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CONCERNING CESARI'S THEOREMS

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Abstract: A version of Schauder's theorem is used for proving assertions slightly generalizing Cesari's results derived in [3]. As an application, the existence of periodic solutions to a parabolic equation is proved.

Key words: Nonlinear equations, periodic solutions of parabolic equations.

AMS: 47H15, 35K05, 35B10

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1. Introduction. Some abstract existence theorems modelled by assertions originating in Landesman-Lazer's works have recently been derived by Cesari in [1],[2] and [3]. In this paper a well-known version of Schauder's Theorem is used for proving two theorems which have been inspired by Cesari's results and which have many points in common with very abstract assertions derived in [8].

The first theorem is proved under a bit weaker assumption than the related result of Cesari, and the second theorem extends a result of Cesari as to the growth of the admissible nonlinearity. The equations with a nonlinear term whose growth is either sublinear or asymptotically linear have been investigated in a great deal of papers. Abundant bibliography on this subject can be found in [8].

As an application of the abstract theorem, a simple assertion on the existence of periodic solutions to a parabolic equation is proved.

It turns out that the used version of Schauder's Theorem is an appropriate tool for proving assertions of Landesman-Lazer's type since the rather complicated determination of a set which has to be mapped into itself, as required by Schauder's Theorem, is replaced by an assumption which is very easy to be satisfied in this context.

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2. Auxiliary Lemma. Let X and W be Banach spaces and let R_X, R_W be positive numbers. Let the space $X\times W$ be equipped with the norm

$$|||(x,w)||| = \max\,(\,\|x\|_X,\ R_X R_W^{-1}\|w\|_W).$$

Let us denote

$$M = \{(x,w)\in X\times W;\ \|x\|_X \leqq R_X,\ \|w\|_W \leqq R_W\}$$
$$= \{(x,w)\in X\times W;\ |||(x,w)||| \leqq R_X\}.$$

Let ∂M be the boundary of M.

2.1. Lemma. Let $T:M \longrightarrow X\times W$ be continuous and compact. Suppose that

(2.2) $\{(x,w)\in \partial M;\ \exists\lambda > 1$ such that $T(x,w) = \lambda(x,w)\} = \emptyset$.

Then there is $(x,w)\in M$ satisfying $T(x,w) = (x,w)$.

Proof. Let a continuous mapping $\mathcal{P}:X\times W \longrightarrow M$ be defined by

$$\mathcal{P}(x,w) = \begin{cases} (x,w) & \text{if } |||(x,w)||| \leq R_X, \\ R_X(x,w)/\,|||(x,w)||| & \text{otherwise.} \end{cases}$$

By Schauder's Theorem there exists $(x,w) \in M$ satisfying $\mathcal{P}T(x,w) = (x,w)$. In virtue of (2.2) we have $\mathcal{P}T(x,w) = T(x,w)$ and this completes the proof.

3. Abstract theorems. The notation used in this section is taken over from [3]. Let X be a real Hilbert space whose inner product is denoted by $\langle\, ,\, \rangle$ and norm by $\|\cdot\|$.

Let $E: D(E) \longrightarrow X$, $D(E) \subset X$, be a linear operator, $1 \leq \dim W < +\infty$, $W = \ker E$ and let N be a continuous (nonlinear) operator. Let P be the orthogonal projection onto W. Let us denote $X_1 = (I-P)X$. Let $H: X_1 \longrightarrow D(E) \cap X_1$ be a linear operator, compact as a mapping of X_1 into X. Let us suppose that E, P and H satisfy:

(i) $X_1 = \{Ex;\ x \in D(E)\}$,

(ii) if $x \in D(E)$, then $HEx = (I-P)x$.

(iii) If $x \in X_1$, then $EHx = x$.

Let $A: X \longrightarrow X$ be a continuous operator satisfying $\|Ax\| \leq \omega(\|x\|)$ for all $x \in X$ with some nondecreasing function $\omega: R^+ \longrightarrow R^+$. Finally, setting $L = \|H\|$, we have the theorem.

