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ON BICOMPACTA IN Σ – PRODUCTS AND RELATED SPACES

N. N. YAKOVLEV

Abstract: In the present article we study the topological properties of bicompacta which are embedded in Σ-products of separable metric spaces. We prove that every Corson bicompactum is hereditarily metalindelöf, while every bicompactum which is embedded in a σ-product of compacta has a closure-preserving covering of compact sets (CPC). We also study the properties of hereditarily metalindelöf bicompacta and of the bicompacta with CPC.

Key words: Bicompacta, Σ-products of spaces, metalindelöf spaces, closure-preserving covering.

Classification: 54D30

In this note we study the Σ-products of metric spaces (the compacta in general). The bicompact subsets of these Σ-products are interesting because every Eberlein bicompactum (weakly bicompact subset of a Banach space) is homeomorphic to some bicompact subset of Σ-products of segments. The aim of this note is to give some exclusively topological, key properties of Corson (and Eberlein) bicompacta, so that the spaces with these properties well enough topologically approximate the properties of bicompact subsets of Σ-products of compacta.

We adopt the terminology of [1]. The word "compactum" will always denote a metrizable bicompact space, while "bicompactum" will denote a Hausdorff bicompact space. We shall denote by I the segment $[0,1]$, by $D = \{0,1\}$ in discrete topology, by N - the natural numbers. In this paper we use also the next notation:

$\Sigma(I,\Gamma) = \{y \in \prod\{I_\alpha : \alpha \in \Gamma\} : |\{\alpha \in \Gamma : y(\alpha) \neq 0\}| < \aleph_1\}$,

$\Sigma_*(I,\Gamma) = \{y \in \prod\{I_\alpha : \alpha \in \Gamma\} : \forall \varepsilon > 0\ |\{\alpha \in \Gamma : y(\alpha) \geq \varepsilon\}| < \aleph_0\}$,

$\sigma(I,\Gamma) = \{y \in \prod\{I_\alpha : \alpha \in \Gamma\} : |\{\alpha \in \Gamma : y(\alpha) \neq 0\}| < \aleph_0\}$.

The topologies of all of these spaces are generated by the product $\prod\{I_\alpha : \alpha \in \Gamma\}$.

It is easy to check that $\Sigma_*(I,\Gamma)$ is the space $c_0(\Gamma)$ in the topology of pointwise convergence on Γ.

It is well-known [3] that the space $\mathcal{X}$ is an Eberlein bicompactum iff it is homeomorphic to some closed subset of $\Sigma_*(I,\Gamma)$. Every bicompactum that is homeomorphic to some closed subset of $\Sigma(I,\Gamma)$ is called a Corson bicompactum [2]. Rosenthal [4] proved that a bicompactum is an Eberlein bicompactum iff it has a σ-point-finite separating family of open F_σ-subsets (where a family $\mathcal{F}$ of subsets is called separating, if given any $x \neq y$ in $\mathcal{X}$, there is an $F \in \mathcal{F}$ such that either $x \in F$ and $y \notin F$, or $y \in F$ and $x \notin F$). As in [5] we say that $\mathcal{X}$ is a strong Eberlein bicompactum iff it is homeomorphic to a bicompact subset of $\Sigma_*(D,\Gamma)$ (which is in fact $\sigma(D,\Gamma)$), or equivalently, $\mathcal{X}$ has a point-finite separating family of closed-open sets [5]. We can also prove (by the method in [6]) that $\mathcal{X}$ is a Corson bicompactum iff it has a point-countable separating family of open F_σ-subsets.

According to [7] a bicompactum $\mathcal{X}$ is called monolithic, if for each cardinal τ, and $A \subseteq \mathcal{X}$ such that $|A| \leq \tau$ it follows that $\omega([A]) \leq \tau$.

A space $\mathcal{X}$ is called metalindelöf (σ-metacompact), if every open covering of $\mathcal{X}$ can be refined by an open point-countable (σ-point-finite) covering.

A family $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is called closure-preserving, if for every $B \subseteq A$ $\cup \{[F_\alpha] : \alpha \in B\} = [\cup \{F_\alpha : \alpha \in B\}]$.

§ 1. Nor $\Sigma(I,\Gamma)$, neither $\Sigma(D,\Gamma)$ are metalindelöf spaces, since they are countably compact, but not bicompact, however,

Theorem 1. Every bicompact subset of $\Sigma(I,\Gamma)$ is hereditarily metalindelöf.

We need the next (see [8])

Lemma 1. Let $\mathcal{B} = \{B\}$ be an uncountable family of subsets of Γ, such that $|B| \leq n$ for some $n \in N$ and all $B \in \mathcal{B}$. Then there is $C \subset \Gamma$ and an uncountable subfamily $\mathcal{B}' \subseteq \mathcal{B}$ such that if $B_1, B_2 \in \mathcal{B}'$ and $B_1 \neq B_2$, then $B_1 \cap B_2 = C$.

Let $\{V_n : n \in N\}$ be a countable base of $I \setminus \{0\}$. $W_n = \{x \in I : x < 1/n\}$. Let $\{V_n^r, n \in N\}$ be a countable base of $I \setminus (W_r \cup \{1/r\})$. Let $k \in N$ and $A = \bigcup_{k=1}^{\infty} N^k$; let $\mathcal{K} = \{1,\dots,k\}$ and $\bar{\Gamma}^k$ be the set of all one-to-one mappings of $\mathcal{K}$ to Γ. Let $\mathcal{B} = \bigcup_{k=1}^{\infty} \bar{\Gamma}^k$. For every $B = \{\gamma_1,\dots,\gamma_k\} \in \mathcal{B}$ and $\bar{n} = \{n_1,\dots,n_k\} \in A$ such that $|B| = |\bar{n}|$, define

$$V(B,\bar{n}) = ((V_{n_1})_{\gamma_1} \times \dots \times (V_{n_k})_{\gamma_k} \times \Pi\{I_\beta : \beta \in \Gamma \setminus B\}) \cap \Sigma(I,\Gamma)$$

$$V(B,\bar{n},r) = ((V^r_{n_1})_{\gamma_1} \times \dots \times (V^r_{n_k})_{\gamma_k} \times \Pi\{I_\beta : \beta \in \Gamma \setminus B\}) \cap \Sigma_*(I,\Gamma)$$

$$W(B,m) = (W_m)_{\gamma_1} \times \dots \times (W_m)_{\gamma_k} \times \Pi\{I_\beta : \beta \in \Gamma \setminus B\}.$$

Proof of the theorem 1: I. The family $\{V(B,\bar{n}), B \in \mathcal{B}, \bar{n} \in A : |B| = |\bar{n}|\}$ is point-countable, for if $x \in V(B,\bar{n})$, then $B \subset \Gamma(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0$, while both $\Gamma(x)$ and A are countable sets.

