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ON THE REGULARITY OF WEAK SOLUTIONS TO NONLINEAR ELLIPTIC SYSTEMS VIA LIOUVILLE'S TYPE PROPERTY

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Abstract: Let u be a weak solution with bounded gradient of a nonlinear elliptic system. In the present paper it is proved that the first derivatives of u are Hölder-continuous if the system satisfies a Liouville's type condition. This condition, roughly speaking, means that every solution defined on the whole R^n to the system and with bounded gradient is a polynomial of at most first degree.

Key words: Regularity, weak solution, nonlinear elliptic system, Liouville's property, Sobolev space.

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§ 1. Introduction. Let $\Omega \subset R^n$, $n \geqq 2$, be a bounded domain and let us consider a nonlinear elliptic system

$$(1.1)\quad -\frac{\partial}{\partial x_i}[a_i^r(x,u,\nabla u)] + a^r(x,u,\nabla u) = -\frac{\partial f_i^r}{\partial x_i} + f^r,$$

$r = 1,2,\dots,m$, where $u \in [W^{1,\infty}(\Omega)]^m$, ∇u is the set of the derivatives $\frac{\partial u_s}{\partial x_j}$, $a_i^r(x,\xi,\eta)$, $\frac{\partial a_i^r}{\partial x_\ell}$, $\frac{\partial a_i^r}{\partial \xi_s}$, $\frac{\partial a_i^r}{\partial \eta_j^s}$ are continuous functions on $\Omega \times R^m \times R^{nm}$, $f_i^r \in W^{1,p}(\Omega)$, $f^r \in W^{1,\frac{p}{2}}(\Omega)$, $p > n$,

$$(1.2)\quad \frac{\partial a_i^r}{\partial \eta_j^s}(x,\xi,\eta)\eta_i^r\eta_j^s > 0 \text{ for } \eta \neq 0,$$

and the summation convention is used.

Here $W^{k,p}(\Omega)$ denotes, as usual, the Sobolev space of

$L^p(\Omega)$ functions whose derivatives up to order k are also $L^p(\Omega)$ functions.

We say that (1.1).(1.2) is a <u>regular system</u> (R) if a weak solution u belongs to the space $[C^{1,\alpha}(\Omega)]^m$, where, of course, $C^{1,\alpha}(\Omega)$ is the space of continuously differentiable functions in Ω whose derivatives are locally α-Hölder continuous,

The history of the regularity problem is described in the book by O.A. Ladyženskaja, N.N. Ural'ceva [1], in the paper by Ch.B. Morrey [2] abd elsewhere. It is well known from the result of E. De Giorgi [3] that, for m = 1, the single equation (1.1),(1.2) is regular. By virtue of a counter example of J. Nečas [4], there exist systems (1.1),(1.2) which are not regular for $n \geq 5$; this question is still open for n = = 3,4. Sufficient conditions for the regularity are also of interest, see M. Giaquinta [5], J. Nečas [6]. Since the examples of the regularity are $[W^{1,\infty}(R^n)]^m$ solutions to a system

$$(1.3) \qquad -\frac{\partial}{\partial x_i}[a_i^r(\nabla u)] = 0$$

of the type

$$(1.4) \qquad |x - x_0|\, g\left(\frac{x-x_0}{|x-x_0|}\right),$$

and in virtue of a trivial fact that a $C^1(R^n)$ vector function of the type (1.4) is a polynomial of at most first degree, we see that the regularity implies weak Liouville's property: we say that the system (1.1),(1.2) has <u>weak Liouville's property</u> (WL), if for every $x^0 \in \Omega$, $\xi \in R^m$, every function v with a bounded gradient of the type (1.4), solving in R^n the system

$$(1.5)\qquad -\frac{\partial}{\partial x_i}[a_i^r(x^0, \xi, \nabla v)] = 0,$$

is a polynomial (more exactly, a vector of polynomials) of at most first degree. We speak about Liouville's property (L) if the same is true without supposing (1.4).