3.1. Theorem. Let $0 < k < 1$. Besides the hypotheses listed above let the following assumptions be satisfied.

(i) There are constants $J_0 \geq 0$, $J_1 \geq 0$ such that $\|Nx\| \leq J_0 + J_1 \|x\|^k$ for all $x \in X$.

(ii) There are constants $R_o \geqq 0$, $\varepsilon > 0$, $K_o > J_o$, $K_1 > J_1$ such that $\langle N(w+Hx), w\rangle \leqq -\varepsilon \|w\|$ for all $x \in X$ and $w \in W$ satisfying $\|w\| \geqq R_o$ and $\|x\| \leqq K_o + K_1 \|w\|^k$.

Then there exists $\alpha_o > 0$ and $C > 0$ such that, for every real α, $|\alpha| \leqq \alpha_o$, the equation

(3.2) $$Ex + \alpha Ax = Nx$$

has a solution $x \in D(E)$ with $\|x\| \leqq C$.

3.3. Remark. Theorem 3.1 is a slightly changed and weakened form of Theorem IV of [3].

Proof of Theorem 3.1. It is easy to see, cf.[2], that instead of solving (3.2) one can investigate the existence of a fixed point of the mapping $T = (T_1, T_2) : X \times W \longrightarrow X \times W$ given by

(3.4) $$T_1(x,w) = w + H(I-P)(-\alpha Ax + Nx),$$

(3.5) $$T_2(x,w) = w + P(-\alpha Ax + NT_1(x,w)).$$

Let $\varkappa > 0$ and $\nu > 0$ satisfy $K_o \geqq \varkappa + J_o$ and $J_1(1+\nu)^k \leqq K_1$. Let us choose R_X such that

$$R_X(1+\nu)^{-1} \geqq R_o, \quad R_X(1+\nu)^{-1} + LK_o + LK_1 R_X^k \leqq R_X$$

and denote $R_W = R_X/(1+\nu)$. Let

(3.6) $$\alpha_o = (\omega(R_X))^{-1} \min(\varkappa, \varepsilon).$$

Let W be equipped with the Hilbert space structure induced by X with the same notation of the norm and the inner product. We denote

$$M = \{(x,w) \in X \times W;\ \|x\| \leqq R_X,\ \|w\| \leqq R_W\}.$$

We will show that the mapping T satisfies (2.2). Let us suppose there is $(x,w) \in \partial M$ and $\lambda > 1$ such that

(3.7) $\quad T_1(x,w) = \lambda x,$

(3.8) $\quad T_2(x,w) = \lambda w.$

Then the two cases will be distinguished:

A) $\|x\| = R_X, \ \|w\| \leqq R_W$, B) $\|x\| < R_X, \ \|w\| = R_W$.

In case (A) we have

$$\|T_1(x,w)\| \leqq R_W + \alpha_0 L\omega(R_X) + LJ_0 + LJ_1 \|x\|^k \leqq$$
$$\leqq R_W + L\varkappa + LJ_0 + LK_1 R_X^k \leqq R_X(1+\nu)^{-1} +$$
$$+ LK_0 + LK_1 R_X^k \leqq R_X.$$

This contradicts (3.7). In case (B) we have $T_1(x,w) = w + Hy$, where

$$\|y\| \leqq \alpha_0 \omega(R_X) + J_0 + J_1 \|x\|^k \leqq$$
$$\leqq \varkappa + J_0 + J_1 R_X^k \leqq$$
$$\leqq K_0 + J_1(1+\nu)^k [R_X(1+\nu)^{-1}]^k \leqq$$
$$\leqq K_0 + K_1 R_W^k = K_0 + K_1 \|w\|^k.$$

Thus, by the assumption (ii) we obtain

$$\langle T_2(x,w), w\rangle = \|w\|^2 - \alpha\langle Ax, w\rangle + \langle NT_1(x,w), w\rangle \leqq$$
$$\leqq R_W^2 + \alpha_0 R_W \omega(R_X) - \varepsilon R_W \leqq R_W^2,$$

since $\alpha_0 \leqq \varepsilon/\omega(R_X)$. This relation contradicts (3.8). Hence T satisfies (2.2) and Lemma 2.1 can be applied to the mapping T on the set M. This completes the proof.

