leqq |m - f|_M |g| + |n - f|_N \, |1 - g| <$$

$$< \check{e}.k(P\cup Q,j) + \check{e}.[k(P\cup Q,j) + 1] \leqq e,$$

and Theorem is then proven.

Remark 1. Notice that we have proven a little more than we had stated: the approximation suggested in the hypothesis of Theorem is adopted, in its proof, for a triple of conveniently chosen sets only.

Remark 2. A little related results are those of Badé and Curtis [3, 4] (cf. also Burckel [5], Chapter VI); they required, roughly speaking, the e in the conditions of Theorem to be only a fixed one, $e < 1/2$, but the k (= k(F,f)) to be an absolute constant independent of the set K (in a strengthened version of theire theorem any point in X has a neighbourhood U and the constant k(U) which dominates g whenever F is a compact subset of U) and came to the same conclusion.

I should like to express here my gratefulness to RNDr. Jaroslav Fuka, CSc., for a series of discourses and fruitful criticism.

R e f e r e n c e s

[1] K. HOFFMAN, I.M. SINGER: Maximal algebras of continuous functions, Acta Math. 103(1960), 217-241.

[2] J. FUKA: Zametka k maksimal'nosti nekotorych algebr funkcij (Russian), Čas. pěst. mat. 93(1968), 346-348.

[3] W. BADÉ, P. CURTIS Jr.: Embedding theorems for commutative Banach algebras, Pacif. J. Math. 18(1966), 391-409.

[4] ———————————— : Banach algebras on F-spaces, Function Algebras, edit. F.T. Birtel, Scott, Foresman and Co., Tulane 1966, 90-92.

[5] R.B. BURCKEL: Characterizations of C(X) among its subalgebras, Lecture Notes 6, Dekker, New York 1972.

Matematicko fyzikální fakulta

Universita Karlova

Sokolovská 83, 18600 Praha 8

Československo

(Oblatum 10.7.1978)