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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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POLYNOMIALS OF THE EIGENVALUES AND POWERS OF MATRICES

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Abstract: Explicit formulae are derived for the entries of any power of the companion matrix of a polynomial p, regarded as functions of the roots of p. The formulae are applied to yield an upper bound for the norm of a power of any matrix in terms of its spectral radius.

Key words: Eigenvalue, recurrence relation, critical exponent, norms of matrices.

AMS: Primary 15
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1. Introduction. It is well known that the m-th power for $m \times n$ of an $n \times n$ matrix A can be represented as a linear combination of the lower powers of A. The coefficients in this combination are known polynomials of the coefficients appearing in the characteristic equation of A (cf. [1,3,6,8]). The last coefficients being elementary symmetric polynomials of the eigenvalues of A, we can write

$$A^m = \sum_{i=1}^{n} w_{i,m} A^{i-1},$$

where $w_{i,m}$ are polynomials of the eigenvalues of A. The polynomials $w_{i,m}$ proved to be useful in studying the relations between the norm of iterates and the spectral radius ([2,4]).

Professor V. Pták conjectured that the sign of the coefficients $w_{i,k}$ ($k \geq n$) depends on i only; at his request, the late Professor V. Knichal supplied a proof of this conjecture. This result was used in an essential manner by V. Pták to characterize contractions A on an n-dimensional Hilbert space which maximize $|A^n|$ under the condition $|A|_\sigma \leq r$. Another application of this result was given by the present author [2]. Knichal's proof was not published. Quite recently three independent proofs were given by N.J. Young [9], V. Pták [5] and the present author which also yield explicit expressions for the $w_{i,k}$.

It is the purpose of the present paper to give such explicit formulae and to apply them to obtain estimates for the norm of iterates of contractions on an n-dimensional normed space; these estimates are independent of the choice of the norm.

2. <u>Definitions and preliminaries</u>. Let n be an arbitrary but fixed positive integer. For i = 1,...,n, we shall define the polynomials

$$E_i = E_i(x_1,\dots,x_n) = \sum_{e_j \in \{0,1\}} x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$$

and

$$a_i = a_i(x_1,\dots,x_n) = (-1)^{n-i} E_{n-i+1}(x_1,\dots,x_n),$$

where $x_1,\dots,x_n$ are considered as indeterminates. Hence

(1) $$(x-x_1)(x-x_2)\dots(x-x_n) = x^n - a_1 - a_2 x - \dots - a_n x^{n-1}$$

and

(2) $(1-x_1x)(1-x_2x)\dots(1-x_nx) = 1-a_nx-a_{n-1}x^2 - \dots -a_1x^n.$

For each i, $1 \leq i \leq n$, and $k \geq 0$, we shall define the polynomials $w_{ik} = w_{ik}(x_1,\dots,x_n)$ by the recursive relations

(3) $w_{i,k+n} = a_1w_{i,k} + a_2w_{i,k+1} + \dots + w_{i,k+n-1}$

with initial conditions

(4) $w_{i,k}(x_1,x_2,\dots,x_n) = \delta_{i,k+1}$, $0 \leq k \leq n - 1$.

To avoid exceptions, we put $w_{i,k} = 0$ for $i < 1$.

To prove that $w_{i,k}$ are the polynomials spoken about in the introduction, suppose that A is a linear operator on an n-dimensional linear space, and that the eigenvalues of A are $\rho_1,\dots,\rho_n$. Note that the polynomial

$$p(x) = x^n - \sum_{i=1}^{n} a_i(\rho_1,\dots,\rho_n)x^{i-1}$$

is the characteristic polynomial of A and that, for i = 1,... ...,n, $w_{i,n} = a_i$. Hence we have, by the Cayley-Hamilton theorem,

(5) $$A^n = \sum_{i=1}^{n} a_i(\rho_1,\dots,\rho_n)A^{i-1}$$

and

(6) $$A^k = \sum_{i=1}^{n} w_{i,k}(\rho_1,\dots,\rho_n)A^{i-1}$$

for k = 0,1,...,n.

