Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 3, 415--430

Persistent URL: http://dml.cz/dmlcz/105866

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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,3 (1978)

NORMS OF MATRIX POWERS

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Abstract: Let A be a square matrix satisfying $\|A\| \leq 1$, $\|\cdot\|$ being an arbitrary operator norm. Upper bounds are obtained for $\|A^m\|$ in terms of the eigenvalues of A. The proof involves calculating explicitly the powers of the companion matrix of a polynomial p in terms of the roots of p.

Key words: Norm, eigenvalue, companion matrix, recurrence relation.

AMS: Primary 15A24, 15A42
Secondary 12D10

It sometimes happens in applied linear algebra that one wishes to sum an infinite power series in matrices: for example, the matrix equation of Ljapunov which arises in control theory (see [1])

$$X - A^* X A = Q \tag{1}$$

has the unique solution

$$X = \sum_{s=0}^{\infty} A^{*s} Q A^s \tag{2}$$

provided the eigenvalues of A have absolute value less than 1 (here the matrices involved are $N \times N$ and A^* denotes the conjugate transpose of A). It would obviously be useful to be able to estimate the error involved in replacing the infinite series (2) by the sum of its first m terms, say with respect to an operator norm $\|\cdot\|$ on the algebra of $N \times N$ matrices (i.e. the operator norm derived from some norm on $\mathbb{C}^N$). We have at once the estimate

$$\left\| \sum_{s=m}^{\infty} A^{*s} Q A^s \right\| \leq \frac{\|A\|^m \|Q\| \|A\|^m}{1 - \|A^*\| \|A\|},$$

valid when $\|A^*\| \|A\| < 1$; however, this is plainly an unsatisfactory estimate. $\|A\|$ and $\|A^*\|$ may be large and (2) may still converge rapidly (for example, if A is nilpotent). The spectral radius formula

$$(3) \qquad \lim_{m \to \infty} \|A^m\|^{1/m} = |A|_\sigma$$

when $|A|_\sigma$ denotes $\max \{ |\lambda| : \lambda$ is an eigenvalue of $A \}$, suggests that we should look for an estimate for $\|A^m\|$ in terms of the eigenvalues of A: indeed, (3) shows that $\|A^m\|$ behaves roughly like $|A|_\sigma^m$ for sufficiently large m, but we need something more precise to get an estimate of the type we desire. Such a result can in fact be obtained by quite elementary means. In order to state the result we introduce the quantities $c(\nu, \ell)$, $\nu, \ell \geq 1$, given by the formula

$$(4) \quad c(\nu, \ell) = \begin{cases} \sum_{j=\ell}^{\nu} \binom{\nu}{j} (-1)^{j+1} & \text{if } \ell \leq \nu, \\ 0 & \text{if } \ell > \nu. \end{cases}$$

<u>Theorem 1. Let $\|\cdot\|$ be an operator norm on the algebra $M_N(\mathbb{C})$ of $N \times N$ complex matrices and let $A \in M_N(C)$ satisfy</u> $\|A\| \leq 1$.

(i) If $|A|_g \leq r < 1$ then, for $m \geq N$,

$$\| A^m \| \leq \binom{m}{N-1} r^{m-N+1} + 0\,(r^{m-N+2}).$$

More precisely,

$$(5) \qquad \| A^m \| \leq r^{m-N+1} \sum_{j=0}^{N-1} \binom{m-N+j}{j} (i + r)^j .$$

(ii) If $p(A) = 0$ where

$$p(z) = (z - \rho_1)(z - \rho_2) \ldots (z - \rho_n)$$

and $r_i = |\rho_i| < 1$, $1 \leq i \leq n$, then, for $m \geq n$,

$$(7) \qquad \| A^m \| \leq h_{m-n+1}(r_1, r_2, \ldots, r_n) + O(r^{m-n+2})$$

where $r = \max\{r_1, r_2, \ldots, r_n\}$ and $h_j(r_1, \ldots, r_n)$ denotes the sum of all monomials of degree j in $r_1, \ldots, r_n$.
More precisely,

$$(8) \qquad \| A^m \| \leq (-1)^{m+n+1} \sum_g (-1)^{\partial g} c(\nu(g), \; n + \partial g - m)\, g(r_1, \ldots, r_n)$$

where the sum is taken over all monomials g in n variables whose degree ∂g satisfies $m - n + 1 \leq \partial g \leq m$, and $\nu(g)$ denotes the number of variables occurring in g with positive exponent.