We prove in this paper by the "partial regularity" method, see Ch. B. Morrey [7], E. Giusti, M. Miranda [8], E. Giusti [9], M. Giaquinta [5], that (L) $\Longrightarrow$ (R). In this connection 3! relations can be thought of between (R), (WL), (L) (some are trivial), especially (WL) $\overset{?}{\Longrightarrow}$ (R), (R) $\overset{?}{\Longrightarrow}$ (L).

Considering the solutions to (1.3) in the form (1.4), we can get, see J. Nečas [6], that, for m = 1, n $\geqq$ 2, we have (WL). Because there is still some hope that for n = 3,4 we get (R) for the systems (1.1),(1.2) it is not unthinkable that we have the property (L) for n = 3,4, which would be a way how to prove this conjecture.

Clearly there are many other interesting questions, as, for example, how to avoid the condition $u \in [W^{1,\infty}(\Omega)]^m$; this seems to be possible via some growth conditions.

We also prove (in § 3 of this paper) an easy result that for the systems (1.1).(1.2) and for n = 2, the property (L) is satisfied. So we get once more the known result that for n = 2 we have (R).

§ 2. Lemmas. Let us first introduce some notation: put

$$u_{x_0,R} = \frac{1}{\operatorname{mes} B_R(x_0)} \int_{B_R(x_0)} u(x)dx,$$

where $B_R(x_0)$ is the ball with the center x_0 and the radius R and

$$U(x_0,R) = R^{-n} \int_{B_R(x_0)} |u(x) - u_{x_0,R}|^2\, dx.$$

Let us mention the result of S. Campanato [10]: if

$$u \in [W^{1,2}_{loc}(B(0,1)) \cap L^2(B(0,1))]^m$$

is a weak solution to the equation with constant coefficients

$$\int_{B(0,1)} b^{hk}_{ij} D_i u_h D_j \psi_k \, dx = 0, \qquad \forall \psi \in [\mathcal{D}(B(0,1))]^m,$$

then for every $0 < \rho < 1$ we have

(*) $U(0,\rho) \leq c\,\rho^2\, U(0,1)$, where c depends on $\max|b^{hk}_{ij}|$

and on the constant α of ellipticity:

$$b^{hk}_{ij}\, \eta^h_i\, \eta^k_j \geq \alpha\, |\eta|^2 .$$

First we get a modification of the main lemma from [8], [9],[5].

<u>Lemma 2.1.</u> Let $v \in [L^\infty(\Omega)]^N \cap [W^{1,2}(\Omega)]^N$ be a weak solution to the system

$$(2.1) \quad \int_\Omega [A^{hk}_{ij}(x,v) D_i v_h D_j \varphi_k + A^{hk}_i(x,v) D_i v_h \varphi_k] dx =$$

$$= \int_\Omega [g^k_j D_j \varphi_k + g^k \varphi_k] dx,$$

where $\Omega \subset R^n$ is a bounded domain, $A^{hk}_{ij}(x,\xi)$, $A^{hk}_i(x,\xi)$ are continuous functions in $\bar\Omega \times R^N$, $g^h_j \in L_p(\Omega)$, $g^k \in L_{\frac{p}{2}}(\Omega)$, $p > n$,

$$(2.2) \qquad A^{hk}_{ij}(x,\xi)\, \eta^h_i\, \eta^k_j > 0 \text{ for } \eta \neq 0.$$

If $x_0 \in \Omega$ and $R \leq \operatorname{dist}(x_0,\partial\Omega)$ we put $v = v^* + w$, where $w \in H^1_0(B(x_0,R))$ is a solution to

$$(2.2') \quad \int_{B(x_0,R)} [A^{hk}_{ij}(x,v) D_i w_h D_j \varphi_k + A^{hk}_i(x,v) D_i w_h \varphi_k] dx =$$

$$= \int_{B(x_0,R)} [g_j^k D_h \varphi_k + g^k \varphi_k] dx.$$

Then for every τ, $0 < \tau < 1$, there exist $\varepsilon_0 = \varepsilon_0(\tau, |v|_\infty)$, $R_0 = R_0(\tau, |v|_\infty)$ such that if $R \leqq \min(R_0, \mathrm{dist}(x_0, \partial\Omega))$ and if

$$V^*(x_0, R) < \varepsilon_0^2 \tag{2.3}$$

then

$$V^*(x_0, \tau R) \leqq 2c\, \tau^2\, V^*(x_0, R), \tag{2.4}$$

where the constant c is from ($*$).