To prove (6) for $k > n$ by induction, suppose that $m > n$ and that (6) is satisfied for k = 0,1,...,m-1. Put $s = m - n$, $\alpha_i = a_i(\rho_1,\dots,\rho_n)$ and $\nu_{i,k} = w_{i,k}(\rho_1,\dots,\rho_n)$. If we multiply (5) by A^s and use the induction hypothesis, we successively get

$$A^m = \sum_{i=1}^{n} \alpha_i A^{s+i-1} = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \nu_{j,s+i-1} A^{j-1} =$$

$$= \sum_{j=1}^{n} (\sum_{i=1}^{n} \alpha_i \nu_{j,s+i-1}) A^{j-1} = \sum_{j=1}^{n} \nu_{j,m} A^{j-1}.$$

If

$$W_k = \begin{bmatrix} w_{1,k} & w_{2,k} & \cdots & w_{n,k} \\ w_{1,k+1} & w_{2,k+1} & \cdots & w_{n,k+1} \\ \cdot & \cdot & \cdots & \cdot \\ w_{1,k+n-1} & w_{2,k+n-1} & \cdots & w_{n,k+n-1} \end{bmatrix}$$

and

(7) $$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \cdot \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix},$$

we have

$$W_{k+1} = TW_k.$$

Since $W_o = (\delta_{i,j})$, we have

$$W_k = T^k$$

and

(8) $$W_{k+1} = T^{k+1} = T^k T = W_k T.$$

<u>Lemma 1</u>. If $1 \leq i \leq n$ and $k \geq 1$, then

(9) $$w_{i\ k+1} = w_{i-1,k} + a_i w_{n,k}.$$

Proof: To obtain (9), it is enough to compare entries in the first row and i-th column of the matrices W_{k+1} and $W_k T$ in (8).

Lemma 2. Let $1 \leq i \leq n$ and $k \geq i - 1$. Then

$$w_{i,k} = \sum_{j=0}^{i-1} a_{j+1} w_{n,k-i+j}. \tag{10}$$

Proof: We get (10) by repeated application of (9).

3. General expression. For $k \geq 0$, put

$$h_k = h_k(x_1,\ldots,x_n) = \sum_{\substack{e_1+\ldots+e_n=k \\ e_i\in\{0,1,\ldots,k\}}} x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}.$$

Lemma 3. Let $k \geq n - 1$. Then

$$w_{n,k} = h_{k-n+1}.$$

Proof: Define the generating function (cf.[7]) for $w_{n,k}$ by

$$f(z) = w_{n,0} + w_{n,1}z + w_{n,2}z^2 + \ldots .$$

If we multiply this equation by $1, -a_n z, \ldots, -a_1 z^n$ and sum up, we get, with help of (3) and (4), that

$$f(z)(1 - a_n z - \ldots - a_1 z^n) = z^{n-1}.$$

Thus

$$\begin{aligned} f(z) &= z^{n-1}(1 - a_n z - \ldots - a_1 z^n)^{-1} = \\ &= z^{n-1}(1 - x_1 z)^{-1} \ldots (1 - x_n z)^{-1} = \\ &= z^{n-1}(1+x_1 z+x_1^2 z^2+\ldots) \ldots (1+x_n z+x_n^2 z^2+\ldots) = \end{aligned}$$

$$= z^{n-1}(h_0 + h_1 z + h_2 z^2 + \dots) =$$

$$= h_0 z^{n-1} + h_1 z^n + h_2 z^{n+1} + \dots .$$

We complete the proof by comparing the coefficients at the corresponding powers of z.