In particular in (ii) we can take p to be the characteristic polynomial of A to get an estimate involving only the eigenvalues of A.

A closely related extremal problem was investigated by V. Pták in an important paper [3]. He discussed the maximum possible value of $\| A^N \|$ for $N \times N$ matrices A satisfying $\|A\| \leq 1$ and $|A|_{\sigma} \leq r < 1$, where now $\|\cdot\|$ denotes the operator norm on N-dimensional Hilbert space, and found an operator for which this maximum is obtained (the restriction of a unilateral shift to a certain N-dimensional subspace of ℓ^2). A significant step in the proof consists in showing that one can take the extremal operator A to have the unique eigenvalue r; both this and the proof of Theorem 1 above depend upon certain technical facts about powers of companion matrices which we now describe.

Let T be the companion matrix of the complex polynomial of degree n, $n \geq 1$, whose roots are $\rho_1, \ldots, \rho_n$. That is, if p is the polynomial given by (6) and p is written

$$(9) \quad p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{m-2} + \ldots + a_0,$$

then in the case n = 1, $T = [\rho_1]$, and for $n \geq 2$,

$$(10) \quad T = \begin{bmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \cdot & \cdot & \cdot \ldots & \cdot \\ -a_0 & -a_1 & -a_2 \ldots & -a_{n-1} \end{bmatrix}$$

Here a_i is a function of the ρ's - in fact $(-1)^{n-i}$ times the elementary symmetric function of degree n - i + 1 in $\rho_1, \ldots, \rho_n$ - and we regard T as a matrix-valued function of $\rho_1, \ldots, \rho_n$.

It can be seen from the form of T that, for any matrix A, the first n - 1 rows of TA are the last n - 1 rows of A, so that when we form the sequence T, T^2, T^3, ... we introduce only one new row at each step. Introduce the $\infty \times n$ matrix T^{∞} whose i-th row is the first row of T^{i-1} ($i = 1,2,\ldots$, $T^0 = I$). One finds that, for any $m \geq 0$, T^m is obtained by taking rows $m + 1$ to $m + n$ of T^{∞}.

Denote by t_{ik} the (i,k) entry of T^{∞} $(1 \leq i < \infty$, $1 \leq k \leq n)$, t_{ik} is a scalar function of $(\rho_1,\ldots,\rho_n)$. Pták made use of the following fact.

Lemma. For $i \geq n + 1$, $(-1)^{n-k}\, t_{ik}$ is a homogeneous polynomial of degree $i - k$ in $\rho_1,\ldots,\rho_n$ with non-negative coefficients.

This Lemma has a slightly curious history. Pták proved the case $k = 1$ and conjectured that it was true in general. At his request the late Professor V. Knichal provided a proof of the conjecture, but since the proof was not published nor even circulated privately, the result has had a somewhat unsatisfactory status subsequent to Knichal´s death. Several mathematicians have asked how it was proved, and Professor Pták therefore suggested finding a proof and placing it on record. In the course of the proof which follows we obtain relations which yield explicit expressions for the entries of T^{∞}, and also lead to the inequalities in Theorem 1.

The fact that t_{ik} is a homogeneous polynomial of degree $i - k$ (for $i \geq n + 1$) can be proved directly from the definition of T^{∞} by induction on i. We shall concern ourselves with the non-trivial part, the assertion about the signs of the coefficients.

