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ENDOMORPHIC UNIVERSES AND THEIR STANDARD EXTENSIONS

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Abstract: This paper is meant as a contribution to the development of mathematics in alternative set theory. In particular, a procedure in some aspects similar to Robinson's non-standard methods is created using specific means of alternative set theory.

Key words: Alternative set theory, ultraproduct, non-standard methods, endomorphic universe, standard extension.

Classification: Primary 02K10, 02K99
Secondary 02H20, 02H13

The classical calculus of Leibniz and Newton is based on the existence of the natural extension of real functions on infinitely small quantities. Robinson's non-standard analysis has tuned this assumption with the mathematics in Cantor's set theory. Moreover, Robinson's non-standard methods have brought a great deal of additional important applications.

This article deals with analogical questions in alternative set theory (AST). We do not transfer the construction of non-standard models to AST word for word - although even this approach is possible - but the method described in the paper is based on specific properties of AST. Let us note that from the point of view of non-standard methods the whole of AST behaves as a non-standard model.

An endomorphic universe is a copy of the universal class conveniently put into the universal class. There are many different endomorphic universes in AST. In many cases there is room enough for the natural extension of endomorphic universe. Let us note that this natural extension is defined inside AST (and it is not considered as a primitive notion). These extensions are so natural that it would be absurd (from the point of view of AST) to call them non-standard extensions (though they correspond to Robinson's non-standard extensions). This is the reason why they are said standard extensions.

The more experienced reader notices that the approach we have chosen enables us to eliminate from AST the method of ultraproduct and to replace it equivalently by another procedure which seems to fit more in AST. For such readers let us note that in AST we can prove that the ultrapower of the universal class is isomorphic to the universal class.

The method described in the paper has a lot of applications; in particular, we can imitate in this way a great deal of results obtained by non-standard methods. Some applications which are in a way specific to AST can be found in [S-V2].

In the first section we investigate basic properties of general endomorphic universes, in particular we show the existence of endomorphic universes with special properties. The second section deals with properties of standard extensions on an endomorphic universe. Especially, we prove that standard extension is determined uniquely. The last section is devoted to the proofs of the existence of endomorphic universes which have standard extensions.

The whole article can be considered as an immediate continuation of [V]. When referring to that book we shall cite only the section and the chapter in question.

The first author showed in AST that the ultrapower of the universal class is isomorphic to the universal class. On the base of this result the second author suggested the conception of this paper. Its concrete realization was carried out by both authors.

§ 1. Basic properties of endomorphic universes

A class is called an endomorphic universe iff it is similar to the universal class. Therefore, a class A is an endomorphic universe iff there is an endomorphism F with $rng(F) = A$.

Let us recall that a function F is an endomorphism iff for every set-formula $\varphi(z_1,\dots,z_n)$ and for every $x_1,\dots,x_n \in V = dom(F)$ we have $(\varphi(x_1,\dots,x_n) \equiv \varphi(F(x_1),\dots,F(x_n)))$. Further, let us remind that φ^A denotes the formula resulting from φ by the restriction of all quantifiers binding set variables to the elements of A and all quantifiers binding class variables to the subclasses of A.

<u>Theorem</u>. The following properties of a class A are equivalent:

(1) A is an endomorphic universe

(2) If $\varphi(Z_1,\dots,Z_n)$ is a normal formula of the langu-

age FL_A (i.e. the elements of A are admitted as parameters) then for every $X_1,\dots,X_n$ the subclasses of A which are at most countable we have

$$\varphi^A(X_1,\dots,X_n) \equiv \varphi(X_1,\dots,X_n)$$

(3) A satisfies the following two conditions

(a) If $\varphi(z)$ is a set-formula of the language FL_A then we have $(\exists x)\,\varphi(x) \rightarrow (\exists x \in A)\,\varphi(x)$

(b) For every countable $F \subseteq A$ there is $f \in A$ with $F \subseteq f$

(4) If $\{\varphi_n(z); n \in FN\}$ is a sequence of set-formulas of the language FL_A then we have

$(\exists x)(\forall n)\,\varphi_n(x) \rightarrow (\exists x \in A)(\forall n)\,\varphi_n(x)$.

Proof. (1) $\rightarrow$ (2). Let $Y \subseteq A$ be a countable class containing all constants occurring in φ and such that $X_i \subseteq Y$ for every $1 \leq i \leq n$. By the second theorem of § 1 ch. V it is sufficient to construct an endomorphism which is identical on Y and the range of which is A. Let G be an endomorphism with $rng(G) = A$. Then $G^{-1} \upharpoonright Y$ is a countable similarity and therefore it can be extended to an automorphism F (see § 1 ch. V). Hence the composition of G and F is an endomorphism we looked for.

The implication (2) $\rightarrow$ (3) is a trivial consequence of the prolongation axiom.

(3) $\rightarrow$ (4). Let $\{\varphi_n; n \in FN\}$ be a sequence of set-formulas of the language FL_A and let c be a set such that for every $n \in FN$ the formula $\varphi_n(c)$ holds. According to (a) we can choose for every $n \in FN$ a set $x_n \in A$ such that $\varphi_1(x_n)$ & & ... & $\varphi_n(x_n)$. By (b) there is $f \in A$ with $f(n) = x_n$ and hence there is a set $x \in A$ such that $\{x_n; n \in FN\} \subseteq x$. For every

$n \in FN$ we put $y_n = \{z \in x;\ \varphi_1(z) \& \dots \& \ \varphi_n(z)\}$ and we have obviously $y_n \subseteq A \& y_n \neq 0$. Using our assumptions we can choose $g \in A$ such that $dom(g) \in N \& (\forall n)(g(n) = y_n) \& (\forall \alpha)(\alpha + 1 \in \in dom(g) \rightarrow g(\alpha + 1) \subseteq g(\alpha))$. Let β be the smallest α for which holds $g(\alpha + 1) = 0 \vee \alpha + 1 \notin dom(g)$. Then $\beta \in A$ and $\beta \notin FN$. We get $A \cap g(\beta) \neq 0$ as a consequence of (a) and of the statement $g(\beta) \neq 0 \& g(\beta) \in A$. Moreover, the formula $(\forall y \in g(\beta))(\forall n)\ \varphi_n(y)$ follows from the construction of the function g and this proves our statement.