Lemma 4. Let $k \geq 0$, $1 \leq i \leq n$. Then

$$E_i h_k = \sum_{\substack{e_1+e_2+\dots+e_n=i+k \\ e_j \in \{0,1,\dots,i+k\}}} \binom{q(e_1,\dots,e_n)}{i} x_1^{e_1} \dots x_n^{e_n}, \tag{11}$$

where $q(e_1,\dots,e_n)$ denotes the number of e_j different from zero.

Proof: If we multiply E_i and h_k, then the result is the sum of products, the first factor of which is the term of E_i, the second one is the term of h_k. Each such product may be written in the form

$$x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}, \tag{12}$$

where the exponents are non-negative integers whose sum is equal to i + k. A product (12) with given exponents $e_1,\dots$ $\dots,e_n$ whose sum equals i + k is obtained by multiplying a term of E_i by a term of h_k. The number of terms of E_i which yield the given product is exactly $\binom{q(e_1,\dots,e_n)}{i}$.

Theorem 1. Let $1 \leq i \leq n$ and $k \geq n - 1$. Then

$$w_{i,k} = (-1)^{n-i} \sum_{\substack{e_1+\dots+e_n=k-i+1 \\ e_j \in \{0,1,2,\dots\}}} \binom{q(e_1,\dots,e_n)-1}{n-i} x_1^{e_1} \dots x_n^{e_n}. \tag{13}$$

Proof: Applying successively (10) and (11), we get

$$w_{i,k} = \sum_{j=0}^{i-1} a_{j+1} w_{n,k-i+j} = \sum_{j=0}^{i-1} (-1)^{n-j-1} g_{n-j} h_{k-i+j-n+1} =$$

$$= \sum_{j=0}^{i-1} (-1)^{n-j-1} \sum_{\substack{e_1+\dots+e_n=k-i+1 \\ e_j \in \{0,1,2,\dots\}}} \binom{q(e_1,\dots,e_n)}{n-j} x_1^{e_1} \dots x_n^{e_n} =$$

$$= \sum_{\substack{e_1+\dots+e_n=k-i+1 \\ e_j \in \{0,1,2,\dots\}}} (-1)^{n-i} \left(\sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{q(e_1,\dots,e_n)}{n-j} \right) x_1^{e_1} \dots x_n^{e_n}$$

$$= (-1)^{n-i} \sum_{\substack{e_1+\dots+e_n=k-i+1 \\ e_j \in \{0,1,2,\dots\}}} \binom{q(e_1,\dots,e_n)-1}{n-i} x_1^{e_1} \dots x_n^{e_n}.$$

In the last step, we have used the identity

$$\binom{s-1}{i} = \binom{s}{i} - \binom{s}{i-1} + \binom{s}{i-2} - \dots \pm \binom{s}{i-i},$$

which is an immediate consequence of

$$\binom{s}{i} = \binom{s-1}{i} + \binom{s-1}{i-1}.$$

4. Evaluation of $w_{i,k}(r,\dots,r)$. Having obtained Theorem 1 it is now comparatively easy to compute $w_{i,k}(r,\dots,r)$. First, counting the number of terms in the sum (13), we have for $k \geq n$

(14) $$w_{i,k}(r,\dots,r) = (-1)^{n-i} r^{k-i+1} \sum_{q=n-i+1}^{\min(n,k-i+1)} \binom{n}{q} \binom{k-i}{q-1} \binom{q-1}{n-i}.$$

Since [7]

$$\binom{n}{m}\binom{m}{p} = \binom{n}{p}\binom{n-p}{n-m},$$

we have

$$\binom{k-i}{q-1}\binom{q-1}{n-i} = \binom{k-i}{n-i}\binom{k-n}{k-i+1-q}$$

and

$$w_{i,k}(r,\dots,r) = (-1)^{n-1} r^{k-i+1} \binom{k-i}{n-i} \sum_{q=n-i+1}^{\min(n,k-i+1)} \binom{n}{q}\binom{k-n}{k-i+1-q}.$$

The last sum may be simplified by use of a variant of the Vandermonde convolution formula [7]

$$\binom{n+p}{m} = \sum \binom{n}{k}\binom{p}{m-k}.$$

Thus

$$(15)\quad w_{i,k}(r,\dots,r) = (-1)^{n-1}\binom{k-i}{n-i}\binom{k}{i-1} r^{k-i+1}.$$

5. <u>Applications</u>. Let X_n be an n-dimensional Banach space, let $L(X_n)$ denote the algebra of all linear operators on X_n and let the operator norm and the spectral radius of $A \in L(X_n)$ be denoted by $|A|$ and $|A|_\sigma$ respectively.

