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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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A NOTE ON COFINAL EXTENSIONS AND SEGMENTS

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Abstract: We work with an extension U^{ω} of the theory U, where U is the theory of the directed antisymmetric relation with an arbitrary large transitive element.

We present a necessary and sufficient condition for a cofinal Δ_0-extension of a model of U^{ω} to be its elementary extension. We also show that the segment determined by an elementary submodel of a model of U^{ω} is elementarily equivalent with them. Finally, we give a necessary and sufficient condition for the existence of an elementary cofinal extension of a model of U^{ω}. We also present an extension T of U with the following property: each model of U, which is a cofinal Δ_0-extension of a model of T is its elementary extension.

Key words: Cofinal extension, elementary extension, segment, schema H (induction schema).

AMS: 02H05, 02H15

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§ 0. Introduction. In [3] we studied the theories U and S. S is the theory of a discrete linear ordering with the least element and without the last element. We obtained relations between the extension U^{ω} (S^{ω} resp.) and the theory U (S resp.) extended by the induction schema, and a necessary and sufficient condition for the existence of some types of end-extensions of countable models of the theory U (S resp.).

This note extends the results from [3] by the ones mentioned in the abstract. Variants of these results also hold for the theory S.

Note that the Zermelo-Fraenkel set theory ZF can be viewed as an extension of the theory U^{ω}, and the Peano arithmetic P as an extension of the theory S^{ω}. The results following for these theories from the theorems presented can be strengthened by using some further special properties of these theories. (See for ex. [1].) We mention some results for these theories in § 4.

§ 1. Notations and terminology. By a language we mean a first-order predicate language with =. Strings of variables are denoted by x, y,... . Writing $\bar{a} \in A$ we mean that $\bar{a}$ is a string of elements of the set A. i, j, k, m, n are variables for natural numbers and ω is the set of natural numbers.

If $T, \Gamma \subseteq Fm(L)$ we put $\Gamma^{T} = \{\varphi \in Fm(L);$ there is a $\psi \in \Gamma$ such that $T \vdash \varphi \equiv \psi\}$. Usually we identify Γ with $\Gamma^{log.ax.}$.

For $T, S \subseteq Fm(L)$ we write $T < S$ to indicate that $T \vdash \varphi$ implies $S \vdash \varphi$. Writing $T \equiv S$ we mean $T < S$ and $S < T$.

For a mapping $F: Fm(L) \longrightarrow Fm(L)$ and a set $\Gamma \subseteq Fm(L)$ we put $F(\Gamma) = \{F(\varphi); \varphi \in \Gamma\}$.

Let C be a set. Then L(C) is the language L augmented by a new individual constant $\underline{c}$ for each $c \in C$. Let $\Gamma \subseteq Fm(L)$. We put $\Gamma(C) = \{\varphi(\underline{c}_1, \dots); \varphi(x_1, \dots) \in \Gamma$, $c_1 \in C, \dots$ and $x_1, \dots$ are free in $\varphi\}$.

By $A \models L$ we mean that A is a structure (or model) for L. We often use the same symbol for a model of a language L and for its universum. Let C be a subset of the universe of a model A of L. Writing $C \models L$ we mean that there is a substructure of A with the universe C. Writing $a \in A$ ($\bar{a} \in A$ resp.) we indicate that a is an element of the universe of A ($\bar{a}$ is a string of elements of the universe of A resp.).

Assume that $A \models L$. Let X be a subset of the universe of A. Then $(A,a)_{a \in X}$ is the usual expansion of A to a structure for L(X). We shall identify the members of X with their names. If there is no danger of confusion we write A instead of $(A,a)_{a \in A}$.

Let A, B be structures for L and let $\Gamma \subseteq Fm(L)$. We say that A is a Γ-substructure of B if A is a substructure of B and, $A \models \varphi$ iff $B \models \varphi$ for each sentence $\varphi \in \Gamma(A)$. Writing $A \subset B$ we mean that A is a substructure of B (and B is an extension of A) while writing $A \prec B$ we mean that A is an elementary substructure of B (and B is an elementary extension of A).

Let L be a language containing a binary predicate $<$. Let $\bar{x}$ be a string $x_1, \ldots x_n$ of variables and x a variable. We write $(\exists \bar{x} < x)\varphi$ for $(\exists x_1 < x) \ldots (\exists x_n < x)\varphi$. Similarly with $\forall$. Let $A \models L$, $\bar{a} \in A$ and $a \in A$. Writing $\bar{a} < a$ we mean that the relation $b <^A a$ holds for each member b of the string $\bar{a}$.