II. Let $\{\mathcal{U}_\gamma\}$ be an arbitrary open family in bicompact $\mathfrak{X}$. Let us index the points of $\cup\mathcal{U}_\gamma$ as $\{y_\beta : \beta < \tau\}$, where τ is the first ordinal with the same cardinality as $\cup\mathcal{U}_\gamma$. Let $x_1 = y_1$. There exists an index γ_1 and an elementary neighbourhood of x_1 $V'(B_1,\bar{n}_1) = V(B_1,\bar{n}_1)\cap W(B_1',m_1)\cap \cap\mathfrak{X}$ such that $B_1'\cap\Gamma(x) = \emptyset$, $B_1\subseteq\Gamma(x)$ and $V'(B_1,\bar{n}_1) \subseteq \subseteq \mathcal{U}_{\gamma_1}$.

Suppose that for every $\nu < \mu < \tau$ we have defined the sequence of indexes $\{\gamma_\nu\}$, points $\{x_o\}$ and neighbourhoods in $\mathfrak{X}$ $\{V'(B_\nu,\bar{n}_\nu)\}$ such that

a) $x_\nu \in V'(B_\nu,\bar{n}_\nu) \subseteq \mathcal{U}_{\gamma_\nu}$

b) $x_\nu \in \cup\mathcal{U}_\gamma \setminus \bigcup_{\alpha<\nu} V'(B_\alpha,\bar{n}_\alpha)$ and x_ν is the first point with this property

c) $V'(B_\nu,\bar{n}_\nu) = V(B_\nu,\bar{n}_\nu)\cap W(B_\nu'\ \bar{n}_\nu)\cap\mathfrak{X}$ and $B_\nu \subset \Gamma(x_\nu)$ but $B_\nu'\cap\Gamma(x_\nu) = \emptyset$.

Let us consider $\bigcup_{\nu<\mu} V'(B_\nu,\bar{n}_\nu)$. If $\bigcup_{\nu<\mu} V'(B_\nu,\bar{n}_\nu) = \cup\mathcal{U}_\gamma$, then put $x_\mu = \emptyset$, $V'(B_\mu,\bar{n}_\mu) = \emptyset$. But if $P = \cup\mathcal{U}_\gamma \setminus \bigcup_{\nu<\mu} V_\nu'(B_\nu,\bar{n}_\nu) \neq \neq \emptyset$, then let x_μ be the first point of P. Now there exists an index γ_μ and the neighbourhood of the point x_μ : $:V'(B_\mu,\bar{n}_\mu) = V(B_\mu,\bar{n}_\mu)\cap W(B_\mu',m_\mu)\cap\mathfrak{X}$ such that $B_\mu \subseteq \subseteq\Gamma(x_\mu)$, $B_\mu'\cap\Gamma(x_\mu) = \emptyset$ and $V'(B_\mu,\bar{n}_\mu) \subset \mathcal{U}_{\gamma_\mu}$.

Obviously, the conditions a) - c) are satisfied. We shall prove now that $\bigcup_{\mu<\tau} V'(B_\mu,\bar{n}_\mu) = \cup\mathcal{U}_\gamma$. Let $y \in \cup\mathcal{U}_\gamma$, then $y = y_{\mu_0}$, for some μ_0 and $x_{\mu_0} = y_\mu$ for some μ . It is clear that $\mu_0 \leq \mu$. But if $y_{\mu_0} \in \cup\mathcal{U}_\gamma \setminus \bigcup_{\mu<\mu_0} V'(B_\mu,\bar{n}_\mu)$, then $\mu_0 \geq \mu$, thus $y_{\mu_0} = y_\mu = x_{\mu_0}$, and $y_{\mu_0} \in$

$\in \bigcup_{\mu \leq \mu_0} V'(B_\mu, \bar{n}_\mu) \subseteq \cup \mathcal{U}_\gamma$.

III. Let us prove that for every $\mu_0 < \tau$ there exist only countably many different $\nu < \tau$ such that $V(B_\nu, \bar{n}_{\nu_0}) = V(B_{\mu_0}, \bar{n}_{\mu_0})$. On the contrary, suppose there exist E: $|E| = \aleph_1$ and $V(B_\mu, \bar{n}_\mu) = V(B_\nu, \bar{n}_\nu)$ for every $\mu, \nu \in E$. Then $B_\mu = B_\nu = B^0$, $\bar{n}_\mu = \bar{n}_\nu = \bar{n}^0$. $V'(B_\mu, \bar{n}_\mu) \subseteq V(B_\mu, \bar{n}_\mu)$ and $V'(B_\mu, \bar{n}_\mu) \neq V'(B_\nu, \bar{n}_\nu)$ according to the construction. We may consider the case $|B'_\mu| = |B'_\nu|$, $m_\nu = m_\mu = m_0$ and $B'_\mu \neq B'_\nu$ because of the uncountability of E. Then there exist $C \subset \Gamma$ and uncountable $E' \subset E$ such that $B'_\mu = C \cup B''_\mu$ for each $\mu \in E'$ and $B''_\mu \cap B''_\nu = \emptyset$ for every $\mu \neq \nu$ and $\mu, \nu \in E'$ (Lemma 1) and we may consider $\{\mu : \mu \in E' \subset \tau\}$ simply isomorphic to ω_1. As $\mathcal{X}$ is a bicompactum and $x_\mu \in \mathcal{X}$ for every $\mu \in E$, then there exists $y \in \mathcal{X}$ - a complete accumulation point of the set $\cup\{x_\mu : \mu \in E'\}$. $|\Gamma(y)| \leq \aleph_0$. The family $\{B''_\mu : \mu \in E'\}$ is disjoint. That is why there exists $\mu_0 \in E'$ such that for every $\mu \geq \mu_0$ $B''_\mu \cap \Gamma(y) = \emptyset$. According to a) $x_\mu \in V'(B_\mu, \bar{n}_\mu) = V(B^0, \bar{n}^0) \cap W(C, m_0) \cap W(B''_\mu, m_0) \cap \mathcal{X}$ and according to b) $x_\mu \notin V'(B_{\mu_0}, \bar{n}_0) = V(B^0, \bar{n}^0) \cap W(C, m_0) \cap W(B''_{\mu_0}, m_0) \cap \mathcal{X}$ for all $\mu > \mu_0$. It follows that for all $\mu > \mu_0$ $x_\mu \notin W(B''_{\mu_0}, m_0)$ but $W(B''_{\mu_0}, m_0) \ni y$, because $\Gamma(y) \cap B''_\mu = \emptyset$. This contradicts the conception of the complete accumulation point.