We begin by noting that the columns of T^{∞} are solutions of the recurrence relation with characteristic polynomial p; that is, for $1 \leq k \leq n$ and $i \geq 1$,

$$(11) \qquad t_{i+n,k} + a_{n-1}\, t_{i+n-1,k} + \ldots + a_0 t_{ik} = 0 \; .$$

Indeed, the left hand side of (11) is the (1,k) entry of $p(T)T^{i-1}$, so that (11) follows from the fact that p is the characteristic polynomial of T.

Let f_k be the generating function of the k-th column of T^{∞}:

$$(12) \qquad f_k(z) = \sum_{i=1}^{\infty} t_{ik}\, z^i \, .$$

We can suppose that $|\rho_j| < 1$, $1 \leq j \leq n$, so that $T^m \longrightarrow 0$ as $m \longrightarrow \infty$, which implies that the t_{ik} are bounded, for fixed values of the ρ's, and hence that (12) defines an analytic function in the open unit disc. Multiplying (12) by $a_{n-j}z^j$ we obtain

$$a_{n-j} z^j f_k(z) = a_{n-j} \sum_{i=j+1}^{\infty} t_{i-j,k} z^i$$

for $0 \leq j \leq n$, a_n being defined to be 1. Sum this from j = 0 to n: by virtue of (11), terms in z^i with $i > n$ vanish and we obtain

$$(13) \qquad \begin{aligned} q(z)\, f_k(z) &= \sum_{i=1}^{n} z^i \sum_{j=0}^{i-1} a_{n-j}\, t_{i-j,k} \\ &= \sum_{i=1}^{n} z^i \sum_{u=1}^{i} a_{n-i+u}\, t_{uk} \end{aligned}$$

for $1 \leq k \leq n$, where

$$(14) \qquad q(z) = z^n\, p(1/z) = 1 + a_{n-1}\, z + \ldots + a_0 z^n .$$

Since $t_{jk} = \delta_{jk}$ (the Kronecker symbol) for $1 \leq j \leq n$, (13) reduces to

$$(15) \qquad f_k(z) = q(z)^{-1}(a_n z^k + a_{n-1}z^{k+1} + \dots + a_k z^n)$$

$$= q(z)^{-1} z^k (q(z) - a_{k-1}z^{n-k+1} - \dots - a_o z^n)$$

$$(16) \qquad = z^k - z^{n+1} q(z)^{-1}(a_{k-1} + a_{k-2}z + \dots + a_o z^{k-1}).$$

Consider the function

$$(17) \qquad F(z) = \sum_{k=1}^{n} (-1)^{n-k} (f_k(z) - z^k).$$

From (16) we have

$$F(z) = -z^{n+1}q(z)^{-1} \sum_{k=1}^{n} (-1)^{n-k}(a_{k-1} + \dots + a_o z^{k-1})$$

$$= -z^{n+1}q(z)^{-1} \sum_{i=0}^{n-1} (-1)^{n-i-1} a_i (1 - z + z^2 - \dots + (-z)^{n-i-1})$$

$$= z^{n+1}q(z)^{-1}(1+z)^{-1} \sum_{i=0}^{n-1} (-1)^{n-i} a_i (1 - (-z)^{n-i})$$

$$(18) \qquad = z^{n+1}q(z)^{-1}(1+z)^{-1}\left(\sum_{i=0}^{n} a_i(-1)^{n-i} - \sum_{i=0}^{n} a_i z^{n-i}\right)$$

$$= z^{n+1}q(z)^{-1}(1+z)^{-1}\{q(-1) - q(z)\}$$

$$= z^{n+1}(1+z)^{-1}\{q(-1)\,q(z)^{-1} - 1\}.$$

Now $q(z) = (1 - \rho_1 z)(1 - \rho_2 z) \dots (1 - \rho_n z)$, so that $q(-1)\,q(z)^{-1} = x_1 x_2 \dots x_n$ where $x_j = (1 + \rho_j)/(1 - \rho_j z)$. Writing $x_1 x_2 \dots x_n - 1 = \sum_{j=1}^{n} x_1 x_2 \dots x_{j-1}(x_j - 1)$ and observing that $(1+z)^{-1}(x_j - 1) = \rho_j/(1 - \rho_j z)$ we obtain from (18)