We denote by Δ_0 the set of limited (w.r.t. $<$) formulas of the language L. We put $\Pi_0 = \Sigma_0 = \Delta_0$ and define by induction:

$$\Pi_{n+1} = \{(\forall \bar{x})\varphi ; \varphi \in \Sigma_n\},$$

$$\Sigma_{n+1} = \{(\exists \bar{x})\varphi ; \varphi \in \Pi_n\}.$$

Let A, B be models of L. We write $A \subset_n B$ to indicate that A is $\Pi_n \cup \Sigma_n$ substructure of B.

Let C be a subset of the universe of A. It is said to be a <u>segment in</u> A if it is closed under $<^A$. It is said to be cofinal in A if for each $a \in A$ there exists $c \in C$ such that $a <^A c$.

B is an end-extension of A if B is a proper extension of A and the universe of a is a segment in B. B is a cofinal extension of A if B is a proper extension of A and A is cofinal in B.

The set $\Gamma \subseteq Fm(L)$ is closed under limited quantification ($Clq(\Gamma)$) if $\varphi \in \Gamma$ implies $(\exists x<y)\varphi \in \Gamma$ and $(\forall x<y)\varphi \in \Gamma$. Evidently, $Clq(\Gamma)$ implies $Clq(\Gamma \cup \neg(\Gamma))$.

Let φ be a formula. Writing g.c.φ we mean the general closure of φ.

§ 2. Some properties of the theory U.

2.0.0. Let L be a language with a binary predicate $<$. We denote by $Tr(x)$ the formula $(\forall y<x)(\forall z<y)(z<x)$ (x is transitive). U (more precisely U(L)) is the theory in L with the axioms:

$$(\forall x,y)(\exists z)(x<z \mathbin{\&} y<z)$$
$$(\forall x)(\exists y)(x<y \mathbin{\&} Tr(y))$$
$$(\forall x,y)(x<y \rightarrow \neg(y<x)).$$

We have $U \vdash x<y \rightarrow x \neq y$ and, for each $\varphi \in Fm(L)$,

$$U \vdash (\forall \bar{x})\varphi \equiv (\forall x)(\forall \bar{x}<x)\varphi$$
$$U \vdash (\exists \bar{x})\varphi \equiv (\exists \bar{x})(\exists \bar{x}<x)\varphi .$$

Let φ be a formula of L, let $\bar{x},y$ be free in φ. We denote by $H(\varphi(\bar{x},y))$ the general closure of the formula

$$(\forall u)((\forall \bar{x}<u)(\exists y)\varphi \rightarrow (\exists v)(\forall \bar{x}<u)(\exists y<v)\varphi)$$

where u,v do not occur in φ. Writing $H(\varphi)$ we mean $H(\varphi(x,y))$ with some x,y free in φ.

For $n \in \omega$ and each theory T in L we put

$T^n = T \cup H(\Pi_n)$ and $T^{\omega} = \bigcup\{T^n; n \in \omega\}$.

2.0.1. Lemma. Let $n \geq 0$. Then $\Pi^{U^n}_{n+1}$ is closed under limited quantification (i.e. $Clq(\Pi^{U^n}_{n+1})$).

Proof. By induction on n. For n = 0. If $\varphi \in \Sigma_1$ then there is a $\psi \in \Delta_0$ such that $U \vdash \varphi \equiv (\exists y)\psi$. We have

$$U^0 \vdash (\forall x<u)\varphi \equiv (\forall x<u)(\exists y)\psi \equiv (\exists v)(\forall x<u)(\exists y<v)\psi,$$

and consequently $(\forall x<u)\varphi \in \Sigma^{U^0}_1$. The relation $(\exists x<u)\varphi \in \Sigma^{U^0}_1$ immediately follows. Suppose the proposition is true for some n. For $\varphi \in \Sigma_{n+2}$ we have some $\psi \in \Pi_{n+1}$ such that $U^n \vdash \varphi \equiv (\exists y)\psi$. This follows from the induction hypothesis. Thus,

$$U^{n+1} \vdash (\forall x<u)\varphi \equiv (\forall x<u)(\exists y)\psi \equiv (\exists v)(\forall x<u)(\exists y<v)\psi.$$

From this and from the induction hypothesis we obtain $(\forall x<u)\varphi \in \Sigma^{U^{n+1}}_{n+2}$. Now $(\exists x<u)\varphi \in \Sigma^{U^{n+1}}_{n+2}$ immediately follows.