IV. Thus the family $\{V'(B_\mu, \bar{n}_\mu) : \mu \in \tau$ is point-countable, for if $x \in V'(B_\mu, \bar{n}_\mu)$, $\mu \in E$ and E is uncountable, then the set of distinct $V(B_\mu, \bar{n}_\mu)$ which contain x is also uncountable, because of III. and the fact that

$V'(B_\mu, \bar{n}_\mu) \subset V(B_\mu, \bar{n}_\mu)$. That is impossible, according to I. $\{V'(B_\mu, \bar{n}_\mu): \mu < \tau\}$ refines $\{\mathcal{U}_\gamma\}$ according to the condition a). Theorem 1 is proved.

In the case of Σ_*-products the situation is simpler:

Theorem 2. $\Sigma_*(I,\Gamma)$ is hereditarily σ-metacompact.

Proof: Let $\omega(\bar{n},r) = \{V(B,\bar{n},r): B \in \mathcal{B}$ and $|B| = |\bar{n}|\}$ $\omega = \bigcup\{\omega(\bar{n},r): \bar{n} \in A = \bigcup_{k=1}^{\infty} N^k,\ r \in N\}$.

I. The family $\omega(\bar{n},r)$ is point-finite, since $x \in V(B,\bar{n},r)$ then $B \subset \Gamma(x,r) = \{\gamma : x(\gamma) \geq 1/r\}$, and $|\Gamma(x,r)| < \aleph_0$.

II. Let $\{\mathcal{U}_\gamma\}$ be an arbitrary family of open sets in $\Sigma_*(I,\Gamma)$. Index the points of $\cup\mathcal{U}_\gamma$ as $\{y_\beta : \beta < \tau\}$, where τ is the first ordinal with the same cardinality as $\cup\mathcal{U}_\gamma$. Let $x_0 = \emptyset$, $V'_0 = \emptyset$, $\gamma_0 = 0$. By a transfinite induction we shall define the sequences of indexes $\{\gamma_\mu : \mu < \tau\}$, points $\{x_\mu : \mu < \tau\}$ and neighbourhoods $\{V'(\Gamma(x_\mu, k_\mu), \bar{n}_\mu, k_\mu + 1): \mu < \tau\}$. Suppose that for all $\nu < \mu < \tau$ we have constructed such sequences with the following conditions:

a) $x_\nu \in V'(\Gamma(x_\nu,k_\nu),\bar{n}_\nu,k_\nu + 1) \subseteq \mathcal{U}_{\gamma_\nu}$

b) $x_\nu \in \cup\mathcal{U}_\gamma \setminus \bigcup_{\alpha<\nu} V'(\Gamma(x_\alpha,k_\alpha),\bar{n}_\alpha,k_\alpha + 1)$ and x_ν is the first point with this property

c) $V'(\Gamma(x_\nu,k_\nu),\bar{n}_\nu,k_\nu + 1) = V(\Gamma(x_\nu,k_\nu),\bar{n}_\nu,k_\nu + 1) \cap W(B'_\nu,k_\nu)$ and $B'_\nu \cap \Gamma(x_\nu) = \emptyset$.

Let us consider $P = \bigcup_{\nu<\mu} V'(\Gamma(x_\nu,k_\nu),\bar{n}_\nu,k_\nu + 1)$. If $\cup\mathcal{U}_\gamma = P$, then $\gamma_\mu = 0$, $x_\mu = \emptyset$, $V'_\mu = \emptyset$. Otherwise, let x_μ be the first point of $\cup\mathcal{U}_\gamma \setminus P$. Then there exists an index γ_μ and the neighbourhood of the point $x_\mu : V(B,\bar{n},r) \cap W(B',m) \subset$

$\subset \mathcal{U}_{\gamma_\mu}$ such that $B \subset \Gamma(x_\mu)$, $B' \cap \Gamma(x_\mu) = \emptyset$. Let $k_\mu = \max\{r,m\}$, $B'_\mu = B'$. Then $\Gamma(x_\mu,k_\mu) \supset B$, therefore we may find $\bar{n} = \bar{n}_\mu$ such that $V'(\Gamma(x_\mu,k_\mu),\bar{n}_\mu,k_\mu + 1) = V(\Gamma(x_\mu,k_\mu),\bar{n}_\mu,k_\mu + 1) \cap W(B'_\mu,k_\mu) \subset \mathcal{U}_{\gamma_\mu}$. The conditions a) - c) are obviously satisfied, $\bigcup_{\mu<\tau} V'(\Gamma(x_\mu,k_\mu),\bar{n}_\mu,k_\mu + 1) = \bigcup \mathcal{U}_\gamma$. It may be checked as in the proof of Theorem 1.

III. Let us prove that if $\mu \neq \nu$, then $V_\mu = V(\Gamma(x_\mu,k_\mu),\bar{n}_\mu,k_\mu + 1) \neq V(\Gamma(x_\nu,k_\nu),\bar{n}_\nu,k_\nu + 1) = V_\nu$. Let $\mu > \nu$ and $V_\mu = V_\nu$, then $\Gamma(x_\nu,k_\nu) = \Gamma(x_\mu,k_\mu)$; $\bar{n}_\nu = \bar{n}_\mu$; $k_\nu = k_\mu$. According to a) $x_\mu \in V_\mu \cap W(B'_\mu,k_\mu)$ and according to b) $x_\mu \notin V_\nu \cap W(B'_\nu,k_\nu)$, because of $\mu > \nu$. It follows that $x_\mu \notin W(B'_\nu,k_\nu) = W(B'_\mu,k_\mu)$ and there exists $\gamma \in B'_\nu$ such that $x_\mu(\gamma) \geq 1/k_\mu$, but then $\gamma \in \Gamma(x_\mu,k_\mu) = \Gamma(x_\nu,k_\nu)$, i.e. $B'_\nu \cap \Gamma(x_\nu) \neq \emptyset$ and this contradicts c).

IV. Thus the family $\vartheta = \{V'(\Gamma(x_\mu,k_\mu),\bar{n}_\mu,k_\mu + 1): \mu < \tau\}$ indexly refines $\{V(\Gamma(x_\mu,k_\mu),\bar{n}_\mu,k_\mu + 1): \mu < \tau\}$ and the last family is σ-point finite (part I). According to a) ϑ refines $\{\mathcal{U}_\gamma\}$.

Remark 1. Theorem 1 and Theorem 2 are true for the corresponding Σ-products of the arbitrary separable metric spaces. The proof is the same.