$$F(z) = z^{n+1} \sum_{j=1}^{n} (1 + \rho_1) \ldots (1 + \rho_{j-1}) \ \rho_j (1 -$$

$$(19) \qquad - \rho_1 z)^{-1} \ldots (1 - \rho_j z)^{-1} .$$

From this it is clear that $F(z)$ can be written as a power series in z in which the coefficient of z^i, $i \geq n + 1$, is a polynomial in $\rho_1, \ldots, \rho_n$ with non-negative coefficients. Now it is clear from the definitions (17) and (12) of F and f_k that the coefficient of z^i in F, $i \geq n + 1$, is $\sum_{k=1}^{n} (-1)^{n-k} t_{ik}$. It follows that the latter sum is a polynomial in the ρ's with non-negative coefficients. If we group together the terms of degree $i - k$ in this polynomial we will get precisely $(-1)^{n-k} t_{ik}$, for we know that $t_{i1}, \ldots, t_{in}$ are all homogeneous polynomials of differing degrees. Thus $(-1)^{n-k} t_{ik}$ is a polynomial with non-negative coefficients, as claimed.

With a little extra effort we can find the entries of T^{∞} precisely.

Theorem 2. The entries of T^{∞} are

$$(20) \qquad t_{ik} = \begin{cases} \delta_{ik} & \text{if } 1 \leq i \leq n \\ \sum_{g} c(\nu(g), n - k + 1) g & \text{if } i \geq n + 1, \end{cases}$$

the sum being taken over all monomials g in $\rho_1, \ldots, \rho_n$ of degree $i - k$.

Recall that the symbols $\nu(g)$, $c(\nu, \ell)$ were defined earlier (see (4) and Theorem 1 (ii); see also the note on page 429).

Proof. The case $1 \leq i \leq n$ is trivial. For $i \geq n+1$, return to equation (18) and note that

$$q(z)^{-1} = \prod_{j=1}^{n}(1 - \rho_j z)^{-1} = \prod_{j=1}^{n}(1 + \rho_j z + \rho_j^2 z^2 + \ldots$$
$$= \sum_{j=0}^{\infty} h_j(\rho_1, \rho_2, \ldots, \rho_n) z^j$$

where $h_j(\rho_1, \ldots, \rho_n)$ denotes the sum of all monomials of degree j in $\rho_1, \ldots, \rho_n$ ($h_0 \equiv 1$). (18) thus yields

$$(21) \quad F(z) = z^{n+1}(1 - z + z^2 - \ldots)\{q(-1) \sum_{j=0}^{\infty} h_j z^j - 1\}.$$

Equating the coefficients of z^i in (21) we obtain, for $i \geq n+1$,

$$(22) \quad \sum_{k=1}^{n} (-1)^{n-k} t_{ik} = (-1)^{i-n} + q(-1) \sum_{j=0}^{i-n-1} (-1)^{i-n-1-j} h_j.$$

As in the previous proof we use the fact that the t_{ik}, $k = 1, \ldots, n$, do not interfere with each other: the right hand side of (22) is a polynomial in the ρ's and the sum of the terms of degree $i - k$ must equal $(-1)^{n-k} t_{ik}$. To calculate this sum note that

$$q(-1) = \sigma_0 + \sigma_1 + \ldots + \sigma_n$$

where σ_j is the elementary symmetric function of degree j in $\rho_1, \ldots, \rho_n$. We wish to find the sum of the terms of a given degree ∂ in the polynomial

$$(23) \quad (-1)^{i-n-1}(\sigma_0 + \sigma_1 + \ldots + \sigma_n)(h_0 - h_1 + \ldots$$
$$\ldots + (-1)^{i-n-1} h_{i-n-1}).$$

Let us think first which products $\sigma_j h_{\partial-j}$ of degree ∂ are defined. We must clearly have $0 \leq \partial \leq i - 1$, and then j must satisfy $0 \leq j \leq n$, $0 \leq \partial - j \leq i - n - 1$, which is equivalent to

(24) $\quad \max\{0, \partial - i + n + 1\} \leq j \leq \min\{\partial, n\}$.

