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REGULAR SOCIETIES WITHOUT SHORT CYCLES

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Abstract: For every triple (n,m,k) of integers bigger than 1 a society is constructed such that each its team has n points, every point lies in m teams and it has not cycles with length $\leq k$.

Key words: Society, graph, cycle, team.

Classification: 05C99, 05C35

In this note all sets are finite. A society (or hypergraph) is a couple (X,R) where X is a set called an underlying set and R is a set of subsets of X called teams of the society. The notion of a society came into being as a generalization of that of a (symmetrical) graph - viewed as a society which has only two-point teams. Graphs were investigated in many papers. A special role among graphs is played by the regular ones. A graph (X,R) is k-regular if for each $x \in X$, card $\{A \in R; x \in A\} = k$. We generalize this notion as follows. A society (X,R) is (n,m)-regular if every team has n points and every point $x \in X$ lies exactly in m teams (n, m are natural numbers, $n,m > 0$). Thus an m-regular graph is a (2,m)-regular society. Another important notion in the theory of graphs is that of a cycle. We generalize this notion for a society in a natural way: a one-to-one sequence $(A_1, A_2, \ldots, A_r)$ of teams, $r > 1$, is a cycle of length r if there are distinct points $(x_1, x_2, \ldots, x_r)$ such that $x_1 \in A_1 \cap A_2$, $x_2 \in A_2 \cap A_3, \ldots, x_r \in A_r \cap A_1$. All simple examples of regular societies (or graphs) have short cycles. The question of existence of k-regular graphs without short cycles was solved in [2],[3]. A k-regular graph has been produced with girth $> n$ - the girth of a society with a cycle is the length of the shortest cycle in it, otherwise the girth is ∞ ($\infty > n$ for every natural number n). We can formulate this result as:

Proposition 1: There is (2,n)-regular society with girth $> k$, for every couple of natural numbers n, k, $k > 1$.

The other important class of graphs are bipartite graphs. In this note a bipartite graph is a triple (X,Z,R) where (X,R) is a graph, $Z \subset X$ such that for every $A \in R$, $Z \cap A \neq \emptyset \neq (X - Z) \cap A$. If we want to generalize the notion of n-regular graph for the class of bipartite graphs then we can do this as follows: a bipartite graph (X,Z,R) is (n,m)-regular if for each $x \in X - Z$, $\operatorname{card}\{A \in R; x \in A\} = n$, and for each $x \in Z$, $\operatorname{card}\{A \in R; x \in A\} = m$. Then the well-known theorem on representatives can now be restated as follows:

Theorem 2: For a society (X,R) define a bipartite graph $\Phi(X,R) = (X \cup R, X, S)$ where $(x,A) \in S$ iff $x \in A$. Then Φ is a bijective correspondence between societies and bipartite graphs such that:

a) a society (X,R) is (n,m)-regular iff $\Phi(X,R)$ is an (n,m)-regular bipartite graph;

b) a society (X,R) has a cycle of length k iff $\Phi(X,R)$

has a cycle of length 2k.

We now reformulate one of the results from [1]:

Proposition 3: For every couple n, k of natural numbers there is an (n,n)-regular society with girth $> k$.

The aim of this note is to prove the following generalization of Propositions 1 and 3.

Theorem 4: For every triple n,m,k of integers bigger than 1 there is an (n,m)-regular society with girth $> k$.

Using Theorem 2 we get:

Corollary 5: For every triple n,m,k of integers bigger than 1 there is an (n,m)-regular bipartite graph with girth $> k$.

The proof of Theorem 4 is based on the following idea. We construct societies $\mathscr{C}(n,m,s)$ without cycles and such that every team contains n points, every point is contained either in one or in exactly m teams. The parameter s characterizes the size of the society (see the introductory definitions and Lemma 6). We take a disjoint union $\mathscr{L}(n,m,s)$ of m copies $\mathscr{C}_i(n,m,s)$. $i=1,2,\dots,m$, of $\mathscr{C}(n,m,s)$ and glue them together by the equivalence generated by a suitable sequence φ of bijections $\varphi_i : B_i(n,m,s) \longrightarrow B_{i+1}(n,m,s)$, where $B_i(n,m,s)$ denotes the set of elements of $\mathscr{C}_i(n,m,s)$ contained only in one team. The resulting society $\mathscr{L}(n,m,s,\varphi)$ is (n,m)-regular (Lemma 7). With the aid of Lemma 9 we can, under certain assumptions on φ, and s, replace φ by another sequence ψ of bijections $\psi_i : B_i \longrightarrow B_{i+1}$ yielding a new society $\mathscr{L}(n,m,s,\psi)$ whose girth is three times bigger than that of $\mathscr{L}(n,m,s,\varphi)$. Repeated blowing up of girth finally proves Theorem 4.