(4) $\rightarrow$ (1). At first we are going to show that for every set b and for every similarity F_0 which is at most countable and the range of which is a subclass of A, there is $a \in A$ such that the function $F_0 \cup \{\langle a,b\rangle\}$ is again a similarity. Under our assumptions there is a set c such that $F_0 \cup \cup \{\langle c,b\rangle\}$ is a similarity (see the third theorem of § 1 ch. V). Let us choose $a \in A$ such that for every set-formula $\varphi(z)$ of the language $FL_{rng(F_0)}$ we have $\varphi(a) \equiv \varphi(c)$. Thus $F_0 \cup \{\langle a,b\rangle\}$ is a similarity and thence we have proved our claim.

Let $\{a_\alpha;\ \alpha \in \Omega\}$ ($\{b_\alpha;\ \alpha \in \Omega\}$ respectively) be an enumeration of A (of V respectively). Using the previous claim and the third theorem of § 1 ch. V we are able to construct by transfinite induction a sequence $\{F_\alpha;\ \alpha \in \Omega\}$ of similarities which are at most countable and such that $\bigcup\{F_\beta;\ \beta \in \in \alpha \cap \Omega\} \subseteq F_\alpha$, $a_\alpha \in rng(F_\alpha) \subseteq A$ and $b_\alpha \in dom(F_\alpha)$. Thus $\bigcup\{F_\alpha;\ \alpha \in \Omega\}$ is an endomorphism we looked for.

Let us note that according to (2) of the last theorem, for every endomorp[illegible] 'erse A and for every finite set x we have $x \in A \equiv x \subseteq A$.

Theorem. Let $\mathfrak{M}$ be a codable class of endomorphic universes such that for every $\mathfrak{N} \subseteq \mathfrak{M}$ which is at most countable there is $A \in \mathfrak{M}$ with $\bigcup \mathfrak{N} \subseteq A$. Then $\bigcup \mathfrak{M}$ is an endomorphic universe.

Proof. To prove the property (4) of the previous theorem let us suppose that $\{\varphi_n(z); n \in FN\}$ is a sequence of set-formulas of the language $FL_{\bigcup \mathfrak{M}}$. Then there is $\mathfrak{N} \subseteq \mathfrak{M}$ which is at most countable and such that $\{\varphi_n; n \in FN\}$ is a sequence of set-formulas of the language $FL_{\bigcup \mathfrak{N}}$. Thus by our assumption there is $A \in \mathfrak{M}$ such that every φ_n is a set-formula of the language FL_A. If there is a set y such that for every $n \in FN$ we have $\varphi_n(y)$ then there is $x \in A$ such that for every $n \in FN$, the formula $\varphi_n(x)$ holds. Since $x \in \bigcup \mathfrak{M}$ our statement is proved.

Theorem. Let $\{A_n; n \in FN\}$ be a sequence of endomorphic universes such that $A_n \subset A_{n+1}$. Then $\bigcup \{A_n; n \in FN\}$ is no endomorphic universe.

Proof. We can choose a function F such that $dom(F) = FN$ and such that for every $n \in FN$ the formula $F(n) \in A_{n+1} - A_n$ holds. If $\bigcup \{A_n; n \in FN\}$ would be an endomorphic universe then there would be $f \in \bigcup \{A_n; n \in FN\}$ with $F \subseteq f$ and thence we would have $f \in A_m$ for some $m \in FN$. Since $m \in A_m$ it would hold $F(m) \in A_m$, which is a contradiction to our assumption $F(m) \in A_{m+1} - A_m$.

For arbitrary class A and arbitrary set d we put

$A[d] = \{f(d); f \in A \,\&\, d \in dom(f)\}$

Theorem. Let A be an endomorphic universe and let $d \in \bigcup A$. Then $A[d]$ is the smallest endomorphic universe subclass of which is the class $A \cup \{d\}$.

Proof. Obviously the class $A[d]$ is a subclass of every endomorphic universe subclass of which is $A \cup \{d\}$. To show that $A \cup \{d\} \subseteq A[d]$ let us fix v with $d \in v \in A$. Let $x \in A$ be given. The function f for which the formula $\mathrm{dom}(f) = v \,\&\, (\forall y \in v)(f(y) = x)$ holds is an element of A and moreover $f(d) = x$. Hence $A \subseteq A[d]$ has been proved. If id denotes the identity on v then $\mathrm{id} \in A$ and $\mathrm{id}(d) = d$ from which $d \in A[d]$ follows.

Therefore it remains to prove that $A[d]$ is an endomorphic universe. To show this we are going to verify the conditions (a) and (b) from the first theorem of this section.

(a) Let $\varphi(z_0, z_1, \ldots, z_n)$ be a set-formula of the language FL and let us assume that for functions $f_1, \ldots, f_n \in A$ the formula $d \in \mathrm{dom}(f_1) \cap \ldots \cap \mathrm{dom}(f_n) \,\&\, (\exists x)\, \varphi(x, f_1(d), \ldots, f_n(d))$ holds. We have to construct $f \in A$ with $\varphi(f(d), f_1(d), \ldots$ $\ldots, f_n(d))$. Put

$u = \{y \in \mathrm{dom}(f_1) \cap \ldots \cap \mathrm{dom}(f_n);\ (\exists x)\, \varphi(x, f_1(y), \ldots, f_n(y))\}$.

Evidently $d \in u \in A$ and moreover there is a function g for which the statement $\mathrm{dom}(g) = u \,\&\, (\forall y \in u)\, \varphi(g(y), f_1(y), \ldots, f_n(y))$ holds. Since A is an endomorphic universe there must be a function f with the above mentioned property and which is moreover an element of A. Thus we get $\varphi(f(d), f_1(d), \ldots, f_n(d))$ as a consequence of $d \in u$.

(b) Let $F \subseteq A[d]$ be a countable function. Then there is a sequence $\{f_n; n \in FN\}$ of elements of A such that $d \in \bigcap \{\mathrm{dom}(f_n); n \in FN\}$ and $F = \{f_n(d); n \in FN\}$. Without loss of generality, we can suppose that $\mathrm{dom}(f_n) = v$ for every $n \in FN$ because if $g \in A \,\&\, d \in \mathrm{dom}(g) \,\&\, h = (g \restriction v) \cup \{0\} \times (v - \mathrm{dom}(g))$ then $h \in A \,\&\, \mathrm{dom}(h) = v \,\&\, g(d) = h(d)$. Since A is an endomorphic universe there is a prolongation $\{f_\alpha ; \alpha \in \beta\} \in A$ of our sequence for which the implication $\alpha \in \beta \rightarrow \mathrm{dom}(f_\alpha) = v$ holds. For every $y \in v$ we define

$$g(y) = \{f_\alpha (y); \alpha \in \beta \mathbin{\&} \mathrm{Fnc}(\{f_\gamma (y); \gamma \leq \alpha\})\}$$

We have evidently $g \in A$ and $d \in v = \mathrm{dom}(g)$ and therefore $g(d) \in \in A[d]$. Moreover $F \subseteq g(d)$ because $F = \{f_n(d); n \in FN\}$ is a function.