§ 3. The main results and their corollaries.

3.0.0. Theorem. Let A, B be models of L. Let B be a cofinal Δ_0-extension of A and $A \models U^{\omega}$. Then

$$A \prec B \text{ iff } B \models U^{\omega}$$

This theorem is an immediate consequence of the following proposition.

3.0.1. Proposition. Let A, B be models of U and let B be a cofinal Δ_0-extension of A. Then

(0) $A \subset_1 B$,

(1) if $A \models U^0$ then $A \subset_2 B$.

(2) Let $B \models U^0$. Then for each $n \geq 0$ holds: if $A \models U^{n+1}$ then $A \subset_{n+3} B$ iff $B \models U^n$.

Proof. First, we shall prove the

3.0.2. <u>Lemma</u>. Let $n \geq 0$ and $\varphi(\bar{x},y,\bar{z}) \in \Pi_n$. Then

$$U^n \vdash H(\varphi(\bar{x},y)).$$

Proof by induction on the length of $\bar{x}$. Suppose the statement holds for $\bar{x}$ of a length m. Let $\varphi(x,\bar{x},y,\bar{z}) \in \Pi_n$ be a formula, where $\bar{x}$ has the length m. Let u,v,w do not occur in φ. Assume that $U^n \vdash (\forall x,\bar{x}< u)(\exists y)\varphi(x,\bar{x},y,\bar{z})$. From this and by using the induction hypothesis we obtain $U^n \vdash (\forall x< u)(\exists w)(\forall \bar{x}< u)(\exists y< w)\varphi$. Now, $(\forall \bar{x}< u)(\exists y< w)\varphi(x,\bar{x},y,\bar{z}) \in \Pi_n^{U^n}$. Thus,

$$U^n \vdash (\exists v)(\forall x< u)(\exists w< v)(\forall \bar{x}< u)(\exists y< w)\varphi(x,\bar{x},y,\bar{z})$$

holds. From this and by using the axiom $(\forall x)(\exists y)(x< y \,\&\, Tr(y))$ of the theory U we deduce that

$$U^n \vdash (\exists v)(\forall x< u)(\forall \bar{x}< u)(\exists y< v)\varphi(x,\bar{x},y,\bar{z}).$$

We shall prove the proposition. (0) Let $\psi \in \Sigma_1(A)$ be a sentence. Then there is a formula $\varphi(x) \in \Delta_0(A)$ such that $A \models \psi \equiv (\exists x)\varphi(x)$, $B \models \psi \equiv (\exists x)\varphi(x)$. If $A \models (\exists x)\varphi(x)$ then $B \models (\exists x)\varphi(x)$. Assume $B \models (\exists x)\varphi(x)$. Then there is an element $a \in A$ such that $B \models (\exists x< a)\varphi(x)$ and, consequently $A \models (\exists x< a)\varphi(x)$. Thus $A \models (\exists x)\varphi(x)$ holds. (1) Let $\varphi(\bar{x},\bar{y}) \in \Delta_0(A)$ be a formula with only free variables $\bar{x},\bar{y}$. Assume $A \models (\forall \bar{x})(\exists \bar{y})\varphi(\bar{x},\bar{y})$. Let $\bar{b} \in B$. Let $a \in A$ be such that $B \models \bar{b}< a$. We have $A \models (\forall \bar{x}< a)(\exists y)(\exists \bar{y}< y)\varphi(\bar{x},\bar{y})$. From this and 3.0.2 we deduce that there is a $c \in A$ such that $A \models (\forall \bar{x}< a)(\exists y< c)(\exists \bar{y} < y)\varphi(\bar{x},\bar{y})$. The last formula is a $\Delta_0(A)$-formula and, consequently, it holds true in B. Thus, $B \models (\forall \bar{x}< a)(\exists \bar{y})\varphi(\bar{x},\bar{y})$. Now, $B \models (\exists \bar{y})\varphi(\bar{b},\bar{y})$ follows immediately. Assume $A \models (\exists \bar{x})(\forall \bar{y})\varphi(\bar{x},\bar{y})$. Then there is an $\bar{a} \in A$ such that $A \models (\forall \bar{y})\varphi(\bar{a},\bar{y})$. Let $\bar{b} \in B$. Let $a \in A$ be such that $B \models \bar{b}< a$. We have $A \models (\forall \bar{y} <$