Proof of Theorem 4: First we give some definitions. We shall assume that n, m, s is a triple of positive integers. Define:

$$P(n,m,s) = \{\{i_j\}_{j=1}^{2t+1};\ t \leq s,\ i_1 \in \{0,1,\dots,n\},\ \text{for } j=1,2,\dots,t\ \ i_{2j} \in \{1,2,\dots,m\},\ i_{2j+1} \in \{1,2,\dots,n\}\};$$

$$Q(n,m,s) = P(n,m,s) \times \{1,2,\dots,m\};$$

$$B(n,m,s) \subseteq P(n,m,s),\ \{i_j\}_{j=1}^{2t+1} \in B(n,m,s) \text{ iff } t = s;$$

$$C(n,m,s) = B(n,m,s) \times \{1,2,\dots,m\};$$

$$T(n,m,s) = \{Z \subset P(n,m,s);\ \exists \{i_j\}_{j=1}^{2t+1} \in P(n,m,s) - B(n,m,s),\ \exists q \in \{1,2,\dots,m\},$$
$$Z = \{\{i_j\}_{j=1}^{2t+1}\} \cup \{\{a_j\}_{j=1}^{2t+3};\ a_j = i_j \text{ for } j \leq 2t+1,\ a_{2t+2} = q,\ a_{2t+3} \in \{1,2,\dots,n\}\}\} \cup \{\{0,1,\dots,n\}\};$$

$$U(n,m,s) = \{Z \times \{q\};\ Z \in T(n,m,s),\ q \in \{1,2,\dots,m\}\};$$

For $\{i_j\}_{j=1}^{2s+1} \in B(n,m,s)$ and for $w < s$ define $\pi_w(\{i_j\}_{j=1}^{2s+1}) = \{i_j\}_{j=1}^{2w+1}$ $\sigma_w(\{i_j\}_{j=1}^{2s+1}) = \{i_j\}_{j=2w+2}^{2s+1}$. For $(x,q) \in C(n,m,s)$ where $x \in B(n,m,s)$ put $\pi_w(x,q) = (\pi_w(x),q)$, $\sigma_w(x,q) = (\sigma_q(x),q)$. Put $\mathscr{C}(n,m,s) = (P(n,m,s),\ T(n,m,s))$, $\mathscr{L}(n,m,s) = (Q(n,m,s),\ U(n,m,s))$. Then it is clear:

Lemma 6: Every team of $\mathscr{L}(n,m,s)$ or $\mathscr{C}(n,m,s)$ has exactly n+1 points. Every point of $P(n,m,s) - B(n,m,s)$ or $Q(n,m,s) - C(n,m,s)$ lies exactly in m+1 teams. Every point of $B(n,m,s)$ or $C(n,m,s)$ lies exactly in one team. The girth of $\mathscr{C}(n,m,s)$ or $\mathscr{L}(n,m,s)$ is bigger than any natural number.

We say that a mapping $\varphi: C(n,m,s) \longrightarrow B(n,m,s)$ fulfils (*) if for every $q \in \{1,2,\dots m\}$ the restriction φ on $B(n,m,s) \times \{q\}$ is a bijection. Let $\sim_\varphi$ be an equivalence on

C(n,m,s) such that a $\sim_\varphi$ b iff $\varphi(a) = \varphi(b)$. (The smallest equivalence on Q(n,m,s) merging the same points as $\sim_\varphi$ will be denoted $\sim_\varphi$, too.) We define a society $\mathcal{L}(n,m,s,\varphi)$ as follows: the underlying set is $Q(n,m,s,\varphi) = Q(n,m,s)/\sim_\varphi$, the teams are $U(n,m,s,\varphi) = \{Z/\sim_\varphi ; Z \in U(n,m,s)\}$. Then it is easy that:

Lemma 7: $\mathcal{L}(m,n,s,\varphi)$ is an (n+1,m+1)-regular society whenever φ fulfils (*).

In the following we want to choose s and φ fulfilling (*) such that the girth of $\mathcal{L}(n,m,s,\varphi) \geq k$.