At the end of this section we shall see that the assumption $d \in \bigcup A$ in the previous theorem is essential.

Let us note that if A is an endomorphic universe and if $d_1, d_2 \in \bigcup A$ then $\{d_1, d_2\} \in \bigcup A$ and $A[\{d_1, d_2\}]$ is the smallest endomorphic universe subclass of which is $A \cup \{d_1, d_2\}$ and hence it is even the smallest endomorphic universe subclass of which is the class $A[d_1] \cup A[d_2]$. The analogical statement holds for arbitrary finite number of elements of $\bigcup A$.

<u>Theorem</u>. Let A be an endomorphic universe and let $c, d \in \in \bigcup A$. Then $A[c] = A[d]$ iff there is a one-one mapping $f \in A$ with $c = f(d)$.

Proof. Let $A[c] = A[d]$. There are $g, h \in A$ such that $c = = g(d)$ and $d = h(c)$ because of $c \in A[d] \mathbin{\&} d \in A[c]$. Put $u = = \{y; h(g(y)) = y\}$. We have evidently $d \in u \in A$. Moreover putting $f = g \restriction u$ we get $f(d) = c$ and $f \in A$. We have to prove that f is a one-one mapping. Let $x, y \in u$ and $x \neq y$. If we would have $f(x) = f(y)$ it would hold the statement $g(x) = g(y)$, and hence we would obtain $h(g(x)) = h(g(y))$. Further from the assumption $x, y \in u$ we would get $x = y$ which is a contradiction. The opposite implication is trivial.

Let us mention the almost trivial fact that if A is an endomorphic universe then $\bigcup A = \bigcup\{\overline{P}_{\alpha};\alpha \in A\} = \bigcup\bigcup A =$ $= P(\bigcup A)$. This is an immediate consequence of the obvious statements $\alpha \in A \equiv \overline{P}_{\alpha} \in A$, $x \in A \rightarrow \tau(x) \in A$ and $\alpha \in A \rightarrow$ $\rightarrow \alpha + 1 \in A$ (for $\overline{P}_{\alpha}$ see § 1 ch. II).

<u>Theorem</u>. Let A and B be endomorphic universes. Then there is the smallest endomorphic universe a subclass of which is $A \cup B$.

Proof. Using the previous fact we have either $A \subseteq \bigcup B$ or $B \subseteq \bigcup A$. Let us suppose that the second inclusion holds. Put $\mathfrak{M} = \{A[d];d \in B\}$. It is sufficient to prove that $\bigcup\mathfrak{M}$ is an endomorphic universe because for every endomorphic universe C the implication $A \cup B \subseteq C \rightarrow \bigcup\mathfrak{M} \subseteq C$ is satisfied. Let $\mathfrak{N} = \{A[d_n];n \in FN\}$ be a countable subclass of $\mathfrak{M}$. Since B is an endomorphic universe, there is $f \in B$ with $(\forall n)(f(n) = d_n)$. Thus $A[d_n] \subseteq A[f] \in \mathfrak{M}$ for every $n \in FN$. The use of the second theorem of this section finishes the proof.

Let us recall that a class X is called revealed if for each countable $Y \subseteq X$ there is a set u such that $Y \subseteq u \subseteq X$.

<u>Theorem</u>. If A is an endomorphic universe then $\bigcup A$ is a revealed endomorphic universe.

Proof. If $\{x_n;n \in FN\} \subseteq \bigcup A$ then there is a sequence $\{y_n;n \in FN\} \subseteq A$ such that for every $n \in FN$ we have $x_n \in y_n$. Since A is an endomorphic universe we can choose $\overline{P}_{\alpha} \in A$ so that $\{y_n;n \in FN\} \subseteq \overline{P}_{\alpha}$ and thus it is $\{x_n;n \in FN\} \subseteq \overline{P}_{\alpha} \subseteq \bigcup A$. We have proved that $\bigcup A$ is a revealed class.

For every f and d with $f \in A \,\&\, d \in dom(f)$ we have $f(d) \in$ $\in rng(f) \in A$ and therefore the formula $f(d) \in \bigcup A$ holds. As a consequence we get $\bigcup A = \bigcup\{A[d];d \in \bigcup A\}$. Thus it is sufficient to show that the class $\bigcup\{A[d]; d\in \bigcup A\}$ is an endomorphic universe. If $\{d_n; n\in FN\}\subseteq \bigcup A$ then there is $f\in \bigcup A$ with $(\forall n)(f(n) = d_n)$. This means, however, that for every $n\in FN$ we have $A[d_n]\subseteq A[f]$ because the statement $d_n\in A[f]$ holds for every $n\in FN$. To finish the proof we use the second theorem of this section.

The theorem we proved just now makes it possible to restrict the investigation of endomorphic universes to the following two types: The first type consists of endomorphic universes which are not semisets (in other words the union of which is the universal class). To the second type belong endomorphic universes which are transitive (a class X is transitive iff $\bigcup X\subseteq X$). In fact if A is an endomorphic universe then $\bigcup A$ is an endomorphic universe of the second type and A becomes an endomorphic universe of the first type if we consider $\bigcup A$ as the universal class.

Theorem. A revealed class A is an endomorphic universe iff for every set-formula $\varphi(z)$ of the language FL_A we have

$(\exists x)\,\varphi(x) \longrightarrow (\exists x\in A)\,\varphi(x)$.

Proof. We are going to verify the condition (4) of the first theorem of this section. Let $\{\varphi_n(z); n\in FN\}$ be a sequence of set-formulas of the language FL_A and let the formula $(\exists x)(\forall n)\,\varphi_n(x)$ hold. Put $X_n = \{y\in A;\ \varphi_1(y)\ \&\ldots\&\ \varphi_n(y)\}$. Then $\{X_n; n\in FN\}$ is a descending sequence of non-empty revealed classes and hence $\bigcap\{X_n; n\in FN\}\neq 0$ by § 5 ch. II.

Theorem. If A is an endomorphic universe such that there is an infinite $u\subseteq A$ then A is revealed.

Proof. Let X be a countable subclass of A and let F be a one-one mapping of X into u. Then there is $f\in A$ with $F\subseteq f$ &

& $Fnc(f^{-1})$ and thus $X \subseteq f^{-1}"u \subseteq A$.

Theorem (A. Vencovská). If $\mathfrak{M}$ is a class of revealed endomorphic universes which is at most countable then $\bigcap \mathfrak{M}$ is a revealed endomorphic universe.