<u>Theorem 2</u>. Let $0 < r < 1$. If $A \in L(X_n)$, $|A| \leqq 1$ and $|A|_\sigma \leqq r$, then for each $k \geqq n$

$$(16)\quad |A^k| \leqq \sum_{i=1}^{n} \binom{k-i}{n-i}\binom{k}{i-1} r^{k-i+1}$$

and

$$(17)\quad \lim_{k\to\infty} \sum_{i=1}^{n} \binom{k-i}{n-i}\binom{k}{i-1} r^{k-i+1} = 0.$$

Proof: Let r, k and A satisfy the assumptions of the theorem. All the eigenvalues $\rho_1,\dots,\rho_n$ of A being less than or equal to r in absolute value, we have by (13)

$$|w_{i,k}(\rho_1,\dots,\rho_n)| \leqq |w_{i,k}(r,\dots,r)|,\quad i = 1,\dots,n.$$

Hence

$$|A^k| = |\sum_{i=1}^{n} w_{i,k}(\rho_1,\dots,\rho_n)A^{i-1}| \leq$$

$$\leq \sum_{i=1}^{n} |w_{i,k}(r,\dots,r)| = \sum_{i=1}^{n} \binom{k-i}{n-i}\binom{k}{i-1} r^{k-i+1}.$$

To prove (17), note that $|T(r,\dots,r)|_6 = r$, where the matrix ~~T is defined~~ by (7), and that

$$(w_{i,k}(r,\dots,r),\dots, w_{n,k}(r,\dots,r))$$

is the first row of the matrix $T(r,\dots,r)^k$ (cf. (8)). For an nxn matrix (a_{ij}) put

$$|(a_{ij})|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| .$$

Now, relation (17) simply follows from

$$\sum_{i=1}^{n} |w_{i,k}(r,\dots,r)| \leq |T(r,\dots,r)^k|_\infty$$

and

$$\lim_{k\to\infty} |T(r,\dots,r)^k|_\infty = 0.$$

The point of the Theorem 2 is that there is an upper bound for $|A^k|$ independent of the choice of the norm.

Denote by $B_{n,\infty}$ the complex n-dimensional vector space, the norm $|x|_\infty$ of the vector $x = (x_1,\dots,x_n)$ being defined by the formula

$$|x|_\infty = \max_{i=1,\dots,n} |x_i| ,$$

and for $0 < r < 1$ and $k \geq 0$ put

$$C(B_{n,\infty},r,k) = \sup\{|A^k|_\infty : A \in L(B_{n,\infty}),\ |A|_\infty \leq 1 \text{ and } |A|_6 \leq r\},$$

In [2] we gave a partial answer to the question of Professor V. Pták [4] about the values of $C(B_{n,\infty},r,k)$; now we are able to give an explicit formula for the case studied in [2].

Theorem 3. Let $0<r\leq 2^{1/n}-1$ and $k\geq n$. Then

$$C(B_{n,\infty},r,k) = \sum_{i=1}^{n} \binom{k-i}{n-i}\binom{k}{i-1} r^{k-i+1}. \tag{18}$$

Proof: We have proved in [2] that under the assumptions

$$C(B_{n,\infty},r,k) = \sum_{i=1}^{n} | w_{i,k}(r,\dots,r)| .$$

Theorem 3 shows that for small r, the formula (17) gives the best possible bound.

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