Theorem 3. a) $\sigma(I,\Gamma)$ has the closure-preserving covering of compact sets;

b) let S_γ be the closed sequence, converging to zero, then $\sigma(S_\gamma,\Gamma)$ in addition, has the σ-closure-preserving covering of finite sets;

c) $\sigma(D,\Gamma)$ has a closure-preserving covering of finite sets.

Proof: a) Let $B \in \mathcal{B}$ and $B = \{\gamma_1,\dots,\gamma_k\}$, then $K(B) = (I)_{\gamma_1} \times \dots \times (I)_{\gamma_k} \times \prod_{\alpha \notin \beta} \{0\}_\alpha \subset \sigma(I,\Gamma)$. Let $\mathcal{K} = \{K(B): B \in \mathcal{B}\}$. Obviously, $\mathcal{K}$ is a covering of $\sigma(I,\Gamma)$. We shall prove now that $\mathcal{K}$ is closure-preserving. Let $x_i \in K(B_i)$ and $x_i \longrightarrow x_0$. If $x_0 \notin \bigcup_{i=1}^{\infty} K(B_i)$ then $x_0 \notin K(B_i)$ for all $i \in N$, that is why there exists $\alpha_i \in \Gamma(x_0)$ such that $\alpha_i \notin B_i$, but $\Gamma(x_0)$ is finite, therefore there are infinitely many different $i(n)$ and also there exists $\alpha_0 \in \Gamma(x_0)$ such that $\alpha_{i(n)} = \alpha_0$, but then $x_{i(n)}(\alpha_0) = 0$(because $\alpha_0 = \alpha_{i(n)} \notin B_{i(n)}$), and $x_0(\alpha_0) \in \Gamma(x_0)$, but this means that $x_0(\alpha_0) \neq 0$, the last is a contradiction, because $x_i \longrightarrow x_0$.

b) and c) may be proved similarly. In the case b) $\mathcal{K} = \bigcup\{\mathcal{K}_n : n \in N\}$, where $\mathcal{K}_n = \{K(B,n), B \in \mathcal{B}\}$ and $K(B,n) = (S_{\gamma_1} \setminus W_n)_{\gamma_1} \times \dots \times (S_{\gamma_k} \setminus W_n)_{\gamma_k} \times \prod_{\alpha \notin B} \{0\}_\alpha \cup \bar{0}$, where $\bar{0} = \{0_\alpha\}_{\alpha \in \Gamma}$.
Obviously $K(B,n)$ is a finite set.

Remark 2. Theorem 3 a) is true also in the case of σ-products of arbitrary compacta. The proof is the same.

Corollary 1. $\sigma(D,\Gamma)$ is hereditarily metacompact.

It follows from a theorem in [9] that each space with the closure-preserving covering of compact sets (we shall denote this as CPC) is metacompact.

Corollary 2. a) Every Corson bicompactum is hereditarily metalindelöf;

b) every Eberlein bicompactum is hereditarily σ-metacompact;

c) every strong Eberlein bicompactum has the closure-preserving covering of finite sets, is hereditarily metacompact and scattered;

d) every Eberlein bicompactum which is embedded in $\sigma(I,\Gamma)$ has CPC.

The scattering in c) was proved in [5].

Remark 3. Independently of the author, E.G. Pytkeev proved that every Eberlein bicompactum is hereditarily metalindelöf.

It is impossible to receive metacompactness in the theorems 1 and 2. It follows from

Theorem 4. There exists a zero-dimensional Eberlein bicompactum which is not hereditarily metacompact.

Construction: Let τ be the regular cardinal, $\tau > \aleph_0$. Let $\Gamma_n = \Gamma_m$ and $|\Gamma_n| = \tau$. Let $T = \cup\{\Gamma_n : n \in N\}$; $S = \{0, 1, \ldots, {}^1/n, \ldots\}$, $D_n = \{0, {}^1/n\}$. Let us denote by $\mathcal{X}$ the product $\prod_{m\in N}\prod_{\alpha\in\Gamma_m}(D_n)_\alpha$ (it is easy to check that $\mathcal{X}$ is homeomorphic to $\prod_{\alpha\in\Gamma} D_\alpha$). Let $F = \{x \in \mathcal{X} :$ for every $n \in N$ $|\{\gamma \in \Gamma_n : x(\gamma) \neq 0\}| \leq 1\}$. F is closed in $\mathcal{X}$. Really, if $x \notin F$, then there exists n_0 such that $|\{\gamma \in \Gamma_{n_0} : x(\gamma) \neq 0\}| > 1$ i.e. there exist $\gamma_1, \gamma_2 \in \Gamma_{n_0} : x(\gamma_1) = {}^1/n_0$ and $x(\gamma_2) = {}^1/n_0$ $(\gamma_1 \neq \gamma_2)$. Now $W(x) = ({}^1/n_0)_{\gamma_1} \times ({}^1/n_0)_{\gamma_2} \times \prod_{\alpha\notin\{\gamma_1,\gamma_2\}} D_\alpha$ is an open neighbourhood of x and $W(x) \cap F = \emptyset$. This implies that F is a bicompactum. Obviously, $F \subset \Sigma(D,T)$, but $F \subset \Sigma_*(S,T)$, too, and therefore F is a zero-dimensional Eberlein bicompactum.

F is not hereditarily metacompact. For every $n \in N$ and $\gamma \in \Gamma_n$ let x_γ be a point of F such that $\Gamma(x_\gamma) = \gamma$, $x_\gamma(\gamma) = {}^1/n$. Let $V_\gamma = (({}^1/n)_\gamma \times \prod_{\alpha \neq \gamma} D_\alpha) \cap F$, let $\Phi = \cup \{x_\gamma : \gamma \in T\}$ and $\vartheta(\Phi) = \{V_\gamma : \gamma \in T\}$. $\vartheta(\Phi)$ is the family of open sets in F. Suppose, $\mathcal{W}$ refines $\vartheta(\Phi)$ and has the same "body" (it means $\cup \mathcal{W} = \cup \vartheta(\Phi)$). We may assume that $\mathcal{W} \supset \{W_\gamma : \gamma \in T\}$, where $W_\gamma = V_\gamma \cap W(B(\gamma), m_\gamma)$ and $B(\gamma) \cap \{\gamma\} = \emptyset$ and $B(\gamma)$ is a finite subset of T.

<u>Lemma 2</u>. If $k \in N$, $\Gamma \subset \Gamma_k$ and $|\Gamma| = \tau$; $\Gamma_n' \subset \Gamma_n$ and $|\Gamma_n'| = \tau$, then there exists $\gamma_0 \in \Gamma$ and a sequence $\{F_n : n \geq k+1\}$ such that $F_n \subset \Gamma_n'$, $|F_n| = \tau$ and for all $\gamma \in \cup \{F_n : n \in N\}$ $B(\gamma) \not\ni \gamma_0$.