Proof. Obviously $\bigcap \mathfrak{M}$ is a revealed class. To prove the condition mentioned in the last theorem let us assume that $\varphi(z)$ is a set-formula of the language $FL_{\bigcap \mathfrak{M}}$. Let us choose a set-theoretically definable one-one mapping F of N onto V and let x be the set for which the formula $\varphi(x)$ & & $(\forall y)(\varphi(y) \rightarrow F^{-1}(y) \geq F^{-1}(x))$ holds. Then $\varphi \in FL_A$ for all $A \in \mathfrak{M}$ and hence $x \in A$ for all $A \in \mathfrak{M}$ from which $x \in \bigcap \mathfrak{M}$ follows.

Theorem. There is a transitive endomorphic universe which is the intersection of countably many sets.

Proof. In the second section of chapter V it is shown that there is an endomorphis F and $\alpha_0 \in N$ such that $F"V \subseteq \overline{P}_{\alpha_0}$. Put $A_0 = \bigcup F"V$, $A_{n+1} = \bigcup F"A_n$, $\alpha_{n+1} = F(\alpha_n)$ and $A = \bigcap \{A_n; n \in FN\}$. Then $\{A_n; n \in FN\}$ is a sequence of transitive revealed endomorphic universes and thence A is a transitive revealed endomorphic universe. Moreover, for every $n \in FN$ we have $A_{n+1} \subseteq$ $\subseteq \overline{P}_{\alpha_{n+1}} \subseteq A_n$.

Theorem. For every set x there is a transitive endomorphic universe $x \ni A$ which is a semiset.

Proof. Let F be an endomorphism such that F"V is a semiset. Put $y = F(x)$. Since $\{\langle x,y \rangle\}$ is a similarity, there is an automorphism G with $G(y) = x$ by § 1 ch. V. Putting $A =$ $= G"F"V$ we obtain an endomorphic universe which is a semiset (because G is an automorphism and because F"V is a semiset) and moreover $x \in A$. Thus $\bigcup A$ has all desired properties.

<u>Theorem</u>. If A is an endomorphic universe and $d \notin \bigcup A$ then there is no minimal endomorphic universe a subclass of which is $A \cup \{d\}$.

Proof. Let B be an endomorphic universe such that $A \cup \cup \{d\} \subseteq B$. Applying the last theorem (and substituting B for V) we obtain a class C such that $A \cup \{d\} \subseteq C \subset B$ and C is an endomorphic universe in the sense of B and hence C is an endomorphic universe by the definition of endomorphic universe.

§ 2. Standard extension

Let A be an endomorphic universe. An operation Ex defined for all subclasses of A is called a standard extension on A iff for arbitrary normal formula $\varphi(Z_1,\dots,Z_n)$ of the language FL_A and arbitrary $X_1,\dots,X_n \subseteq A$ we have

$$\varphi^A(X_1,\dots,X_n) \equiv \varphi(Ex(X_1,\dots,Ex(X_n)).$$

<u>Theorem</u>. An operation Ex defined for all subclasses of an endomorphic universe A is a standard extension on A iff for arbitrary normal formula $\varphi(z_1,\dots,z_k,Z_1,\dots,Z_n)$ of the language FL_A and for arbitrary $X_1,\dots,X_n \subseteq A$ we have

$$Ex(\{\langle x_1,\dots,x_k\rangle \in A;\ \varphi^A(x_1,\dots,x_k,X_1,\dots,X_n)\}) =$$
$$= \{\langle x_1,\dots,x_k\rangle;\ \varphi(x_1,\dots,x_k,Ex(X_1),\dots,Ex(X_n))\}.$$

Proof. At first let us suppose that Ex is a standard extension on A. Put $Y = \{\langle x_1,\dots,x_k\rangle \in A;\ \varphi^A(x_1,\dots,x_k,X_1,\dots,X_n)\}$. Obviously

$(\forall y \in A)(y \in Y \equiv (\exists x_1,\dots,x_k \in A)(y = \langle x_1,\dots,x_k\rangle$ &

& $\varphi^A(x_1,\dots,x_k,X_1,\dots,X_n)))$ holds and since Ex is a standard extension we get $(\forall y)(y \in Ex(Y) \equiv (\exists x_1,\dots,x_k)(y = \langle x_1,\dots,x_k\rangle \ \& \ \varphi(x_1,\dots,x_k,Ex(X_1),\dots,Ex(X_n))))$. From here we obtain $Ex(Y) = \{\langle x_1,\dots,x_k\rangle ; \varphi(x_1,\dots,x_k,Ex(X_1),\dots \dots,Ex(X_n))\}$.

Conversely let Ex have the property mentioned in the theorem. Then for $X_1,\dots,X_n \subseteq A$ it holds $\varphi^A(X_1,\dots,X_n) \equiv \equiv \{x \in A; \ \varphi^A(X_1,\dots,X_n)\} = A = \{x \in A; x = x\} \equiv \equiv \{x; \ \varphi(Ex(X_1,\dots,Ex(X_n))\} = V = \{x; x = x\} \equiv \equiv (\forall x) \ \varphi(Ex(X_1,\dots,Ex(X_n)) \equiv \varphi(Ex(X_1),\dots,Ex(X_n))$. Thus Ex is a standard extension on A.

Up to the end of this section let A denote an endomorphic universe different from V and let Ex denote a standard extension on A.

As immediate consequences of the definition of standard extension or of the previous theorem we obtain statements of the following list in which X and Y denote subclasses of A.

$X \subseteq Ex(X)$

$X = A \cap Ex(X)$

$X \subseteq Y \equiv Ex(X) \subseteq Ex(Y)$

$Ex(A) = V$

$Ex(0) = 0$

$Ex(X \cap Y) = Ex(X) \cap Ex(Y)$

$Ex(X \cup Y) = Ex(X) \cup Ex(Y)$

$Ex(X - Y) = Ex(X) - Ex(Y)$

$X \cap Y = 0 \equiv Ex(X) \cap Ex(Y) = 0$

The following statements follow from the fact that $\langle x,y\rangle \in A \equiv x,y \in A$.