Proof. Let us suppose the contrary. Then $\exists \tau$, $x_\nu \in \Omega$, $\varepsilon_\nu \longrightarrow 0$, $R_\nu \longrightarrow 0$, $v^\nu \in [H^1(\Omega)]^N$, $|v^\nu|_{L_\infty} \leqq |v|_{L_\infty}$, such that $V^{*(\nu)}(x_\nu, R_\nu) = \varepsilon_\nu^2$, $V^{*(\nu)}(x_\nu, \tau R_\nu) > 2c\, \tau^2 \varepsilon_\nu^2$.
Put $x = x_\nu + R_\nu y$, $s^\nu(y) = \varepsilon_\nu^{-1} [v^{*\nu}(x_\nu + R_\nu y) - v^{*\nu}_{x_\nu, R_\nu}]$.
We have $\int_{B(0,1)} |s^\nu(y)|^2 dy = S^\nu(0,1) = 1$,

$$S^\nu(0, \tau) > 2c\, \tau^2. \tag{2.5}$$

Put further $t^\nu(y) = \omega^\nu(x_\nu + R_\nu y)$ $(v^\nu = v^{*\nu} + \omega^\nu)$. Then we can suppose $x_\nu \longrightarrow x_0 \in \bar{\Omega}$, $s^\nu \rightharpoonup s$ in $L_2(B(0,1))$, $\varepsilon_\nu s^\nu(y) \longrightarrow 0$ almost everywhere in $B(0,1)$. We have

$$v^\nu(x_\nu + R_\nu y) = s^\nu(y)\varepsilon_\nu + v^{*\nu}_{x_\nu, R_\nu} + t^\nu(y). \tag{2.6}$$

Since

$$\int_{B(x_\nu, R_\nu)} |\omega^\nu|^2 dx \leqq c_1 R_\nu^2 \int_{B(x_\nu, R_\nu)} D_i \omega_h^\nu D_i \omega_h^\nu dx, \tag{2.7}$$

we first get that (2.2) is uniquely solvable for R_ν small enough. We further get from (2.7) and (2.2) that

$$\int_{B(0,1)} |t^\nu(y)|^2 dy \leqq c_2 R^{2(1-\frac{n}{p})}, \tag{2.8}$$

so we can also suppose that $t^{\nu}(y) \longrightarrow 0$ almost everywhere. Hence from (2.6) it follows that we can suppose $v^{*\nu}_{x_\nu, R_\nu} \longrightarrow$ $\longrightarrow \xi \in R^N$ and therefore

$$A^{hk}_{ij}(x_\nu + R_\nu y, s^\nu(y)\varepsilon_\nu + v^{*\nu}_{x_\nu, R_\nu} + t^\nu(y)) \longrightarrow A^{hk}_{ij}(x^0, \xi)$$

almost everywhere in $B(0,1)$. Hence we get that $s^\nu \longrightarrow s$ in $[W^{1,2}_{loc}(B(0,1))]^N$ and that

$$(2.9) \qquad \int_{B(0,1)} A^{hk}_{ij}(x^0, \xi) D_i s_h D_j \psi_k \, dy = 0$$

$$\forall \psi \in [\mathcal{D}(B(0,1))]^N.$$

Thus we have

$$(2.10) \qquad S(0, \tau) \leq c\,\tau^2\, S(0,1) \leq c\,\tau^2,$$

which is a contradiction with

$$(2.11) \qquad S(0, \tau) > 2c\,\tau^2$$

obtained from (2.5).

<u>Lemma 2.2.</u> Under the conditions of Lemma 2.1, for every point $x_0 \in \Omega$ such that $V^*(x_0, R) < \varepsilon_0^2$, there exists a $B(x_0, R_1) \subset \Omega$ such that $v \in C^{\alpha}(\overline{B(x_0, R_1)})$ with $\alpha = \min(\frac{1}{2}, 1 - \frac{n}{p})$.