<u>Proof:</u> For every $\gamma_0 \in \Gamma$ and each $n \geq k+1$ let $F_n(\gamma_0) = \{\gamma \in \Gamma_n' : B(\gamma) \not\ni \gamma_0\}$. Suppose that for every $\gamma_0 \in \Gamma$ there exists $n_0(\gamma_0) \geq k+1$ such that $|F_{n_0(\gamma_0)}(\gamma_0)| < \tau$. As $|\Gamma| = \tau > \aleph_0$, then we may find $\Gamma' \subset \Gamma$ and n_0 such that for all $\gamma_0 \in \Gamma'$ $|F_{n_0}(\gamma_0)| < \tau$. Let $\{\gamma_n : n \in N\}$ be the sequence of distinct indexes of Γ'. Then $\Gamma_{n_0}' \setminus \bigcup_{m=1}^{\infty} F_{n_0}(\gamma_n) \neq \emptyset$, because $|\bigcup_{m=1}^{\infty} F_{n_0}(\gamma_n)| < \tau$. Let $\beta \in \Gamma_{n_0}' \setminus \bigcup_{m=1}^{\infty} F_{n_0}(\gamma_n)$, then $\beta \notin \bigcup_{m=1}^{\infty} F_{n_0}(\gamma_n)$ and $\beta \in \Gamma_{n_0}'$, therefore $B(\beta) \ni \gamma_n$ for all $n \in N$, that is impossible, because $B(\beta)$ is finite.

<u>Lemma 3</u>. There exists a sequence of distinct indexes $\{\gamma_n : n \in N\}$ such that $\gamma_n \in \Gamma_n$ and $B(\gamma_n) \not\ni \gamma_m$ if $n \neq m$.

<u>Proof</u>: Let $k = 1$, $\Gamma = \Gamma_1$, $\Gamma_n' = \Gamma_n$ $(n \geq 2)$. Then according to Lemma 2, there exists $\gamma_1 \in \Gamma_1$ and a sequence $\{F_n^1, n \geq 2\}$ such that $F_n^1 \subset \Gamma_n'$, $|F_n^1| = \tau$ and $B(\gamma) \not\ni \gamma_1$ for every $\gamma \in \cup \{F_n^1, n \geq 2\}$. Let $k = n_0 - 1$ and we have alre-

ady defined $\{\gamma_k\}_{k<n_o}$; $\{F_n^{n_o-1}, n \geq n_o\}$ such that $F_n^{n_o-1} \subset \Gamma_n$, $|F_n^{n_o-1}| = \tau$; $B(\gamma_i) \not\ni \gamma_j$, $i,j < n_o$ $(i \neq j)$ and for every $\gamma \in \bigcup\{F_n^{n_o-1}, n \geq n_o\}$ and each $k < n_o$ $B(\gamma) \not\ni \gamma_k$. Then according to Lemma 2, if $k = n_o$ $\Gamma = F_{n_o}^{n_o-1} \setminus \bigcup_{i=1}^{n_o-1} B(\gamma_i)$ and $\Gamma_n' = F_n^{n_o-1} \setminus \bigcup_{i=1}^{n_o-1} B(\gamma_i)$ there exists $\gamma_{n_o} \in \Gamma$ and a sequence $\{F_n^{n_o}, n \geq n_o + 1\}$ such that $|F_n^{n_o}| = \tau$ and $B(\gamma) \not\ni \gamma_o$ for every $\gamma \in \bigcup_{n \geq n_o+1} F_n^{n_o}$. Obviously, $B(\gamma_{n_o}) \not\ni \gamma_k$ for each $k < n_o$ (as $\gamma_{n_o} \in F_{n_o}^{n_o-1}$). $B(\gamma_k) \not\ni \gamma_{n_o}$ by a definition of Γ. Therefore, by the induction, we receive the required sequence.

With the help of Lemma 3 it is easy to show that the family $\mathcal{W}$ is at least point-countable.

Let $y \in F$ be a point such that if $\gamma \neq \gamma_n$, then $y(\gamma) = 0$ and if $\gamma = \gamma_n$, then $y(\gamma_n) = {}^1/n$. Now $y \in W_{\gamma_n}$, because $y(\gamma_n) = \frac{1}{n}$ and for every $\gamma \in B(\gamma_n)$ $y(\gamma) = 0$ (because $B(\gamma_n) \not\ni \gamma_m$ if $n \neq m$). Therefore F is not hereditarily metacompact.

§ 2. The hereditary properties, arised in the theorems 1 - 3 are responsible for many other well-known topological properties of bicompacta, contained in Σ-products and sometimes, we are able to specify some of them.

In our consideration we shall denote the hereditarily metalindelöf bicompactum as HM-bicompactum.

Definition. Name a space $\mathcal{X}$ a super-Fréchet space if for every $\mathcal{Y} \subseteq \mathcal{X}$ and $x_o \in [\mathcal{Y}]$, whenever $\Psi(x_o, \mathcal{Y}) = \lambda$, then always there exists a discrete in itself set $A \subseteq \mathcal{Y}$, such that $|A| = \lambda$ and $[A] \setminus A = \{x_0\}$.

Obviously, a super-Fréchet space is a Fréchet-Uryson space.

Theorem 5. Every HM-bicompactum is a super-Fréchet space.

Corollary 3. If $\mathcal{X}$ is a Corson bicompactum, and x_0 is a G_λ-point in $\mathcal{X}$, then there exists an Alexandrov super-sequence, converging to x_0, the length of which is λ.

Theorem 6. Every HM-bicompactum has a dense set of G_δ-points.

Theorems, similar to those of 5 and 6, are true for the hereditary (and not only) properties, more general than HM. We drop the proof of all these facts, because of another direction of our note; they will appear in an article written by the author and E.G. Pytkeev (see this issue).

Theorem 7. Let $\mathcal{X}$ be a scattered bicompactum, then

a) $\mathcal{X}$ is HM iff $\mathcal{X}$ is a Corson bicompactum

b) $\mathcal{X}$ is hereditarily σ-metacompact iff $\mathcal{X}$ is an Eberlein bicompactum.

Every strong Eberlein bicompactum is already scattered, so we have

Theorem 8. The next conditions are equivalent:

a) $\mathcal{X}$ is a strong Eberlein bicompactum,

b) $\mathcal{X}$ is a bicompactum with the closure-preserving covering of finite sets,

c) $\mathcal{X}$ is scattered and hereditarily metacompact.