3.9. <u>Remark</u>. Using (3.4), we easily derive that every solution x to (3.2) given by the preceding theorem satisfies $x-Hz \in W$, $\|x-Hz\| \leqq R_W$ for some $z \in X$, $\|z\| \leqq R_Z \equiv \varkappa + J_o + J_1 R_X^k$. The constants R_W and R_Z depend on the operators E and N but not on the operator A.

Under the assumptions and notations given at the beginning of this section we can prove a theorem which deals with another class of admissible nonlinearities.

3.10. <u>Theorem</u>. Let the following assumptions be satisfied.

(i) There are constants $J_o \geqq 0$, $J_1 > 0$ such that $\|Nx\| \leqq J_o + J_1 \|x\|$ for all $x \in X$.

(ii) There exists $\nu > 1$ such that $LJ_1(1+\nu) < 1$.

(iii) There are constants $R_o \geqq 0$, $\varepsilon > 0$, $K_o > J_o$ such that $(N(w+Hx),w) \leqq -\varepsilon \|w\|$ for all $x \in X$, $w \in W$ satisfying $\|w\| \geqq R_o$ and $\|x\| \leqq K_o + J_1(1+\nu)\|w\|$.

Then there exists $\alpha_o > 0$ and $C > 0$ such that, for every real α, $|\alpha| \leqq \alpha_o$, the equation $Ex + \alpha Ax = Nx$ has a solution $x \in D(E)$ with $\|x\| \leqq C$.

Proof. Let $\varkappa$ satisfy $K_o \geqq \varkappa + J_o$. We choose R_X such that

$$R_X(1+\nu)^{-1} \geqq R_o,$$

$$R_X(1+\nu)^{-1} + LK_o + LJ_1 R_X \leqq R_X.$$

These inequalities can be satisfied since $(1+\nu)^{-1} + LJ_1 \leqq 2(1+\nu)^{-1} < 1$. Putting $R_W = R_X/(1+\nu)$ and proceeding along the same lines as in the proof of Theorem 3.1 we will complete the proof of this theorem.

3.11. Remark. By applying Lemma 2.1 one can also prove the other theorems in [1],[2] and [3] as well as some further theorems of Landesman-Lazer's type, e.g. [4],[5].

3.12. Remark. Numerous abstract results connected with theorems of Landesman-Lazer's type can be found in [8].

4. Example. In this section we will prove the existence of periodic solutions to a parabolic equation by applying Theorem 3.1. We will denote by S^1, Z and Z^+ the unit circle in $\mathbb{R}^2$, the sets of integers and positive integers respectively. For brevity we will write $Q = S^1 \times (0, \pi)$ and $X = L^2(Q)$. By Y we denote the closure of all functions $x = x(t, \xi) \in C^{\infty}(\overline{Q})$ vanishing at $\xi = 0$ and $\xi = \pi$ in the norm

$$\| x \|_Y = (\| x_t \|^2_{L^2} + \| x_{\xi\xi} \|^2_{L^2})^{1/2} .$$

Equivalently, the space Y is the space of all functions x which can be written in the form

$$x(t, \xi) = \sum_{j \in Z, k \in Z^+} x_{j,k} \, e^{ijt} \sin k\xi ,$$

where $x_{j,k} = \bar{x}_{-j,k}$ and the number

$$\pi^2 \sum_{j \in Z, k \in Z^+} (j^2 + k^4) |x^2_{j,k}|^2$$

is finite (and equal to $\|x\|^2_Y$). This implies that the identity map of Y into X is compact and, further, that $Y \subset C(\overline{Q})$ and $\|x\|_C \leq C_S \|x\|_Y$ for all $x \in Y$ with a certain constant C_S.