<u>Proof.</u> We get by a standard argument that if $\sigma > 0$ is small enough, $|\bar{x} - x_0| < \sigma$, and $R_{\bar{x}} = R - |\bar{x} - x_0|$, then $V^*(\bar{x}, R_{\bar{x}}) < \varepsilon_0^2$. If $v = v^* + \omega$ in $B(\bar{x}, R_{\bar{x}})$, we first have

$$(2.12) \qquad \int_{B(\bar{x}, R_{\bar{x}})} |\omega|^2 dx \leq c_1 R^2_{\bar{x}} \int_{B(\bar{x}, R_{\bar{x}})} |D\omega|^2 dx \leq$$

$$\leq c_2 R^2_{\bar{x}} R_{\bar{x}}^{n(1-\frac{2}{p})} \Big[\sum \Big[\int_B |f_i^r|^p dx \Big]^{\frac{2}{p}} +$$

$$+ \sum \Big(\int_B |f^r|^{\frac{p}{2}} dx \Big)^{\frac{4}{p}} \Big] \leq c_3 R_{\bar{x}}^{2+n-\frac{2n}{p}} .$$

Thus

$$(2.13)\quad V(\bar{x}, \tau R_{\bar{x}}) \leq 2V^{*}(\bar{x}, \tau R_{\bar{x}}) + 2\Omega(\bar{x}, \tau R_{\bar{x}}) \leq$$

$$\leq 4c\,\tau^{2} V^{*}(\bar{x}, R_{\bar{x}}) + 2c_3 R_{\bar{x}}^{2(1-\frac{n}{p})} \tau^{-n} \leq$$

$$\leq 8c\,\tau^{2} V(\bar{x}, R_{\bar{x}}) + 8c\,\tau^{2} c_3 R_{\bar{x}}^{2(1-\frac{n}{p})} + 2c_3 \tau^{-n} R_{\bar{x}}^{2(1-\frac{n}{p})}.$$

Choose $\tau \in (0,1)$ such that $8c\tau = \varkappa \leq 1$ and small enough. We get from (2.13) that

$$(2.14)\quad V(\bar{x}, \tau R_{\bar{x}}) \leq 8c\,\tau^{2}\, V(\bar{x}, R_{\bar{x}}) + c_4\, R_{\bar{x}}^{2(1-\frac{n}{p})}.$$

For k being a positive integer, we get from (2.14) that

$$(2.15)\quad V(\bar{x}, \tau^{k} R_{\bar{x}}) \leq \tau^{k}\, V(\bar{x}, R_{\bar{x}}) +$$

$$+ R_{\bar{x}}^{2(1-\frac{n}{p})}\, c_4\, \frac{(\varkappa\tau)^{k} + \tau^{k\,2(1-\frac{n}{p})}}{|\varkappa\tau - \tau^{2(1-\frac{n}{p})}|}\ .$$

If $0 < \rho < R - \delta$ and if we choose k such that $\tau^{k+1} R_{\bar{x}} < \rho \leq \tau^{k} R_{\bar{x}}$, we get $\tau^{n} V(\bar{x}, \rho) \leq \left(\frac{\rho}{\tau^{k} R_{\bar{x}}}\right)^{n} V(\bar{x}, \rho) \leq V(\bar{x}, \tau^{k} R_{\bar{x}}) \leq \frac{\rho}{R_{\bar{x}}\tau} V(\bar{x}, R_{\bar{x}}) + c_4 R_{\bar{x}}^{2(1-\frac{n}{p})} \frac{\frac{\rho}{R_{\bar{x}}\tau} + \left(\frac{\rho}{R_{\bar{x}}\tau}\right)^{2(1-\frac{n}{p})}}{|\varkappa\tau - \tau^{2(1-\frac{n}{p})}|}$, and using [10], we get the result, q.e.d.

§ 3. Main results

<u>Theorem 1</u>. Let $u \in [W^{1,\infty}(\Omega)]^{m}$ be a weak solution to (1.1) and let the conditions on a_i^r, a^r, f_i^r, f^r, Ω, mentioned in § 1, be fulfilled. Let the system (1.1) satisfy the Liouville's property, i.e., for $\forall x^{0} \in \Omega$ and $\forall \xi \in R^{m}$ the only solution to (1.5) defined in the whole R^{n} and possessing a bounded gradient is a polynomial of at most first degree.