$Rel(X) \equiv Rel(Ex(X))$

$\mathrm{Fnc}(X) \equiv \mathrm{Fnc}(\mathrm{Ex}(X))$

$\mathrm{Ex}(X^{-1}) = (\mathrm{Ex}(X))^{-1}$

$\mathrm{Ex}(\mathrm{dom}(X)) = \mathrm{dom}(\mathrm{Ex}(X))$

$\mathrm{Ex}(\mathrm{rng}(X)) = \mathrm{rng}(\mathrm{Ex}(X))$

$\mathrm{Ex}(Y''X) = \mathrm{Ex}(Y)''\mathrm{Ex}(X)$

$\mathrm{Ex}(X \times Y) = \mathrm{Ex}(X) \times \mathrm{Ex}(Y)$

If $\varphi(z)$ is a set-formula of the language FL_A then $\mathrm{Ex}(A \cap \{x;\ \varphi(x)\}) = \{x;\ \varphi(x)\}$ and thus we have in particular

$x \in A \longrightarrow \mathrm{Ex}(x \cap A) = x$

$\mathrm{Ex}(\bigcup X \cap A) = \bigcup \mathrm{Ex}(X) \ (= \{\langle x,y\rangle ;x \in y\}''\mathrm{Ex}(X))$

$\mathrm{Ex}(FN) \subseteq N$

<u>Theorem</u>. Let $\varphi(z,Z_1,\ldots,Z_n)$ be a normal formula of the language FL_A and let $X_1,\ldots,X_n \subseteq A$. Then $(\exists x)\, \varphi(x,\mathrm{Ex}(X_1),\ldots,\mathrm{Ex}(X_n)) \equiv (\exists x \in A)\, \varphi(x,\mathrm{Ex}(X_1),\ldots,\mathrm{Ex}(X_n))$.

Proof. Let us suppose that the formula $(\exists x)\, \varphi(x,\mathrm{Ex}(X_1),\ldots,\mathrm{Ex}(X_n))$ holds. Then we have $(\exists x \in A)\, \varphi^A(x,X_1,\ldots,X_n)$ and therefore we can choose $a \in A$ so that $\varphi^A(a,X_1,\ldots,X_n)$. Hence we have $\varphi(a,\mathrm{Ex}(X_1),\ldots,\mathrm{Ex}(X_n))$ and thus we have proved $(\exists x \in A)\, \varphi(x,\mathrm{Ex}(X_1),\ldots,\mathrm{Ex}(X_n))$. The converse implication is trivial.

<u>Theorem</u>. For every x there is $X \subseteq A$ which is at most countable so that $x \in \mathrm{Ex}(X)$.

Proof. To prove our statement by contradiction let us assume that $a \in V$ and that for every $X \subseteq A$ which is at most countable we have $a \notin \mathrm{Ex}(X)$. Let $\leq$ be an ordering of A of type Ω. Put

$\psi(F,X) \equiv (X \subseteq A \,\&\, F \subseteq A^2 \,\&\, (\forall x \in A)(\forall y \leq x)(y \in F(x) \equiv y \in X))$

We have $(\forall X \subseteq A)(\exists F)\, \psi(F,X)$ because every two disjoint at most countable subclasses of A can be separated by a set which is an element of A. Moreover $a \in Ex(dom(F))$ is a consequence of $\psi(F,X)$. Further let us realize that if $X,Y \subseteq A$ and $X \neq Y$ then $\psi(F,X) \& \psi(G,Y)$ implies that the class $\{x;F(x) = G(x)\}$ is at most countable and hence $a \in Ex(\{x;F(x) \neq G(x)\})$ and therefore $Ex(F)(a) \neq Ex(G)(a)$. Thus if we put

$$Q = \{\langle x,y \rangle ;(\exists X,F)(\psi(F,X) \& Ex(F)(a) = y \& x \in X)\}$$

then Q codes all subclasses of A which contradicts the second theorem of § 5 ch. I.

From the last theorem and from the above summarized results we can conclude

Theorem. If $X \subseteq A$ then $Ex(X) = \bigcup\{Ex(Y);Y \subseteq X \& Y \precsim FN\}$.

Theorem. If $X \subseteq A$ then $X = Ex(X)$ iff X is finite.

Proof. If X is finite then X is a set which is an element of A. Hence the equality $X = Ex(X)$ is obvious in this case. On the other hand let X be an infinite subclass of A with $X = Ex(X)$. If $Y \subseteq X$ then $Ex(Y) \subseteq Ex(X) \subseteq A$ and therefore $Y = A \cap Ex(Y) = Ex(Y)$. Thence we can suppose without loss of generality that X is countable. If $f \in A$ then $Ex(f''X) = f''Ex(X) = f''X$ by the statements mentioned above. Since every countable $Y \subseteq A$ is of the form $f''X$ where $f \in A$, we have $Ex(Y) = Y$ for all such Y and thus $Ex(A) = A$ by the last theorem. This contradicts $Ex(A) = V$.

Theorem. The class $Ex(X)$ is revealed for every $X \subseteq A$.

Proof. By the last but one theorem we can assume in our proof without loss of generality that X is a countable class. At first let us realize that

$$FN = \{\alpha \in A;(\forall u \in A)(u \hat{\approx} \alpha \rightarrow (\exists v \in A)(v = X \cap u))\}.$$

In fact if $u \hat{\approx} n \,\&\, u \in A$ then $X \cap u \in Fin$ and hence $X \cap u \in A$. On the other hand for every $\alpha \in A - FN$ there is $u \in A$ such that $X \subseteq u \,\&\, u \hat{\approx} \alpha$ because A is an endomorphic universe. In this case $X \cap u = X$ is a countable proper class.

Let $Y \subseteq Ex(X)$ and let Y be at most countable. Using the last theorem we can choose $\gamma \in Ex(FN) - FN$ thus by our assumptions the formula $(\forall u)(u \hat{\approx} \gamma \rightarrow (\exists v)(v = Ex(X) \cap u))$ holds. Moreover there is u with $u \hat{\approx} \gamma \,\&\, Y \subseteq u$ and therefore there is v so that $v = u \cap Ex(X)$. Hence we have found v with $Y \subseteq v \subseteq Ex(X)$ which finishes the proof.

We are going to prove a little stronger result.

A class X is called fully revealed if for every normal formula $\varphi(z,Z)$ of the language FL, the class $\{x; \varphi(x,X)\}$ is revealed.

Let us note that every set-theoretically definable class is fully revealed and that each fully revealed class is revealed. Moreover if X is fully revealed and if F is an automorphism then $F''X$ is fully revealed, too.

Theorem. The class $Ex(X)$ is fully revealed for every $X \subseteq A$.

Proof. If a normal formula $\varphi(z,Z)$ of the language FL is given then $\{x; \varphi(x,Ex(X))\} = Ex(\{x \in A; \varphi^A(x,X)\})$ by the first theorem of this section and the class in question is revealed according to the last theorem.

Theorem. Let X be a fully revealed class and let $\varphi(z,Z)$ be a normal formula of the language FL_V (i.e. we admit arbitrary sets as parameters). Then the class $\{x; \varphi(x,X)\}$ is fully revealed.