$< a)\varphi(\bar{a},y)$ and consequently $B\models(\forall\bar{y}< a)\varphi(\bar{a},y)$. Thus $B\models\varphi(\bar{a},\bar{b})$. The proposition (1) is proved. (2) By induction on n. n = 0: we suppose $A\models U^1$, $B\models U^o$, We have to prove that $A\subset_3 B$. Let $\varphi(\bar{x})\in\Sigma^{U^o}_2(A)$ with free variables $\bar{x}$ only. We can suppose that $\varphi(\bar{x})$ is of the form $(\exists y)\psi(\bar{x},y)$ with some $\psi\in\Pi_1(A)$. (By using 2.0.1.) Assume $A\models(\forall\bar{x})\varphi(\bar{x})$. We shall prove that $B\models(\forall\bar{x})\varphi(\bar{x})$. Let $\bar{b}\in B$. Let $a\in A$ be such that $B\models\bar{b}< a$. We have $A\models(\forall\bar{x}< a)(\exists y)\psi(\bar{x},y)$. Thus, there is a $c\in A$ such that $A\models(\forall\bar{x}< a)(\exists y< c)\psi(\bar{x},y)$ (by using 3.0.2). We have $(\forall\bar{x}< a)(\exists y< c)\psi\in\Sigma^{U^o}_1(A)$ (by using 2.0.1), and, consequently $B\models(\forall\bar{x}< a)(\exists y< c)\psi(\bar{x},y)$. We deduce from this $B\models\varphi(\bar{b})$. Assume $B\models(\forall\bar{x})\varphi(\bar{x})$. We shall prove that $A\models(\forall\bar{x})\varphi(\bar{x})$. Let $\bar{a}\in A$. We have $B\models(\exists y)\psi(\bar{a},y)$. We deduce $A\models(\exists y)\psi(\bar{a},y)$ from part (1) of 3.0.1. The case n = 0 is proved. Assume that (2) holds for an n. Let $A\models U^{n+1+1}$ and $B\models U^o$. We shall prove that $A\subset_{n+1+3} B$ implies $B\models U^{n+1}$. First, we obtain $B\models U^n$ from the induction hypothesis. If $\varphi\in\Pi_{n+1}$, then $H(\varphi)\in\Pi^{U^n}_{n+4}$. From this (and by using the hypothesis on A,B) we deduce that $B\models H(\Pi_{n+1})$, and, consequently, $B\models U^{n+1}$. To finish the proof we must show: if $B\models U^{n+1}$ then $A\subset_{n+4} B$.

Let $B\models U^{n+1}$. We deduce from the induction hypothesis that $A\subset_{n+3} B$. Let $\varphi(\bar{x})\in\Sigma^{U^{n+1}}_{n+3}(A)$ be a formula with free variables $\bar{x}$ only. We can suppose that $\varphi(\bar{x})$ is of the form $(\exists y)\psi(\bar{x},y)$ with some $\psi\in\Pi_{n+2}(A)$ (by using 2.0.1). We are going to prove that $A\models(\forall\bar{x})\varphi(\bar{x})$ iff $B\models(\forall\bar{x})\varphi(\bar{x})$. Obviously, if $B\models(\forall\bar{x})\varphi(\bar{x})$ then $A\models(\forall\bar{x})\varphi(\bar{x})$. Assume that $A\models(\forall\bar{x})\varphi(\bar{x})$ and let $\bar{b}\in B$. Let $a\in A$ be such that $B\models\bar{b}< a$. We have $A\models(\forall\bar{x}< a)(\exists y)\psi$. Thus, $A\models(\forall\bar{x}< a)(\exists y< c)\psi(\bar{x},y)$

holds with some $c \in A$. (This follows from $A \models U^{n+2}$ and the 3.0.2.). From this and by using $(\forall \bar{x} < a)(\exists y < c)\psi \in \Pi^{U^{n+1}}_{n+2}(A)$ we obtain $B \models (\forall \bar{x} < a)(\exists y < c)\psi$. Consequently, $B \models (\bar{b})$ holds. The proof is finished.

3.0.3. Corollary. Let $B \models L$ and let $A \models U^{\omega}$. Let B be a cofinal Δ_0-extension of A. Then $A \prec B$ iff $B \models U^{\omega}$.

3.1.0. We shall prove that the segment determined by an elementary submodel of a model of U^{ω} is also an elementary submodel.

Let C be a subset of the universe of the model $A \models L$. We put

$\hat{C} = \{a \in A$, there is a $c \in C$ such that $a \leq^A c\}$.