Then $u \in [C^{1,\alpha}(\Omega)]^m$, $\alpha = \min(\frac{1}{2}, 1 - \frac{n}{p})$.

Proof. Let $x^0 \in \Omega$. Put $u_R(y) = \frac{1}{R}[u(x^0 + Ry) - u(x^0)]$, $x^0 + Ry = x$. If O is the image of Ω we have

$$(3.1) \quad \int_O [a_i^r(x^0+Ry, Ru_R(y)+u(x^0), \nabla_y u_R(y)) \frac{\partial \psi_r(y)}{\partial y_i} +$$

$$+ a^r(x^0+Ry, Ru_R(y) + u(x^0), \nabla_y u_R(y)) R\psi_r(y)] dy =$$

$$= \int_O [f_i^r(x^0+Ry)\frac{\partial \psi_r}{\partial y_i}(y) + f^r(x^0+Ry)R\psi_r(y)]dy.$$

Let $B(0,a) \subset O$. We get in a standard way that

$$(3.2) \quad \int_{B(0,a)} |D^2 u_R(y)|^2 \, dy \leqq c(a).$$

Hence we can choose $R_k \longrightarrow 0$ in such a way, that $u_{R_k} \longrightarrow p$ in $[W^{1,2}(B(0,a))]^m$ $\forall a > 0$. Thus $p \in [W^{1,\infty}(R^n)]^m$ and it is a weak solution to

$$(3.3) \quad \int_{R^n} a_i^r(x^0, u(x^0), \nabla_y p)\frac{\partial \psi_r}{\partial y_i} \, dy = 0$$

$$\forall \psi \in [\mathcal{D}(R^n)]^m.$$

Therefore, by assumption, p is a polynomial of at most first degree. So we have

$$(3.3') \quad 0 \longleftarrow \int_{B(0,1)} |Du_{R_k}(y) - Dp|^2 \, dy = R_k^{-n} \int_{B(x^0, R_k)} |Du(x) - Dp|^2 dx.$$

If $'$ is the $\frac{\partial}{\partial x_t}$ derivative we get from (1.1) the equation in variations

$$(3.4) \int_\Omega [\frac{\partial a_i^r}{\partial \frac{\partial u_s}{\partial x_j}} \frac{\partial u_s'}{\partial x_j} \frac{\partial \varphi_r}{\partial x_i} + \frac{\partial a_i^r}{\partial u_s} u_s' \frac{\partial \varphi_r}{\partial x_i} + \frac{\partial a_i^r}{\partial x_t} \frac{\partial \varphi_r}{\partial x_i} +$$

$$+ \frac{\partial a^r}{\partial \frac{\partial u_s}{\partial x_j}} \frac{\partial u_s}{\partial x_j} \varphi_r + \frac{\partial a^r}{\partial u_s} u_s' \varphi_r + \frac{\partial a^r}{\partial x_t} \varphi_r] dx =$$

$$= \int_\Omega [f_i^{r'} \frac{\partial \varphi_r}{\partial x_i} + f^{r'} \varphi_r] dx .$$

Writing (3.4) for every $\frac{\partial}{\partial x_t}$, $t = 1,2,\dots,n$, removing the terms $\frac{\partial a_i^r}{\partial u_s} u_s' \frac{\partial \varphi_r}{\partial x_i}$, $\frac{\partial a_i^r}{\partial x_t} \frac{\partial \varphi_r}{\partial x_i}$, $\frac{\partial a^r}{\partial u_s} u_s' \varphi_r$, $\frac{\partial a^r}{\partial x_t} \varphi_r$ to the right-hand side of (3.4), and denoting by $v_{\varkappa}$ the derivatives $\frac{\partial u_s}{\partial x_t}$, we get, with $\frac{\partial a_i^r}{\partial \frac{\partial u_s}{\partial x_j}}(x,u(x),v) \equiv b_{ij}^{rs}(x,v)$ (and the same with a_i^r), a system of the type (2.1). The result follows from Lemmas 2.1, 2.2 and from (3.3′), because, in decomposing $v = v^* + \omega$ on $B(x^0,R)$ as in Lemma 2.1, we have $\Omega(x^0,R) \longrightarrow 0$ for $R \longrightarrow 0$, as above, so $V^*(x^0,R) \longrightarrow 0$, q.e.d.