Proof. Evidently it is sufficient to show that all classes of the above described form are revealed. Let $\psi(z,z_1,\dots,z_n,Z)$ be a normal formula of the language FL and let $a_1,\dots,a_n$ be parameters such that $\varphi(z,Z)\equiv\psi(z,a_1,\dots,a_n,Z)$. We have $\{x;\varphi(x,X)\} = \{x;\psi(x,a_1,\dots,a_n,X)\} = \mathrm{rng}(\{\langle x,x_1,\dots,x_n\rangle;\psi(x,x_1,\dots,x_n,X)\}\cap(V\times\{a_1,\dots,a_n\}))$. Therefore the investigated class is revealed as the range of intersection of two revealed classes (cf. § 5 ch. II).

From the previous statement we obtain the following result using an appropriate formula and coding finite sequence of classes by a class.

Consequence. If $\varphi(z,Z_1,\dots,Z_n)$ is a normal formula of the language FL_V (!) and if $X_1,\dots,X_n\subseteq A$ then the class $\{x;\varphi(x,Ex(X_1),\dots,Ex(X_n))\}$ is fully revealed.

<u>Theorem</u>. The class $Ex(X) - X$ is revealed for every $X\subseteq A$.

Proof. Let be given a countable Z with $Z\subseteq Ex(X) - X$. According to the third theorem of this section there is $Y\subseteq X$ so that Y is at most countable and $Z\subseteq Ex(Y)$. Since Ex(Y) is revealed, there is u with $Z\subseteq u\subseteq Ex(Y)$ and with $Y\cap u = 0$. Moreover $Ex(Y)\cap X = Y$ by the second property of Ex in our list and hence $u\cap X = u\cap(Ex(Y)\cap X) = u\cap Y = 0$.

<u>Theorem</u>. Ex(Def) is an endomorphic universe.

Proof. Let $\varphi(z,z_1,\dots,z_n)$ be a set formula of the language FL. Then the formula

$$(\forall x_1,\dots,x_n\in Def)((\exists x\in A)\ \varphi^A(x,x_1,\dots,x_n)\rightarrow$$
$$\rightarrow(\exists x\in Def)\ \varphi^A(x,x_1,\dots,x_n))$$

holds because A is an endomorphic universe. Hence we get

$(\forall x_1,\dots,x_n\in Ex(Def))((\exists x)\ \varphi(x,x_1,\dots,x_n)\rightarrow$

$\longrightarrow (\exists x \in Ex(Def)\ \varphi(x,x_1,\ldots,x_n))$

Thus a use of a statement of the first section and of the fact that Ex(Def) is revealed finishes the proof.

Theorem. If $u \subseteq A$ then u is a finite set.

Proof. Let $u \subseteq A$ be an infinite set and let X be a countable subclass of u. Since Ex(X) is a revealed class, there is v such that $X \subseteq v \subseteq Ex(X)$. Hence $X = A \cap Ex(X) = u \cap Ex(X) = u \cap v$ which contradicts the assumption that X is a proper class.

As an immediate consequence we get $P(A) \subseteq A$ and thence $P^A(A) = P(A)$. Therefore we obtain

Theorem. For every $X \subseteq A$ it is $Ex(P(X)) = P(Ex(X))$.

Theorem. If X is a countable subclass of A then $Ex(X) = \bigcap\{u \in A; X \subseteq u\}$.

Proof. If $X \subseteq u \in A$ then $X \subseteq u \cap A$ and hence $Ex(X) \subseteq Ex(u \cap A) = u$. To prove the converse inclusion let us assume that $y \in Ex(Y) \cap \bigcap\{u \in A; X \subseteq u\}$ and that Y is a subclass of A which is at most countable. There are $u_1, u_2 \in A$ such that $u_1 \cap u_2 = 0\ \&\ X \subseteq u_1\ \&\ (Y - X) \subseteq u_2$. Evidently we have $y \in u_1$. Since $y \notin u_2$, the formula $y \notin Ex(Y - X)$ follows from the first part of the proof. However, this implies $y \in Ex(X)$.

From the last theorem and from a theorem we have proved before we get

Theorem. If $X \subseteq A$ then $Ex(X) = \bigcup\{\bigcap\{u \subseteq A; Y \in u\}; Y \subseteq X\ \&\ Y \preccurlyeq FN\}$.

In particular, there is at most one standard extension on every endomorphic universe. The following result is a consequence of the last theorem and of the formula $Ex(A) = V$.

Theorem. If there is a standard extension on an endomorphic universe A then for every x there is X which is at most countable and which is a subclass of A such that the formula $(\forall u \in A)(X \subseteq u \longrightarrow x \in u)$ holds.

In the next section we are going to prove that the property mentioned in the last theorem is also sufficient for the existence of a standard extension.

Let us realize that there are endomorphic universes which do not have standard extensions. As an example can serve each revealed endomorphic universe different from V (since it has infinite subsets) or each endomorphic universe which is a semiset (according to the last results).

§ 3. Existence of standard extension

If A is an endomorphic universe then for every $X \subseteq A$ we put $E_A(X) = \bigcap \{u \in A; X \subseteq u\}$.

Theorem. Let X and Y be at most countable subclasses of an endomorphic universe A. Then we have

(1) $E_A(X \cup Y) = E_A(X) \cup E_A(Y)$

(2) $X \cap Y = 0 \equiv E_A(X) \cap E_A(Y) = 0$

(3) $E_A(\mathrm{dom}(X)) = \mathrm{dom}(E_A(X))$

(4) $E_A(X \times Y) = E_A(X) \times E_A(Y)$

(5) If $\varphi(z)$ is a set-formula of the language FL_A then the equality $E_A(\{x \in X; \varphi(x)\}) = \{x \in E_A(X); \varphi(x)\}$ holds.

Proof. (1) One inclusion follows from the trivial statement $X \subseteq Y \longrightarrow E_A(X) \subseteq E_A(Y)$. If $x \notin E_A(X) \cup E_A(Y)$ then there are $u, v \in A$ with $X \subseteq u \,\&\, Y \subseteq v \,\&\, x \notin u \cup v$. Since we have $X \cup Y \subseteq u \cup v$ we get $x \notin E_A(X \cup Y)$ which finishes the proof.

(2) If X and Y are disjoint then there are $u,v \in A$ which are also disjoint and such that $X \subseteq u \& Y \subseteq v$. Thus we have $E_A(X) \cap E_A(Y) \subseteq u \cap v = 0$. The converse implication is again trivial.