Let $\mathcal{X}$ be a scattered bicompactum, $\mathcal{X}_1 \subseteq \mathcal{X}$ - a set of all isolated points of $\mathcal{X}$. $\mathcal{X}_\alpha$ - a set of all isolated points of $\mathcal{X} \setminus \cup\{\mathcal{X}_\beta : \beta < \alpha\}$. Then we shall call an ordinal α the index of scattering of $\mathcal{X}$ (is $(\mathcal{X})$) if α is the first ordinal such that $\mathcal{X}_{\alpha+1} = \emptyset$.
Obviously, $\mathcal{X} = \cup\{\mathcal{X}_\beta : \beta \leq \text{is }(\mathcal{X})\}$. $\mathcal{X}_\beta$ is dense in $\cup\{\mathcal{X}_\gamma : \gamma \geq \beta\}$ and $\mathcal{X}_\alpha$ is finite for $\alpha = \text{is }(\mathcal{X})$.

The proof of the theorems 7 and 8 may be done by the same method. Let us prove, for example, Theorem 8.

a) $\Longrightarrow$ b) This is Theorem 3 a). b) $\Longrightarrow$ c) It follows from [9] (while the scattering follows from the fact that $\mathcal{X} = \cup\{F_n : n \in N\}$, where F_n is a scattered bicompactum).

c) $\Longrightarrow$ a) We use the induction. If is $(\mathcal{X}) = 1$, then $\mathcal{X}$ is finite. Let is $(\mathcal{X}) = \beta$ and for every hereditarily metacompact bicompactum $\mathcal{Y}$ such that is $(\mathcal{Y}) < \beta$ it is proved that $\mathcal{Y}$ is a strong Eberlein bicompactum. For every $y \in \mathcal{X}$ there exists $\alpha \leq \beta : y \in \mathcal{X}_\alpha$. Let $O(y)$ be a closed-open bicompact neighbourhood of y such that $O(y) \cap \cup\{\mathcal{X}_\gamma : \gamma \geq \alpha\} = \{y\}$. $\mathcal{X}_\beta$ is finite, $\mathcal{X} \setminus \mathcal{X}_\beta$ is open. Let $\vartheta = \{V\}$ be a point-finite closed-open refining of $\{O(y) : y \in \mathcal{X} \setminus \mathcal{X}_\beta\}$. For every V, $V \subseteq O(y)$ (for some y), therefore is $(V) < \beta$. V is an open bicompactum in $\mathcal{X}$. Let F(V) be a point-finite separating family of closed-open sets in V. Then $\mathcal{F} = \cup\{F(V) : V \in \vartheta\} \cup \{O(y) : y \in \mathcal{X}_\beta\}$ is a point-finite separating family of closed-open sets in $\mathcal{X}$. The theorem is proved.

<u>Remark 4</u>. P. Simon [5] posed a question: is every scattered Eberlein bicompactum a strong Eberlein bicompactum? It is claimed in [12] that every Corson scattered bicompactum is strong Eberlein. If it is so, then all of the conditions

of Theorems 7 and 8 are equivalent.

Theorem 9. For every ordinal $\alpha > 1$ and cardinal $\tau \geq \aleph_0$ such that $\tau \geq |\alpha|$, there exists a strong Eberlein bicompactum $\mathcal{X} = \mathcal{X}(\alpha, \tau)$ such that $w(\mathcal{X}) = \tau$ and is $(\mathcal{X}) = \alpha$.

Proof: Let $\alpha = 2$ and $\tau \geq \aleph_0$. Let A be a set of power τ with the discrete topology on it. Then $\mathcal{X} = \mathcal{X}(2, \tau) = A \cup \{\Theta\}$ is a one-point bicompactification of A. $w(\mathcal{X}) = \tau$ is $(\mathcal{X}) = 2$. Suppose that for every $\beta < \alpha$ and $\tau : \tau \geq |\beta|$ we have constructed the strong Eberlein bicompacta $\mathcal{X}(\beta, \tau)$ with the necessary properties. Let $\tau \geq |\alpha|$ and $\tilde{\mathcal{X}}_\alpha = \sum_{\beta < \alpha} \mathcal{X}(\beta, \tau)$ be a free union of the bicompacta $\mathcal{X}(\beta, \tau)$. Put $\mathcal{X}(\alpha, \tau) = (\sum_{m=1}^{\infty} \tilde{\mathcal{X}}_{\alpha m}) \cup \{\Theta\}$ a one-point bicompactification of a locally bicompact space $\sum_{n=1}^{\infty} \tilde{\mathcal{X}}_{\alpha_n}$. It is easy to see that $\mathcal{X}(\alpha, \tau) = \mathcal{X}$ is a strong Eberlein bicompactum, $w(\mathcal{X}) = \tau$ and is $(\mathcal{X}) = \alpha$.

§ 3. HM-bicompacta and bicompacta in Σ-products have many common properties, but not all. The reason seems to be in the absence of the "monolithness": there exists even hereditarily Lindelöf, separable, but not metrizable bicompactum [11].

On the other hand, every bicompactum admitting CPC is obviously monolithic. Eberlein bicompacta often admit CPC (Theorem 3, Theorem 8) (but not always, as it will be seen later). In this connection let us point also

Theorem 10. Every scattered Corson bicompactum admits a closure-preserving covering of countable compacta.

The sketch of the proof: Using the induction upon the index of scattering and Theorem 7 a), we may prove the existence of a point-countable separating family $\mathcal{F}$ of open-closed sets in $\mathcal{X} = \cup\{\mathcal{X}_\alpha : \alpha \leq \beta\}$ such that for each $y \in \mathcal{X}_\alpha$ there is $F_y \in \mathcal{F} : F_y \cap (\cup\{x_\gamma : \gamma \geq \alpha\}) = \{y\}$. Now, for each y, if $A_1 = \{y\}$, $A_n = \{z : F_z \cap A_{n-1} \neq \emptyset\}$ then $K_y = \cup\{A_n : n \in M\} \cup \cup \mathcal{X}_\beta$ is a countable compactum. A family $\{K_y : y \in \mathcal{X}\}$ is closure-preserving.

We see that the bicompacta admitting CPC deserve a special investigation.

Let $\mathcal{F} = \{F\}$ be a closure-preserving family of compacta in a space $\mathcal{X}$. Let $\{\mathcal{F}_\alpha\}$ be a family of maximal centred subsystems of $\mathcal{F}$. Then for every $\mathcal{F}_\alpha$: $\Phi_\alpha = \cap\{F : F \in \mathcal{F}_\alpha\} \neq \emptyset$, and Φ_α is a compactum. Obviously, if $\alpha \neq \beta$, then $\Phi_\alpha \cap \Phi_\beta = \emptyset$. A family $\{\Phi_\alpha\}$ is discrete in $\mathcal{X}$, since if $x \in \mathcal{X}$, then $V_x = \mathcal{X} \setminus \cup\{F \in \mathcal{F} : F \not\ni x\}$ is an open neighbourhood of x, intersecting at most one Φ_α (only in the case $x \in \Phi_\alpha$). If $\mathcal{X}$ is a bicompactum, then the system $\{\Phi_\alpha\}$ is finite and $\Phi = \cup \Phi_\alpha$ is compact. We shall call the set $\Phi = \cup \Phi_\alpha$ a maximal set for the family $\mathcal{F}$.