Theorem 2. Let us consider the system (1.1),(1.2). Let n be the dimension of the space, n = 2. Then (L) is satisfied.

Proof. Let $v \in [W^{1,\infty}(R^2)]^m$ be a weak solution to the equation

$$(3.5) \qquad \int_{R^2} a_i^r(x^0, \xi, \nabla v) \frac{\partial \psi_r}{\partial y_i} dy = 0.$$

Let $T > 0$ and let $\eta \in \mathcal{D}(B(0,2T))$, $0 \leq \eta \leq 1$, $\eta = 1$ in $B(0,T)$, $|D_i \eta| \leq \frac{c_1}{T}$. We get the equation in variations

$$(3.6) \qquad \int_{R^2} \frac{\partial a_i^r}{\partial \frac{\partial v_s}{\partial x_j}}(x^0, \xi, \nabla v) \frac{\partial v_s'}{\partial y_j} \cdot \frac{\partial \psi_r}{\partial y_i} dy = 0.$$

Putting $\psi_r = v_r' \eta^2$, we get from (3.6), using the boundedness of the gradient, that

(3.7) $\int_{B(0,2T)} |Dv'|^2 \eta^2 \, dy \leq c_2.$

Hence $\int_{R^2} |Dv'|^2 \, dy < \infty$. But there exists $\psi^n \in [\mathcal{D}(R^2)]^m$ such that $D\psi^n \to Dv'$ in $[L^2(R^2)]^{2m}$ (and there exists $\Lambda^n \in R^m$ such that $\Lambda^n + \psi^n \to v'$ in $[L^2_{loc}(R^n)]^m$. Hence

$$\int_{R^2} \frac{\partial a_i^\kappa}{\partial \frac{\partial v_k}{\partial x_j}}(x^\circ, \xi, \nabla v) \frac{\partial v'_k}{\partial y_j} \frac{\partial v'_\kappa}{\partial y_i} \, dy = 0$$

and thus v is a polynomial of at most first degree.

References

[1] O.A. LADYŽENSKAJA, N.N. URAL'CEVA: Linejnye i kvazilinejnye uravnenija elliptičeskogo tipa, Moscow (1973), 2-nd edition,

[2] Ch.B. MORREY: Differentiability theorems for weak solutions of nonlinear elliptic differential equations, BAMS, Vol. 75(1969), 684-705.

[3] E. De GIORGI: Sulla differenziabilità e analiticità delle estremali degli integrali multipli regolari, Mem. Acad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3),3(1957), 25-43.

[4] J. NEČAS: Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity, Theory of Nonlinear Operators, Abhandlungen der Akademie der Wissenschaften der DDR, Jahrg. 1977, Nr. 1N, 197-206.

[5] M. GIAQUINTA: Sistemi ellittici non lineari, Convegno su: Sistemi ellittici non lineari ed applicazioni, Università di Ferrara, Editrice Universitaria, 1978,

[6] J. NEČAS: On the regularity of weak solutions to variational equations and inequalities for nonline-

ar second order elliptic systems, Proceedings of Equadiff IV, Prague 1977, to appear in Springer 1979,

[7] Ch.B. MORREY: Partial regularity results for nonlinear elliptic systems, Journ. Math. and Mech. 17(1968),

[8] E. GIUSTI; M. MIRANDA: Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi lineari, Arch. Rat. Mech. and Anal. 31(1968),

[9] E. GIUSTI: Regolarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario, Ann. Scuola Norm. Sup. Pisa 23(1969),

[10] S. CAMPANATO: Equazioni ellitiche del II ordine e spazi $L^{2,\lambda}$, Ann. Mat. Pura e Appl. 69(1965),

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