Let $h: R \longrightarrow R$ be a continuous function, which for some $\hat{x} > 0$

satisfies

a) h is monotone on $(-\infty,-\hat{x}) \cup (\hat{x},+\infty)$,

b) $h(x)x \geq 0$ for all $x \in (-\infty,-\hat{x}) \cup (\hat{x},+\infty)$,

c) $\lim_{x \to +\infty} h(x) = +\infty$, $\lim_{x \to -\infty} h(x) = -\infty$,

d) for some $0 < k < 1$ and $h_0 > 0$, $h_1 > 0$, $|h(x)| \leq h_0 + h_1 |x|^k$ for all $x \in R$.

Let $a = a(t,\xi,x): Q \times R \longrightarrow R$ be measurable in (t,ξ), continuous in x and satisfy

(4.1) $|a(t,\xi,x)| \leq \tilde{\omega}(|x|)$ for all $(t,\xi,x) \in Q \times R$, where $\tilde{\omega}: R^+ \longrightarrow R^+$ is a nondecreasing function.

We will now prove an assertion which is closely related to an analogous result in [6], Sec.6.6 and which for $\alpha =$ $= 0$ follows from Theorem V.3 of [7] as a very special case.

4.2. Theorem. Let the functions h and a satisfy all the hypotheses formulated above in this section. Let $m \in Z$, $m > 1$. Let $p \in L^2(Q)$. Then to every α sufficiently close to 0, there exists $x \in Y$ satisfying

(4.3) $x_t - x_{\xi\xi} - m^2 x + \alpha a(t,\xi,x) + h(x) = p(t,\xi)$ for almost all $(t,\xi) \in R \times (0,\pi)$,

(4.4) $x(t,0) = x(t,\pi) = 0$, $t \in R$,

(4.5) $x(t+2\pi,\xi) = x(t,\xi)$, $(t,\xi) \in R \times (0,\pi)$.

Proof. Let us denote

$$Ex = x_t - x_{\xi\xi} - m^2 x, \quad W = \text{span}\{\sin mx\}$$

and P the orthogonal projection onto W. Then $X_1 = (I-P)X =$

$= EY$. Given $x \in X_1$, i.e. $x = \sum_{(j,k)\neq(0,m)} x_{j,k}\, e^{ijt} \sin kx$, $x_{j,k} = \bar{x}_{-j,k}$, $\sum |x_{j,k}|^2 < +\infty$, we define a bounded linear operator $H: X_1 \longrightarrow Y$ by $(Hx)(t,\xi) = \sum_{(j,k)\neq(0,m)} (ij+k^2-m^2)^{-1}\, x_{j,k}\, e^{ijt} \sin k\xi$.

Obviously, the mapping H considered as a mapping of X_1 into X satisfies the assumptions (ii) and (iii) of Section 3. This mapping is compact since the identity map of Y into X is compact.

Further we set $Nx = p(t,\xi) - h(x)$. This operator satisfies hypothesis (i) of Theorem 3.1. Let positive numbers K_0', K_1' and $\bar{\varepsilon}$ be chosen arbitrarily. Then hypothesis (ii) will be verified as soon as we have shown that there is $r_0 > 0$ such that, for all $r \geq r_0$, $y \in C(\bar{Q})$, $\|y\|_C \leq K_0' + K_1' r^k$, we have

$$V(r) \equiv \langle h(r \sin m\xi + y(t,\xi)), \sin m\xi \rangle_{L^2} > \bar{\varepsilon}\,, \tag{4.6}$$

$$\langle h(-r \sin m\xi + y(t,\xi)), -\sin m\xi \rangle_{L^2} > \bar{\varepsilon}\,. \tag{4.7}$$