3.1.1. Lemma. Let $A \models U^0$, $B \models U$ and let $A \subset_0 B$. Then

(1) $\hat{A}$ is a segment in B,

(2) $\hat{A} \models L$ (i.e. there is a substructure $\hat{A}$ of B with the universe $\hat{A}$),

(3) $A \subset_0 \hat{A} \subset_0 B$,

(4) $\hat{A} \models U$.

Proof. (1) Let $a \in \hat{A}$ and $b < a$, $b \in B$. Then there is an element $c \in A$ such that $B \models a < c \,\&\, Tr(c)$. Thus, $B \models b < c$ and, consequently, $\hat{A}$ is a segment in B. (2) We shall prove that $\hat{A}$ is closed under each F^B, where F is a function of the language L. Let F be an n-ary function of the language L and let $\bar{c} \in \hat{A}^n$. Let $a \in A$ be such that $B \models \bar{c} < a$. For some $b \in A$ we have: $A \models (\forall \bar{x} < a)(F(\bar{x}) < b)$ (by using $A \models H(\Delta_0)$). Thus, $B \models (\forall \bar{x} < a)(F(\bar{x}) < b)$ and so $B \models F(\bar{c}) < b$. Consequently, $\hat{A}$ is closed under F^B. (3) We shall prove that $A \subset_0 \hat{A} \subset_0 B$. Let $\varphi(x)$ be an $L(\hat{A})$-formula with only free variable x and with the following property:

(*) $\hat{A} \models \varphi(a)$ iff $B \models \varphi(a)$ holds for each $a \in \hat{A}$.

Let $c \in \hat{A}$. Then $\hat{A} \models (\exists x < c)\varphi(x)$ iff $B \models (\exists x < c)\varphi(x)$.

Proof. Suppose $B \models (\exists x < c)\varphi(x)$. Then there is a $b \in B$ such that $B \models b < c \ \& \ \varphi(b)$. We have $b \in \hat{A}$ (by using (1)) and consequently $\hat{A} \models b < c \ \& \ \varphi(b)$. Thus $\hat{A} \models (\exists x < c)\varphi(x)$ holds.

Now, we have $\hat{A} \subset B$ (by using (2)). Thus, (*) holds for each atomic $L(\hat{A})$-formula. We deduce from the facts above that $\hat{A} \subset_0 B$. We suppose $A \subset_0 B$ and, consequently the statement (3) holds. (4) follows easily from (1) - (3).

3.1.2. <u>Theorem</u>. Let $A \models U^{\omega}$ and let $B \models L$.

If $A \prec B$ then $\hat{A} \models L$ and $A \prec \hat{A} \prec B$.

Proof. Assume $A \prec B$. If $\hat{A} = B$ then the statement holds. Suppose $\hat{A} \neq B$. Then $A \subset_0 \hat{A} \subset_0 B$ follows from 3.1.1. We shall prove $\hat{A} \prec B$ by induction on the complexity of formulas. Only the tollowing induction step is not easy:

Let $\varphi(\bar{x},y) \in L$ be a formula with the free variables $\bar{x},y$ only such that for each $\bar{a} \in \hat{A}$, $b \in \hat{A}: \hat{A} \models \varphi(\bar{a},b)$ iff $B \models \varphi(\bar{a},b)$. Then for each $\bar{a} \in \hat{A}$ we have $\hat{A} \models (\exists y)\varphi(\bar{a},y)$ iff $B \models (\exists y)\varphi(\bar{a},y)$.

Let $\bar{a} \in \hat{A}$. Obviously, if $\hat{A} \models (\exists y)\varphi(\bar{a},y)$ then $B \models (\exists y)\varphi(\bar{a},y)$. Assume $B \models (\exists y)\varphi(\bar{a},y)$. Let $\tilde{\varphi}(\bar{x},y)$ be the formula $\varphi(\bar{x},y) \vee (\forall z)\neg\varphi(\bar{x},z)$. We have $(\forall \bar{x})(\exists y)\tilde{\varphi}(\bar{x},y)$. Let $a \in A$ be such that $B \models \bar{a} < a$. From $A \models U^{\omega}$ and 3.0.2 we deduce that

$$A \models (\forall \bar{x} < a)(\exists y)\tilde{\varphi}(\bar{x},y) \longrightarrow (\exists v)(\forall \bar{x} < u)(\exists y < v)\tilde{\varphi}(x,y).$$