Let $w > 1$ be a natural number with $w < s$. Define: $H(n,m,s,w) = \{\sigma_w(x);\ x \in B(n,m,s)\}$, $L(n,m,s,w) = H(n,m,s,w) \times \{1,2,\dots,m\}$. We say that $\psi: L(n,m,s,w) \longrightarrow H(n,m,s,w)$ fulfils (*) if for every $q \in \{1,2,\dots,m\}$ the restriction of ψ to $H(n,m,s,w) \times \{q\}$ is a bijection. Further for $\psi_1: L(n,m,s,w) \longrightarrow H(n,m,s,w)$, $\psi_2: C(n,m,w) \longrightarrow B(n,m,w)$ define $\psi_1 \boxtimes \psi_2 : C(n,m,s) \longrightarrow B(n,m,s)$ by $\sigma_w(\psi_1 \boxtimes \psi_2(x)) = \psi_1(\sigma_w(x))$, $\pi_w(\psi_1 \boxtimes \psi_2(x)) = \psi_2(\pi_w(x))$. Then it is easy to prove

Lemma 8: Let $\psi = \psi_1 \boxtimes \psi_2$. Then ψ fulfils (*) iff ψ_1 and ψ_2 fulfil (*). Moreover the projection from C(n,m,s) to B(n,m,s) fulfils (*).

Now we prove the basic lemma of the proof:

Lemma 9: Let $\psi: L(n,m,s,w) \longrightarrow H(n,m,s,w)$ fulfil (*). Let $h: L(n,m,s,w) \longrightarrow \{2\ell, 2\ell+1, \dots, 2w+1\}$ be a one-to-one mapping. Define $\varphi: L(n,m,s,\ell) \longrightarrow H(n,m,s,\ell)$ as follows: $\sigma_w(\varphi(x)) = \psi(\sigma_w(x))$, and if $x = (\{i_j\}_{j=2\ell}^{2s+1}, q)$ then $\varphi(x) = \{a_j\}_{j=2\ell}^{2s+1}$ where for $j \in \{2\ell, 2\ell+1, \dots, 2w+1\}$, $j \neq h(\sigma_w(x))$

we have $a_j = i_j$, and if $j = h(\sigma_w(x))$ and j is even then $a_j = i_j + 1 \pmod m$, if $j = h(\sigma_w(x))$ and j is odd then $a_j = i_j + 1 \pmod n$.

Then φ fulfils (*). If moreover the girth α of $\mathcal{G}(n,m,s,p \boxtimes \varphi) > \frac{s-w}{2}$ where p is the projection, then for every $\chi : C(n,m,\ell-1) \longrightarrow B(n,m,\ell-1)$ fulfilling (*) it holds that the girth of $\mathcal{G}(n,m,s,\chi \boxtimes \varphi) \geq 3\alpha$.

<u>Proof:</u> The former statement is obvious. We have to prove the latter one. Assume that $A_1, A_2, \ldots, A_r$ is a cycle in $\mathcal{G}(n,m,s,\chi \boxtimes \varphi)$ and $r < 4\alpha$. Choose $\{i_j\}_{j=1}^{2w+1} \in B(n,m,w)$ and define $B_q = \{x; \exists y = (a,u) \in A_q,\ \sigma_w(x) = \sigma_w(y),\ \pi_w(x) = (\{i_j\}_{j=1}^{2w+1}, u)\}$. Since $A_1, A_2, \ldots, A_r$ form a cycle we get that $B_1 \cap B_2 \neq \emptyset$, $B_2 \cap B_3 \neq \emptyset, \ldots, B_r \cap B_1 \neq \emptyset$, moreover we can choose a sequence of distinct points $(x_1, x_2, \ldots, x_r)$ such that $x_1 \in B_1 \cap B_2$, $x_2 \in B_2 \cap B_3, \ldots, x_r \in B_r \cap B_1$. Hence $B_1, B_2, \ldots, B_r$ can be divided into sections which form cycles.

Define $\mu : \{1,2,\ldots,r\} \longrightarrow \{1,2,\ldots,m\}$ as follows: $\mu(q) = u$ if A_q is an image of some $Z \times \{u\}$ by $\sim_{\chi \boxtimes \varphi}$, $Z \in T(n,m,s)$. Since $\chi \boxtimes \varphi$ fulfils (*) by Lemma 8 we get that μ is a correctly defined mapping. By Lemma 6, μ is not constant. Let $\mu(j-1) \neq \mu(j) = \mu(j+1) = \ldots = \mu(\bar{j}) \neq \mu(\bar{j}+1)$. Then from the definition of $\mathcal{C}(n,m,s)$ we get that B_j and $B_{\bar{j}}$ determines all members between j and $\bar{j}$. The analogous statement holds for A_j and $A_{\bar{j}}$. Further if we choose $x, y \in \bigcup_{q=j}^{\bar{j}} A_q - C(n,m,s)$, $x = (\{a_u\}_{u=1}^{2t+1}, \mu(j))$, $y = (\{b_u\}_{u=1}^{2t'+1}, \mu(j))$, then $t, t' > w$ and for $u < 2w+1$, $a_u = b_u$. On the other hand if we choose $z \in A_{j-1} - C(n,m,s)$, $z = (\{c_u\}_{u=1}^{2s-1}, \mu(j-1))$ then there are two indexes $u_1, u_2 < 2w+1$ such that $c_{u_1} \neq a_{u_1}$, $c_{u_2} \neq a_{u_2}$ $(u_1 = h(\sigma_w(x'))$,