Now consider how often a fixed monomial g of degree ∂ arises in the expansion of (23). Each occurrence of g arises from a product $\sigma_j h_{\partial - j}$ for some j satisfying (24), and such an occurrence carries the sign $(-1)^{i-n-1+\partial-j}$. If g contains $\nu(g)$ variables with positive exponent then each selection of j of them corresponds to an occurrence of g in the expansion of $\sigma_j h_{\partial-j}$: it follows that this expansion contains $g \binom{\nu(g)}{j}$ times $\left(\binom{\nu}{j}\right.$ is defined to be zero if $j > \nu$ or $\left. j < 0\right)$, for j satisfying (24). This assertion remains true if $j > \partial$ or $j < 0$, since the binomial coefficient is zero in these cases, and hence the coefficient of g in the expansion of (23) can be written

$$(25) \qquad \sum_{j=\partial-i+n+1}^{\nu(g)} \binom{\nu(g)}{j} (-1)^{i-n-1+\partial-j}$$

Observe that if $\partial - i + n + 1 \leq 0$, $\partial \neq 0$ (i.e. $0 < \partial < i - n$), (25) equals $(1 - 1)^{\nu(g)} = 0$, as we should expect from (22). Putting $\partial = i - k$ in (25) and multiplying by $(-1)^{n-k}$, we deduce from (22) that, for $i \geq n + 1$, $1 \leq k \leq n$,

$$t_{ik} = \sum_g \left(\sum_{j=n-k+1}^{\nu(g)} \binom{\nu(g)}{j} (-1)^{j+1} \right) g$$

where the sum is taken over all monomials g of degree $i - k$. This is precisely (20).

We can deduce particularly simple expressions for the entries in the first and last columns of T^{∞} (these can also be obtained directly from equations (15) and (16)). From (4) we have $c(\nu,n) = \binom{\nu}{n}(-1)^{n+1}$, and (20) then shows that t_{i1} is the sum of all monomials of degree $i - 1$ in $\wp_1, \ldots$ $\ldots \wp_n$ which involve all n variables with a positive exponent. This can be written (for $i \geq n + 1$)

$$(26) \qquad t_{i1} = (-1)^{n-1} \wp_1 \wp_2 \cdots \wp_n h_{i-n-1}(\wp_1, \ldots, \wp_n).$$

And again from (4), $c(\nu,1) = 1$ for all $\nu \leq n$, so that, for $i \geq n + 1$,

$$(27) \qquad t_{in} = h_{i-n}(\wp_1, \ldots, \wp_n).$$

Even without calculating any of the coefficients $c(\nu,k)$ we can make a striking deduction from Theorem 2: in any one column of T^{∞} the coefficient with which any monomial occurs depends only in the number of variables occurring in it and not on its degree. Let us illustrate this odd property by inspecting the second column of T^{∞} in the case $n = 3$. The (4,2) entry is $-(\wp_1\wp_2 + \wp_2\wp_3 + \wp_3\wp_1)$: it follows that the monomial $\wp_1^5 \wp_3^3$, occurring in the tenth row, also has coefficient $- 1$. On the other hand, since the monomial $\wp_1^2$ occurs with coefficient zero, there can be no pure powers of a variable anywhere in the second column. The only remaining type of monomial is one involving all of $\wp_1, \wp_2, \wp_3$; if we calculate the (5,3) entry of T^{∞} we find that it contains a term $-2 \wp_1 \wp_2 \wp_3$, and therefore all monomials of this last type in the second column occur with coefficient $- 2$. And in general, as soon as we know the first $n + k$

entries in the k-th column of T^{∞} we can write down any subsequent entry without calculation.