Lemma 4. Let $\mathcal{X}$ be a bicompactum, $w(\mathcal{X}) = \tau$ and $\aleph_0 < \lambda \leq \tau$. A family $\mathcal{F}$ is a CPC on $\mathcal{X}$. Then there are a bicompactum $F \subset \mathcal{X}$ and a compactum $M \subset F$ such that

1. $V = \mathrm{Int}_{\mathcal{X}} F$ cannot be covered by the subfamily $\mathcal{F}' \subset \mathcal{F}$ such that $|\mathcal{F}'| < \lambda$.

2. For every open O such that $O \supset M$ $\mathrm{Int}_{\mathcal{X}}(F \setminus O)$ can be covered by the subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \lambda$.

Proof: I. Let $F_1 = \mathcal{X}$, $\Phi_1 = \Phi$ - a maximal set for $\mathcal{F}$ in F_1 $V_1 = F_1$, $O_0 = \emptyset$. Assume that for every $n < k$ we have already defined the sequences $\{F_n\}$, $\{\Phi_n\}$, $\{V_n\}$ and $\{O_n\}$ $(n< k)$ such that

a) $F_n = \mathcal{X} \setminus \bigcup_{i<n} O_i$; $V_n = \mathrm{Int}_{\mathcal{X}} F_n$; Φ_n is a maximal set for $\mathcal{F}_n = \{B\cap F_n : B \in \mathcal{F}\}$, $O_{n-1} \supset \Phi_{n-1}$ and is open in $\mathcal{X}$

b) V_n cannot be covered by the subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \lambda$.

Consider $F_{k-1} = \mathcal{X} \setminus \bigcup_{i<k-1} O_i$, $\Phi_{k-1} \subset F_{k-1}$, $V_{k-1} \neq \emptyset$. If there is an open neighbourhood $O(\Phi_{k-1})$ such that $V_k = \mathrm{Int}_{\mathcal{X}} (F_{k-1} \setminus O(\Phi_{k-1})) \neq \emptyset$ and V_k cannot be covered by a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \lambda$, then put $O_{k-1} = O(\Phi_{k-1})$, $F_k = F_{k-1} \setminus O_{k-1} = \mathcal{X} \setminus \bigcup_{i<k} O_i$ and $V_k = \mathrm{Int}_{\mathcal{X}} F_k$. But if for every neighbourhood $O(\Phi_{k-1})$ $\mathrm{Int}_{\mathcal{X}} (F_{k-1} \setminus O(\Phi_{k-1}))$ can be covered by a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \lambda$, then put $F_k = V_k = \Phi_k = \emptyset$, $O_{k-1} = \emptyset$.

II. There exists such a natural $k>0$ that $F_k = V_k = \Phi_k = O_{k-1} = \emptyset$. On the contrary, suppose for every natural k $F_k \neq \emptyset$, $O_{k-1} \neq \emptyset$. Let $x_k \in \Phi_k$. Then $\{x_k\}_{k\in N}$ is a discrete set in $\mathcal{X}$. Really, if $x \notin \bigcup_{k=1}^{\infty} O_k$, then $V_x = \mathcal{X} \setminus \bigcup \{F \in \mathcal{F} : F \not\ni x\}$ is open and $V_x \cap \Phi_k = \emptyset$ because if $x \notin \Phi_\alpha \subset \Phi_k$, then there is $F \in \mathcal{F}_\alpha$ such that $F \not\ni x$. But if $x \in \bigcup_{k=1}^{\infty} O_k$, then there is the first natural k_0 such that $x \in O_{k_0}$, but according to a) $O_{k_0} \cap (\bigcup_{k\geq k_0} F_k) = \emptyset$, so $O_{k_0} \cap (\bigcup \{x_k : k \geq k_0\}) = \emptyset$. Thus $\{x_k\}_{k=1}^{\infty}$ is discrete in a bicompactum; a contradiction.

III. Let k be the least natural number such that $F_k = \emptyset$, then $F_{k-1} \neq \emptyset$, $\Phi_{k-1} \subseteq F_{k-1}$ and $V_{k-1} = \mathrm{Int}_{\mathcal{X}} F_{k-1}$ cannot be covered by a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \lambda$, while for every open $O(\Phi_{k-1})$ $\mathrm{Int}_{\mathcal{X}} (F_{k-1} \setminus O(\Phi_{k-1}))$ can be covered by such a subfamily. Now put $F = F_{k-1}$, $M = \Phi_{k-1}$, $V = \mathrm{Int}_{\mathcal{X}} F$, and Lemma is proved.

Theorem 11. Let $\mathcal{X}$ be a bicompactum, admitting CPC, $\mathcal{U} \subseteq \mathcal{X}$ be open.

Then there is an open $V \subseteq \mathcal{U}$ such that $w(V) \leq \aleph_0$ (and thus, the set of the points of a countable local weight is dense in $\mathcal{X}$).

Proof: If $w([\mathcal{U}_1]) = \tau > \aleph_0$, where $[\mathcal{U}_1] \subset \mathcal{U}$ and $\mathcal{U}_1$ is open in $\mathcal{X}$, then let $V \subseteq [\mathcal{U}_1]$, F and M are chosen as in Lemma 4 (for $\mathcal{X} = [\mathcal{U}_1]$ and $\lambda = \aleph_1$). $V \setminus M \neq \emptyset$ (otherwise $V \subset M$ but $w(V) \geq \aleph_1$, while M is compact) and $V \setminus M$ is open in $[\mathcal{U}_1]$, therefore $(V \setminus M) \cap \mathcal{U}_1 \neq \emptyset$ and open in $\mathcal{U}_1$ and so in $\mathcal{X}$. Let $z \in (V \setminus M) \cap \mathcal{U}_1$, then there are open sets O and W_1 such that $O \supset M$, $W_1 \ni z$ and $O \cap W_1 = \emptyset$. Now $W = W_1 \cap (V \setminus M) \cap \mathcal{U}_1$ is open in $\mathcal{X}$ and $W \subseteq V \setminus O \subseteq F \setminus O$, therefore $W \subseteq \mathrm{Int}_{\mathcal{X}} (F \setminus O)$ and according to Lemma 4 can be covered by a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \aleph_1$. Hence $w([V]) = w(V) \leq \aleph_0$.