$u_2 = h(\sigma_w(x''))$ where $x', x'' \in C(n,m,s)$, $\chi \boxtimes \varphi(x') = \chi \boxtimes \varphi(x'')$ and $A_{j-1} \cap A_j$ contains the class of $\sim_{\chi \boxtimes \varphi}$ containing x' - from the definition of φ we get card $A_{j-1} \cap A_j = 1$). Thus if $\mu(j-1) \neq \mu(j)$ then there is j' such that $\{\mu(j'), \mu(j'-1)\} = \{\mu(j), \mu(j-1)\}$ and $B_{j'-1} \cap B_{j'} = B_{j-1} \cap B_j$. Hence $\{B_{j'-1}, B_{j'}\} = \{B_{j-1}, B_j\}$.

We choose a cycle $B_t, B_{t+1}, \dots, B_{t'}$. Then by the following considerations one of the following two cases is necessary:

1) this cycle is in the sequence $B_1, B_2, \dots, B_r$ once more in the converse ordering. But $A_1, A_2, \dots, A_r$ is a cycle so these two cycles do not exhaust all sets $B_1, B_2, \dots, B_r$. From the rest we can also choose a cycle and again by the foregoing considerations there is still another cycle these. Thus from the sequence $B_1, B_2, \dots, B_r$ it is possible to choose four disjoint cycles, hence $r \gtrsim 4\alpha$ - a contradiction.

2) This cycle is contained in two other cycles (in the converse ordering). Hence $r \gtrsim 3\alpha$.

The lemma is proved.

Assume that it is given k such that the girth of $\mathcal{G}(n,m,s,\varphi) \geq k$. Now we complete the proof by induction. Choose s such that the following considerations are possible. Choose ψ_0 as the projection. Put $k_0 = s - \frac{k}{2}$ (>0). Assume that $k_1 = s - \frac{k}{2} - \frac{(mn)^{k_0}}{2}$ (>0). Then there is a bijection from $L(n,m,s,k_0)$ to $\{2k_1+2, 2k_1+3, \dots, 2k_0\}$ and we can construct ψ_1 by Lemma 9. Since the girth of $\mathcal{G}(n,m,s,\psi_0) = 2$ we get that the girth of $\mathcal{G}(n,m,s,p \boxtimes \psi_1) \geq 6$ (p is the projection). Now we assume that $k_2 = s - k_1 - \frac{(mn)^{k_1}}{2}$ (≥ 0) and we can construct ψ_2 again by Lemma 9. If we repeat this step for a suitable number of times we wind up with the girth of $\mathcal{G}(n,m,s,p \boxtimes \psi_t) \geq k$, which concludes the proof of Theorem 4.

Define

$\varphi(n,m,k) = \{r;$ there is an (n,m)-regular society (X,R) with girth $\geq k$ and card $X = r\}$;

$R(n,m,k) = \min \varphi(n,m,k)$.

Then it is easy to prove:

Proposition 10: $\varphi(n,m,k)$ forms an additive subsemigroup of natural numbers. If $r \in \varphi(n,m,k)$ then $\frac{rm}{n}$ is an integer. Define $\Phi(0) = [\frac{k}{2}]$, $\Phi(i+1) = \Phi(i) + [\frac{(nm)^{\Phi(i)}}{2}]$. Then $\Phi(0) \leq \frac{n}{n+1} R(n,m,k) \leq \Phi(t)$ where t is the smallest number such that $2(3^t) \geq k$.

Proof: The disjoint union of societies preserves (n,m)-regularity and the girth of the disjoint union is the minimum of girths. Hence we get the first statement. The second statement follows from the fact that for (n,m)-regular society (X,R) it holds: card $R \cdot n = m \cdot$ card X. The third one follows from the proof of Theorem 4.

The form of $\varphi(n,m,k)$ or the value of $R(n,m,k)$ is an open problem. These values are known only for simple cases e.g. if $n = 1$ or $m = 1$, or $n = 2 = m$.

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