Thus, there is a $c \in A$ such that $A \models (\forall \bar{x} < a)(\exists y < c)\tilde{\varphi}(\bar{x},y)$. We obtain $B \models (\forall \bar{x} < a)(\exists y < c)\tilde{\varphi}(\bar{x},y)$ by using $A \prec B$. Consequently, $B \models (\exists y < c)\varphi(\bar{a},y)$. Let $b \in B$ be such that $B \models b < c\ \&$ $\&\ \varphi(\bar{a},b)$. We have $b \in \hat{A}$. By using the induction hypothesis we obtain $\hat{A} \models \varphi(\bar{a},b)$ and, consequently, $\hat{A} \models (\exists y)\varphi(\bar{a},y)$. The induction step in question is proved. Now, we have $\hat{A} \prec B$. $A \prec \hat{A}$ results from this and $A \prec B$ immediately.

3.1.3. Proposition. Let $A \models U^{\omega}$. Then A has a cofinal elementary extension iff A has a proper elementary extension which is not an end-extension of A.

Proof. Let B be a proper elementary extension of A which is not an end-extension. By using 3.1.1 we obtain that the model in question is the $\hat{A}$.

3.2.0. Throughout this paragraph we shall work with a countable language L (containing a binary predicate $<$) and with structures with the absolute equality only.

We shall give a necessary and sufficient condition for the existence of a cofinal elementary extension of the models of the theory U(L).

Let $A \models L$ and let $a \in A$. We put $\hat{a} = \{b \in A; A \models b < a\}$.

3.2.1. Proposition. Let $A \models U$. The model A has a Δ_0-extension which is not an end-extension iff there is an $a \in A$ such that $\hat{a}$ is infinite.

Proof. Assume that $\hat{a}$ is finite for each $a \in A$. Let B be a Δ_0-extension of A. Let $a \in A$ and let $b \in B$ be such that $B \models b < a$. Then $B \models (\exists x < a)(x = x)$ and consequently $A \models (\exists x < a)(x = x)$. Thus, $\hat{a} \neq 0$. We have $A \models (\forall z < a) \bigwedge \{z = c;\ c \in \hat{a}\}$. We deduce $B \models (\forall z < a) \bigwedge \{z = c; c \in \hat{a}\}$ and, consequently, $B \models b = c$ for some $c \in \hat{a}$. The model B is and end-extension of A.

Assume that there exists an $a \in A$ such that $\hat{a}$ is infinite.

Let $p(x) = \{x \neq c;\ c \in \hat{a}\} \cup \{x < a\}$. Then p(x) is a set of $L(\hat{a} \cup \{a\})$-formulas which is consistent with the theory of $(A,y)_{y \in \hat{a} \cup \{a\}}$. Then there is an elementary extension B, $A \prec B$, such that p(x) is realized in $(B,y)_{y \in \hat{a} \cup \{a\}}$. Suppose $b \in B$ realizes p(x) in $(B,y)_{y \in \hat{a} \cup \{a\}}$. We have $B \models b < a$. Assume $b \in A$. Then $A \models b < a$ and, consequently, $A \models b = c$ for some $c \in \hat{a}$, which is a contradiction. The proof is finished.

3.2.2. <u>Theorem</u>. Let A be a model of U^{ω}. Then A has a cofinal elementary extension iff then there exists an $a \in A$ such that $\hat{a}$ is infinite.

3.2.3. <u>Corollary</u>. Let A be a countable model of U^{ω} and let $a \in A$ be such that $\hat{a}$ is infinite. Then there exists an elementary end-extension of A and there exists a cofinal elementary extension of A.

Proof. The existence of a cofinal elementary extension follows from the previous theorem and the existence of an elementary end-extension follows from the theorem 2.4 in [3].

3.3.0. Let L be a language containing a binary predicate $<$. Let T be a theory in L and let T be stronger than $U^{\omega}(L)$. Writing T instead of U^{ω} in the theorems 3.0.0, 3.1.2 and in the corollary 3.0.3 we obtain valid proposition.

Moreover, let L be countable. Restricting ourselves to models with the absolute equality we obtain true assertions writing T instead of U^{ω} in 3.2.2 and 3.2.3.

3.4.0. We shall present an important extension of U.

Let L be a language containing the binary predicate $<$ and the constant 0. We denote by S (more precisely by S(L)) the following theory in L:

$<$ is an antisymmetric linear ordering with the least element 0 and without the last element, satisfying moreover $x \neq 0 \rightarrow (\exists y < x)(\forall z < x)(z < y \vee z = y)$.