(3) If $x \in dom(E_A(X))$ then there is y so that $\langle y,x \rangle \in \in E_A(X)$ and moreover there is $v \in A$ with $rng(X) \subseteq v$ because X is at most countable. Let $dom(X) \subseteq u \in A$ be given. Then $X \subseteq v \times \times u$ and hence $\langle y,x \rangle \in v \times u$ from which $x \in u$ follows.

Conversely let us assume $x \in E_A(dom(X))$. Let $F \subseteq X$ be a function satisfying $dom(F) = dom(X)$. Since A is an endomorphic universe there is a function $f \in A$ with $F \subseteq f$. If $u \in A$ satisfying $F \subseteq u$ is given then we have $x \in dom(u \cap f)$ and hence $\langle f(x),x \rangle \in u$. Therefore we have shown that $\langle f(x),x \rangle \in E_A(F)$ which implies $x \in dom(E_A(X))$.

(4) We have $(E_A(X))^{-1} = E_A(X^{-1})$ because $X \subseteq u \longrightarrow X^{-1} \subseteq \subseteq u^{-1}$. Hence $E_A(X \times Y) \subseteq E_A(X) \times E_A(Y)$ follows from (3). To prove the converse inclusion let us suppose that $u \in A$ with $X \times Y \subseteq \subseteq u$ is given. The class $Z = \{u"\{z\}; z \in X\}$ is a subclass of A which is at most countable and moreover $Y \subseteq \bigcap Z$ holds. Since even Y is at most countable there is $v \in A$ such that $Y \subseteq v \subseteq \bigcap Z$ (cf. § 4 ch. I). Put $w = \{z; v \subseteq u"\{z\}\}$. Then $X \subseteq w \in A$ and at the end we get $E_A(X) \times E_A(Y) \subseteq w \times v \subseteq u$ which finished the proof.

(5) Let $y \in E_A(\{x \in X; \varphi(x)\})$ and let $X \subseteq u \in A$. Then $\{x \in X; \varphi(x)\} \subseteq \{x \in u; \varphi(x)\}$ and hence $y \in \{x \in u; \varphi(x)\}$ and the formula $y \in E_A(X) \& \varphi(y)$ is a consequence of the last statement. Conversely let us suppose that the formula $y \in E_A(X) \& \& \varphi(y)$ holds. There is $v \in A$ so that $X \subseteq v$. Put $w = v \cap \{x; \neg \varphi(x)\}$. If $\{x \in X; \varphi(x)\} \subseteq u \in A$ then $X \subseteq u \cup w$ and therefore $y \in u \cup w$. By the definition of w, $\varphi(y)$ implies $y \notin w$ and thence

$y \in u$. Therefore we have proved $y \in E_A(\{x \in X;\ \varphi(x)\})$.

Theorem. An endomorphic universe A has a standard extension iff $V = \bigcup\{E_A(X); X \subseteq A \ \&\ X \preccurlyeq FN\}$.

Proof. In the last section we have proved that from the existence of a standard extension on A the condition mentioned in the theorem follows. To prove the converse implication for every $X \subseteq A$ we put

$Ex(X) = \bigcup\{E_A(Y); Y \subseteq X \ \&\ Y \preccurlyeq FN\}$.

Thus we obtain

(0′) $Ex(A) = V$

Proofs of the following easy consequences of the definition of Ex and of the last theorem are left to the reader (in the case (3′) we use the formula $Y \subseteq dom(X) \longrightarrow (\exists F \subseteq X)\ dom(F) = Y$). For $X, Y \subseteq A$ we have

(1′) $Ex(X \cup Y) = Ex(X) \cup Ex(Y)$

(2′) $X \cap Y = 0 \equiv Ex(X) \cap Ex(Y) = 0$

(3′) $Ex(dom(X)) = dom(Ex(X))$

(4′) $Ex(X \times Y) = Ex(X) \times Ex(Y)$

(5′) If $\varphi(z)$ is a set-formula of the language FL_A then the equality $Ex(\{x \in X;\ \varphi(x)\}) = \{x \in Ex(X);\ \varphi(x)\}$ holds.

By the first theorem of the second section it remains to prove that if $\varphi(z_1,\dots,z_k,Z_1,\dots,Z_n)$ is a normal formula of the language FL_A and if $X_1,\dots,X_n \subseteq A$ then we have

$Ex(\{\langle x_1,\dots,x_k\rangle \in A;\ \varphi^A(x_1,\dots,x_k,X_1,\dots,X_n)\}) =$

$= \{\langle x_1,\dots,x_k\rangle;\ \varphi(x_1,\dots,x_k,Ex(X_1),\dots,Ex(X_n))\}$.

A proof of this equality can be done by induction. If φ is an atomic formula of the form $x_i \in x_j$ or of the form $x_i \in X_j$ then we can use statements (0′),(4′) and (5′). It is sufficient to deal with these atomic formulas only, since the atomic formula $x_i = x_j$ can be reduced to the first type because $x_i = x_j \equiv (\forall z)(z \in x_i \equiv z \in x_j)$. The induction step for negation and for conjunction follows from (1′) and (2′) since using the usual set-theoretical considerations we obtain from these statements equalities $Ex(X - Y) = Ex(X) - Ex(Y)$ and $Ex(X \cap Y) = Ex(X) \cap Ex(Y)$. The induction step for existencial quantifiers is a consequence of (3′).

<u>Theorem</u>. Let A be an endomorphic universe, let X be its subclass which is at most countable and let $d \in E_A(X)$. Then $A[d] \subseteq \bigcup \{E_A(Y); Y \subseteq A \& Y \preccurlyeq FN\}$.

Proof. Let $f \in A$ and $d \in dom(f)$. Then $f''X \subseteq A \& f''X \preccurlyeq FN$ and we are going to prove that $f(d) \in E_A(f''X)$. At first let us realize that $d \notin E_A(X - dom(f))$ because $d \in dom(f)$. Hence according to (1) of the first theorem of this section we get $d \in E_A(X \cap \cap dom(f))$. For every $u \in A$ with $f''X \subseteq u$ we have $X \cap dom(f) \subseteq \subseteq f^{-1}''f''X \subseteq f^{-1}''u \in A$ and therefore $d \in f^{-1}''u$. Thus we have proved our claim and the theorem is its immediate consequence.