Corollary 4. Every non-metrizable bicompactum, admitting CPC is not homogeneous.

Theorem 12. Let $\mathcal{X}$ be a bicompactum admitting CPC, $w(\mathcal{X}) = \tau$ and $\aleph_0 < \lambda \leq \tau$. Then if λ is regular, then there is a family $\{V\}$ of pairwise disjoint open sets with a countable local weight such that $|\{V\}| = \lambda$.

Proof: Let V be chosen as in Lemma 4. Suppose that for every $\alpha < \gamma < \Omega(\lambda)$ we have defined a system $\{V_\alpha\}$ of open sets such that

a) $[V_\alpha]_{\mathcal{X}} \subset V$ and $[V_\alpha]_{\mathcal{X}}$ can be covered by a countable subfamily of $\mathcal{F}$

b) $V_\alpha \cap [\bigcup\{V_\beta : \beta < \alpha\}] = \emptyset$.

Let us construct V_γ . As $\gamma < \Omega(\lambda)$ and because of a)

$\bigcup\{V_\alpha : \alpha < \gamma\}$ can be covered by a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| < \lambda$. But $\mathcal{F}$ is closure-preserving, so $[\bigcup\{V_\alpha : \alpha < \gamma\}]$ also can be covered by such a subfamily. Then $\mathcal{U} = V \setminus [\bigcup\{V_\alpha : \alpha < \gamma\}]_{\mathcal{X}} \neq \emptyset$ is open and $w(\mathcal{U}) = \lambda$. Let $V_\gamma \subseteq \mathcal{U}$ be an open set which can be covered by a countable subfamily of $\mathcal{F}$ (V_γ exists, because of Theorem 11). Obviously, a) and b) are satisfied.

Corollary 5. Let $\mathcal{X}$ be a bicompactum, admitting CPC. Then

a) $c(\mathcal{X}) = w(\mathcal{X})$;

b) $\mathcal{X}$ contains an open dense metrizable subset with the local countable weight.

Theorem 13. Let $\mathcal{X}$ be a bicompactum admitting CPC, $w(\mathcal{X}) = \tau$ and $\aleph_0 < \lambda \leq \tau$ and λ is regular. Then there is a compact set $M \subseteq \mathcal{X}$ which cannot be represented as the intersection of less than λ open sets.

Choose M as in Lemma 4.

Corollary 6. Let $\mathcal{X}$ be a bicompactum admitting CPC. Then, if $\Psi_k(\mathcal{X})$ is a pseudocharacter of compacta in $\mathcal{X}$, then

$$\Psi_k(\mathcal{X}) = c(\mathcal{X}) = w(\mathcal{X}) = s(\mathcal{X}) = |\mathcal{X}|.$$

Answering the question of Rosenthal [4]: does every non-metrizable Eberlein bicompactum contain a compactum which is not G_δ ?

Benyamini, M. Rudin, Wage, recently gave a counter-example.

Their bicompactum does not admit CPC according to Theorem 13. Another bicompactum of this type is $\prod_{i=1}^{\infty} X_i$, where X_i is a "double circumference" of Alexandrov [11] (each X_i is embedded in $\sigma(I,\Gamma)$ and hence admits CPC).

Theorem 14. Every bicompactum $\mathcal{X}$ admitting CPC is a Fréchet-Uryson bicompactum.

It is sufficient to prove that $t(\mathcal{X}) \leq \aleph_0$ (see [7]). If $A \subseteq \mathcal{X}$ then $B = \cup\{[S]:S\subset A$ and $|S| \leq \aleph_0\}$. Let $\mathcal{F}$ be a CPC on $\mathcal{X}$. If $F_\alpha \in \mathcal{F}$, then put $\Phi_\alpha = B \cap F_\alpha$. Φ_α is closed in F_α (because $t(F_\alpha) \leq \aleph_0$) and thus each Φ_α is compact.

If $x_0 \in [\cup\{\Phi_\alpha : \alpha \in \Gamma\}] \setminus \cup\{\Phi_\alpha : \alpha \in \Gamma\}$, then $x_0 \in \cup\{F_\alpha : \alpha \in \Gamma\}$ and so there is $\alpha_0 : x_0 \in F_{\alpha_0} \setminus \Phi_{\alpha_0}$, thus $x_0 \notin B$. It follows that $\{\Phi_\alpha\}$ is also CPC, but only on the set B. Therefore B is metacompact, and B is obviously countably compact, so B is bicompact, thus $[A] = B$ and $t(\mathcal{X}) \leq \aleph_0$.

Every linearly ordered Eberlein bicompactum is metrizable [13].

Theorem 15. Every linearly ordered bicompactum with CPC is metrizable.

Proof: Let $\mathcal{F}$ be a CPC on $\mathcal{X}$. Suppose that $\mathcal{X}$ is not metrizable. Let $F_1 \in \mathcal{F}$. Then $\mathcal{U} = \mathcal{X} \setminus F_1 = \cup (a_\alpha, b_\alpha)$ and $(a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) = \emptyset$ $(\alpha \neq \beta)$. If every interval (a_α, b_α) is metrizable, then $\mathcal{U}$ is metrizable (as a free union of metric spaces). Then $\mathcal{X} = \mathcal{U} \cup F_1$ is a union of two metric spaces, and hence is an Eberlein bicompactum [14], so $\mathcal{X}$ is metrizable [13] and that is not so. Let $(a_1, b_1) \in \{(a_\alpha, b_\alpha)\}$ and (a_1, b_1) be not metrizable. Then $[a_1, b_1]$ is also not metrizable.

Using the induction, we receive a system of segments $[a_n, b_n]$ and of compacta $\{F_n\}$ such that

a) $[a_{n+1}, b_{n+1}] \subset (a_n, b_n)$ $(a_n \neq b_n)$

b) $F_{n+1} \cap (a_n, b_n) \neq \emptyset$, $F_{n+1} \cap (a_{n+1}, b_{n+1}) = \emptyset$.

Let $y_n \in F_{n+1} \cap (a_n, b_n)$ and x_0 be an accumulation point of $\{y_n\}$. Then $x_0 \notin F_n$ and $\mathcal{F}$ is not closure-preserving, a contradiction.

§ 4. Problems

1. Is it true that every bicompactum admitting CPC is embedded in $\sigma(I, \Gamma)$?

2. Is it true that every scattered bicompactum admitting CPC is a strong Eberlein bicompactum?

3. Let K be a compactum, X - HM-bicompactum. Is it true that $t(C_0(K, \mathcal{X})) \leq \aleph_0$? It is true, if X is hereditarily Lindelöf [15] , or Corson bicompact (the last was proved by Pytkeev).

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