Obviously, S(L) is stronger than U(L). We define S^n and S^ω similarly as U^n and U^ω (i.e. $S^n = S \cup H(\Pi_n)$ and $S^\omega = S \cup H(Fm)$).

Let φ be an L-formula and let x have a free occurrence in φ. We denote by $Min(\varphi(x))$ the general closure of the formula

$$(\exists x)\varphi(x) \rightarrow (\exists x)(\varphi(x) \& (\forall y < x)\neg\varphi(y)).$$

Writing $Min(\varphi)$ we mean $Min(\varphi(x))$ with some x having a free occurrence in φ.

In [3] we proved

$$(\Delta) \qquad S \cup Min(Fm) \equiv S^\omega \cup Min(\Delta_0)$$

Obviously, $$U^\omega(L) \prec S^\omega(L).$$

Thus, for the theories from (Δ) we can obtain the results indicated in 3.3.0.

§ 4. Special extension of the theory U. We shall present the language L and the theory T in L stronger than U(L) with the following property: if $A \models T$, $B \models U$ and B is a cofinal Δ_0-extension of A then B is an elementary extension of A.

4.0.0. We say that the formula $\vartheta(x,y,z)$ of the language L with exactly three free variables x,y,z is a universal Σ-selector in the theory T in L, iff

(a) ϑ is a Σ_1-formula of the language L,

(b) $T \vdash (\forall x,y)(\exists! z)\vartheta(x,y,z)$,

(c) for each formula φ of the Language L,

$T \vdash g.c.((\forall x < u)(\exists y)\varphi(x,y) \rightarrow (\exists w)(\forall x < u)(\forall z)(\vartheta(w,x,z) \rightarrow$
$\rightarrow \varphi(x,z))$

(where u,w do not occur in φ, ϑ).

The theory T in L has a universal Σ-selector if there exists a universal Σ-selector in T.

4.0.1. Let L be a language containing a binary predicate $<$ and a constant $\bar{n}$ for each $n \in \omega$.

We denote by ∇ the schema

$(\forall x)(x < \bar{n} \rightarrow x = \bar{0} \vee \ldots \vee x = \overline{n-1})$; $n \in \omega$.

4.0.2. Proposition. Let T be a theory in L and let ϑ be a universal Σ-selector in T.

(1) Let T contain the schema ∇ . Then, for each n,

$$T \vdash (\forall x_0, \ldots, x_n)(\exists w) \bigwedge_{i \leq n} \vartheta(w,i,x_i).$$

(2) Let T be stronger than $U^0(L)$. Then T is stronger than $U^{\omega}(L)$.

Proof. (1) follows immediately from (c) in 4.0.0 with $\varphi(x,y) \equiv \bigwedge_{i \leq n} (x = \bar{i} \,\&\, y = x_i)$ by using the schema ∇.

(2) Let $\varphi(x,y)$ be a formula. We have

$T \vdash g.c.((\forall x < u)(\exists y)\varphi(x,y) \rightarrow (\exists w)(\forall x < u)(\forall z)(\vartheta(w,x,z) \rightarrow$
$\rightarrow \varphi(x,z)).$

In [3] we proved that U^0 is equivalent to $U \cup H(\Sigma_1)$. Thus

$T \vdash (\forall w)(\forall x < u)(\exists z)\vartheta(w,x,z) \rightarrow (\exists v)(\forall x < u)(\exists z < v)\vartheta(w,$
$x,z)).$

From this we deduce that

$T \vdash g.c.((\forall x < u)(\exists y)\varphi(x,y) \rightarrow (\exists w)(\forall x < u)(\exists y < w)\varphi(x,y)).$

4.0.3. Corollary. Let T be a theory in L stronger than $U^0(L) \cup Min(\Delta_0)$ and let T have a universal Σ-selector. Then T is stronger than $U(L) \cup Min(Fm)$.

Proof. In [3] we proved that $U^{\omega}(L) \cup \mathrm{Min}(\Delta_0)$ is stronger than $U(L) \cup \mathrm{Min}(Fm)$. From 4.0.1, (2) we deduce that T is stronger than $U^{\omega}(L) \cup \mathrm{Min}(\Delta_0)$ and, consequently, T is stronger than $U(L) \cup \mathrm{Min}(Fm)$.

4.0.4. <u>Theorem</u>. Let T be a theory in L stronger than $U^0(L) \cup \nabla$ and let T have a universal Σ-selector. Let $A \models T$, $B \models U$ and let B be a cofinal extension of A. Then the statements are equivalent:

(1) B is a Δ_0-extension of A,

(2) B is an elementary extension of A.