To this end we define, $\eta > 0$

$$\Omega_+ = \{(t,\xi) \in Q;\ \sin m\xi > \eta\},$$

$$\Omega_- = \{(t,\xi) \in Q,\ \sin m\xi < -\eta\},$$

$$\Omega_0 = \{(t,\xi) \in Q,\ |\sin m\xi| \leq \eta\}.$$

For brevity we set $\varphi(r) = K_0' + K_1' r^k$ and $\tilde{K} = \max\{|h(x)|, |x| \leq \hat{x}\}$. Since $m > 1$ we can fix $\eta > 0$ such that

$$\min(\text{meas}\, \Omega_+, \text{meas}\, \Omega_-) > \text{meas}\, \Omega_0. \tag{4.8}$$

Let us choose $\hat{r}$ such that $r \geqq \hat{r}$ implies $\eta r - \varphi(r) \geqq \hat{x}$ and $\varphi(r) \geqq \hat{x}$.

Now, let $r \geqq \hat{r}$ and let $y \in C(\overline{Q})$ satisfy $\|y\|_C \leqq \varphi(r)$. As

$$\int_{\Omega_0} \int h(r \sin m\xi + y(t,\xi)) \sin m\xi \, d\xi \, dt \geqq$$

$$\geqq \eta \{\min(-\widetilde{K}, h(-\varphi(r))) - \max(\widetilde{K}, h(\varphi(r)))\} \text{ meas } \Omega_0,$$

we have

$$V(r) \geqq \int_{\Omega_+} \int h(\eta r - |y(t,\xi)|) \eta \, d\xi \, dt -$$

$$- \int_{\Omega_-} \int h(-\eta r + |y(t,\xi)|) \eta \, d\xi \, dt +$$

$$+ \int_{\Omega_0} \int h(r \sin m\xi + y(t,\xi)) \sin m\xi \, d\xi \, dt \geqq$$

$$\geqq \eta \{ h(\eta r - \varphi(r)) \text{ meas } \Omega_+ - \max(\widetilde{K}, h(\varphi(r))) \text{meas } \Omega_0 \} +$$

$$+ \eta \{ -h(-\eta r + \varphi(r)) \text{ meas } \Omega_- + \min(\widetilde{K}, h(-\varphi(r))) \text{meas } \Omega_0 \}.$$

By (4.8) this yields $\lim_{r \to +\infty} V(r) = +\infty$. Hence (4.6) is verified for all sufficiently large r. The estimate (4.7) can be proved similarly.

By Remark 3.9 the solutions of the equation

$$\text{(4.9)} \qquad Ex + \alpha Ax = Nx$$

with the operators E and N defined above will satisfy the estimate

$$\|x\|_C \leqq B \equiv \pi^{-1} R_W + \sup \{\|Hz\|_C;\ \|z\|_X \leqq R_Z\}$$

no matter how the operator A is defined. Hence we can put

$$Ax(t,\xi) = a_B(t,\xi,x(t,\xi))$$

with $a_B(t,\xi,x) = \max(-\widetilde{\omega}(B), \min(\widetilde{\omega}(B), a(t,\xi,x)))$.

The estimate of $\|x\|_C$ just mentioned shows that solutions to equation (4.9) guaranteed by Theorem 3.1 satisfy (4.3) - (4.5). This completes the proof.

4.10. Remark. Let a continuous function $h: R \longrightarrow R$ satisfy hypotheses a) - d) formulated at the beginning of this section and let there exist a positive number A such that $Ah(x) \leqq -h(-x) \leqq A^{-1}h(x)$ for all $x \geqq \hat{x}$. Then Theorem 4.2 holds also for $m = 1$.

4.11. Remark. If h grows linearly, i.e. if it satisfies $|h(x)| \leqq h_0 + h_1|x|$, $x \in R$, $h_0, h_1 > 0$, instead of d), then Theorem 4.2 holds provided h_1 is sufficiently small.

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