<u>Theorem</u>. Let $\mathcal{M}$ be an ultrafilter on the ring of all set-theoretically definable classes. Let F be an endomorphism and let F, $\mathcal{M}$, d be coherent (see § 2 ch. V). Put $A = F''V$. Then we have

$$(\exists X)(X \subseteq A \& X \preccurlyeq FN \& d \in E_A(X)) \equiv$$
$$\equiv (\exists Y)(Y \preccurlyeq FN \& (\forall u)(Y \subseteq u \rightarrow u \in \mathcal{M})).$$

Proof. Let us suppose at first that there is a class X with $X \subseteq A \& X \preccurlyeq FN \& d \in E_A(X)$. Put $Y = F^{-1}''X$. Evidently Y is at most countable. For every u with $Y \subseteq u$ we have $X \subseteq F(u)$ and therefore $d \in F(u)$. Finally we get $u \in \mathcal{M}$ as a consequence of the statement $d \in F(u) \equiv \{x; x \in u\} \in \mathcal{M}$.

Conversely let us assume that there is a class Y such that the formula $Y \precsim FN \,\&\, (\forall u)(Y \subseteq u \rightarrow u \in \mathfrak{M})$ holds. Put $X = F"Y$. Obviously X is a subclass of A which is at most countable. To prove that $d \in E_A(X)$ let us suppose that $X \subseteq u \in A$ is given. Thus we have $Y = F^{-1}"X \subseteq F^{-1}(u)$ and therefore $F^{-1}(u) \in \in \mathfrak{M}$. However, $\{x; x \in F^{-1}(u)\} \in \mathfrak{M}$ implies $d \in u$.

Theorem. Let $\mathfrak{M}$ be an ultrafilter on the ring of all set-theoretically definable classes and let $\mathfrak{M}$ contain a set. Moreover let us suppose that $0, \mathfrak{M}, d$ are coherent. Then there is an endomorphism F such that $F, \mathfrak{M}, d$ are coherent and such that the equality $(F"V)[d] = V$ holds.

Proof. Let us choose $u \in \mathfrak{M}$ and let F_0 be an endomorphism such that $F_0, \mathfrak{M}, d$ are coherent (the existence of such endomorphism was proved in § 2 ch. V). Then we have $\{x; x \in u\} \in \in \mathfrak{M}$ and therefore d is an element of $F_0(u)$. From here the formula $d \in \bigcup(F_0"V)$ follows and hence $(F_0"V)[d]$ is an endomorphic universe by the first section. Thus there is an endomorphism (F_1, say) such that $F_1"V = (F_0"V)[d]$. According to § 1 ch. V we can choose an automorphism F_2 with $F_2(F_1^{-1}(d)) = d$. Let G be the composition of F_2 and F_1^{-1} and let F denote the composition of G and F_0. Evidently G is a similarity, F is an endomorphism and moreover it holds $G(d) = d$.

At first we show that $F, \mathfrak{M}, d$ are coherent. If $\varphi(z, z_0, \ldots, z_n)$ is a set-formula of the language FL then we have $\{x; \varphi(x, x_1, \ldots, x_n)\} \in \mathfrak{M} \equiv \varphi(d, F_0(x_1), \ldots, F_0(x_n)) \equiv$
$\equiv \varphi(G(d), G(F_0(x_1)), \ldots, G(F_0(x_n))) \equiv \varphi(d, F(x_1), \ldots, F(x_n))$.

Thus it remains to prove that $(G"V)[d] = V$. For arbitrary y we have $G^{-1}(y) \in (F_0"V)[d]$ and therefore there is $f \in F_0"V$ so that $f(d) = G^{-1}(y)$. Hence we get $G(f) \in G"F_0"V = F"V$ and moreever $G(f)(d) = y$ because G is a similarity.

The previous statements enable us to obtain a lot of endomorphic universes having standard extension. So for example we can choose a non-trivial ultrafilter $\mathfrak{M}$ on the ring of all set-theoretically definable classes with $\alpha \notin FN \rightarrow \alpha \in \mathfrak{M}$. By § 2 ch. V we are able to choose further d such that $0, \mathfrak{M}$, d are coherent. Thus the last theorem assures the existence of an endomorphism F so that $F, \mathfrak{M}$,d are coherent and such that if we put $A = F"V$ then we have moreover $A[d] = V$. By the last but one theorem there is a class X with $X \subseteq A \& X \precsim FN \& d \in E_A(X)$. Hence according to the third theorem of this section we have $V = A[d] \subseteq \bigcup \{E_A(Y); Y \subseteq A \& Y \precsim FN\}$ and thence at the end there is a standard extension on A as a consequence of the second theorem of this section. Let us note that $A \neq V$ since $\mathfrak{M}$ is non-trivial and therefore $d \notin A$.

Another way how to construct an endomorphic universe having a standard extension is described in the following theorem.

<u>Theorem</u>. If $d \notin Def$ then there is an endomorphic universe such that $d \notin A$, $A[d] = V$ and $d \in E_A(Def)$.

Proof. Let $\mathfrak{N}$ denote the class of all classes of the form $\{x; \varphi(x)\}$ where $\varphi(z)$ is a set-formula of the language FL for which $\varphi(d)$ holds. Obviously if $\mathfrak{M}$ is a non-trivial ultrafilter with $\mathfrak{N} \subseteq \mathfrak{M}$ then $0, \mathfrak{M}$,d are coherent. Moreoever if $\varphi(z)$ is a set-formula such that $\varphi(d)$ is satisfied then the formula $(\exists x) \varphi(x)$ holds, therefore we have $(\exists x \in Def) \varphi(x)$. This enables us to choose a non-trivial ultrafilter $\mathfrak{M}$ satisfying $\mathfrak{N} \subseteq \mathfrak{M}$ and $\{u; Def \subseteq u\} \subseteq \mathfrak{M}$. According to the last theorem there is an endomorphism F such that $F, \mathfrak{M}$,d are coherent and if we put $A = F''V$ then $A[d] = V$. Since $\mathfrak{M}$ is non-trivial we have $d \notin A$. For every $Def \subseteq u \in A$ it is $Def = F^{-1}''Def \subseteq F^{-1}(u) \in \mathfrak{M}$ i.e. $\{x; x \in F^{-1}(u)\} \in \mathfrak{M}$. Using the assumption that $F, \mathfrak{M}, d$ are coherent we get $d \in u$. Hence we have proved $d \in E_A(Def)$.

Consequence. $Def = \bigcap \{A;\ A$ is an endomorphic universe$\}$.

Thus we see that there is no minimal endomorphic universe. Moreover we can construct two endomorphic universes the intersection of which is the class Def and hence the intersection of two endomorphic universes need not be an endomorphic universe.

Theorem. If A is an endomorphic universe having a standard extension then there is an endomorphic universe B so that $A \cap B = Def$.

Proof. Put $B = E_A(Def)$ and use the results of the previous section.

R e f e r e n c e s

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