Proof. Let ϑ be a universal Σ-selector in T. By using 3.0.1 we obtain $A \subset_2 B$. From this we deduce

$$B \models (\forall x_0,\dots,x_n)(\exists v) \bigwedge_{i \leq n} \vartheta(v,\bar{i},x_i)$$

for each $n \in \omega$.

We obtain also $B \models (\forall x,y)(\exists ! z)\vartheta(x,y,z)$.

We denote by L^F the language $L \cup \{F\}$, where F is a new binary function symbol. Let U^F be the following theory in L^F:

$$U \cup \{(\forall x,y)(\exists ! z)\vartheta(x,y,z)\} \cup \{F(x,y) = z \equiv \vartheta(x,y,z)\} \cup$$
$$\cup \{(\forall x_0,\dots,x_n)(\exists v) \bigwedge_{i \leq n} \vartheta(v,\bar{i},x_i);\ n \in \omega\}.$$

We have $A \models U^F$, $B \models U^F$.

Let $\varphi(F(x_1,y_1),\dots,x_1,y_1,\dots,\bar{u})$ be a formula of the language L^F. Then

$$U^F \vdash \varphi(F(x_1,y_1),\dots,x_1,y_1,\dots,\bar{u}) \equiv (\forall z_1,\dots)(\vartheta(x_1,y_1,z_1) \,\&$$
$$\& \dots \rightarrow \varphi(z_1,\dots,x_1,y_1,\dots,\bar{u})).$$

Consequently, for each $n \geq 1$, each Π_n-formula of the language L^F is equivalent in U^F to a Π_n-formula of the language L. We deduce from this that $A \subset_2 B$ for the language L^F. Assume $A \subset_n B$ for the language L^F with some $n \geq 2$. We shall prove

$A \subset_{n+1} B$ for the language L^F. Let $\psi \in \Pi_{n+1}(A)$ be a sentence of the language $L^F(A)$. We may suppose that ψ has the form $(\forall x)(\exists y)\varphi(x,y)$, where $\varphi(x,y)$ is a Π_{n-1}-formula of the language $L^F(A)$ with exactly two free variables x,y. This follows from the fact that U^F enables to contract quantifiers, i.e. if Q is $\forall$ or $\exists$ then $U^F \vdash (Qx_0,\dots,x_n)\varphi(x_0,\dots,x_n) \equiv$ $\equiv (Qx)\varphi(F(x,\bar{0}),\dots,F(x,\bar{n}))$ holds for all n and all L^F-formula φ. To finish the proof we must show that

$$A \models (\forall x)(\exists y)\varphi \quad \text{implies} \quad B \models (\forall x)(\exists y)\varphi .$$

Assume $A \models (\forall x)(\exists y)\varphi(x,y)$. Let $a \in A$. Then $A \models (\forall x < a)(\exists y)\varphi$. Thus, there is an element $c \in A$ such that $A \models (\forall x < a)\varphi(x, F(c,x))$ holds. The last formula is a Π_n-sentence of the language $L^F(A)$ and, consequently, holds in B. We deduce $B \models (\forall x)(\exists y)\varphi$ from the fact that A is cofinal in B.

4.0.5. Corollary. Let T be as in 4.0.4. Let $A \models T$, $B \models U$ and let $A \subset_0 B$. Then the structure $\hat{A}$ is an elementary extension of A.

4.1.0. Let L be the language of the Zermelo-Fraenkel set theory ZF (Peano arithmetic P resp.). We have that ZF is stronger than $U^{\omega}(L)$ (P is stronger than $S^{\omega}(L)$ resp.). Thus, by using 3.3.0 we can immediately deduce the variant of the results presented for the theory ZF (P resp.). For example: Let A, B be models of ZF (P resp.) and let B be a cofinal Δ_0-extension of A. Then $A \prec B$.

4.1.1. The following facts are well-known:

(1) the theory P can be viewed as the extension of $S^0 \cup \nabla$ and P has a universal Σ-selector,

(2) each extension of a model of P, which is a model of P,

is a Δ_0-extension.

Thus, from this and by using 4.0.4 we can deduce the following known proposition (see also [1]):

Let $A \models P$, $B \models S$ and let B be a cofinal extension of A. Then the following are equivalent:

(1) $A \subset_0 B$

(2) $A \prec B$

(3) $B \models P$.

R e f e r e n c e s

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