Proof of Theorem 1. We make use of a simple but surprisingly powerful observation of Z. Dostál [2]. Let A satisfy the hypotheses of Theorem 1 (ii) ($\| A \| \leq 1$, $p(A) = 0$); then, for $m \geq 0$,

$$\| A^m \| \leq \sum_{k=1}^{n} |t_{m+1,k}| . \tag{28}$$

To see this, introduce the $nN \times N$ matrix

$$H(A) = \begin{bmatrix} I_N \\ A \\ \cdot \\ \cdot \\ \cdot \\ A^{n-1} \end{bmatrix}$$

Since $p(A) = 0$, $A^n = - a_o I_N - a_1 A - \dots - a_{n-1} A^{n-1}$, and hence

$$H(A)\, A = \begin{bmatrix} A \\ A^2 \\ \cdot \\ \cdot \\ \cdot \\ -a_o I_N - a_1 A - \dots - a_{n-1} A^{n-1} \end{bmatrix} = (T \otimes I_N) H(A),$$

where $T \otimes I_N$ is the usual Kronecker product of matrices. It follows that $H(A) A^m = (T^m \otimes I_N) H(A)$, and on equating the $N \times N$ blocks in (1,1) position we obtain

$$A^m = t_{m+1,1} I_N + t_{m+1,2} A + \dots + t_{m+1,n} A^{n-1}.$$

Properties of operator norms then immediately yield (28).

Since $(-1)^{n-k} t_{m+1,k}$ is a polynomial in $\wp_1, \dots, \wp_n$

with non-negative coefficients, (28) implies

$$\| A^m \| \leq \sum_{k=1}^{n} (-1)^{n-k} t'_{m+1,k}, \tag{29}$$

the dash indicating evaluation at $(r_1,\dots,r_n)$, $r_i = |\varphi_i|$. By Theorem 2, $t'_{m+1,k}$ is the sum of all monomials in $r_1,\dots$ $\dots,r_n$ with $\partial g = m - k + 1$, each multiplied by the coefficient $c(\nu(g), n - k + 1)$. Writing $k = m - \partial g + 1$, we obtain

$$\sum_{k=1}^{n} (-1)^{n-k} t'_{m+1,k} = \sum_{k=1}^{n} \sum_{g} (-1)^{n+m+\partial g+1} c(\nu(g), n + \partial g - m) g \tag{30}$$

where the inner summation is over all monomials g in $(r_1,\dots,r_n)$ with $\partial g = m - k + 1$. Combining (29) and (30) we obtain (8).

To see (7), observe that the term of lowest degree on the right hand side of (29) occurs when $k = n$, so that

$$\| A^m \| \leq t'_{m+1,n} + O(r^{m-n+2}).$$

(7) is an immediate consequence of this and (27).

<u>Proof of Theorem 1</u> (i). Rather than deducing part (i) from part (ii) we find it simpler to return to inequality (29). Suppose, then, that $|A|_g \leq r$ and take p to be the characteristic polynomial of A, so that $n = N$ and $r_i = |\varphi_i| \leq r$, $1 \leq i \leq N$. Since the right hand side of (29) is a polynomial in $r_1,\dots,r_N$ with positive coefficients, it is not decreased if each r_i is replaced by r; we can therefore obtain an upper bound for $\| A^m \|$ by evaluating $\sum_{k=1}^{N} (-1)^{N-k} t_{m+1,k}$ with each φ_i equal to r. With this

choice of φ_i (19) shows that

$$F(z) = z^{N+1} r \sum_{j=1}^{N} (1 + r)^{j-1} (1 - rz)^{-j}$$

$$= z^{N+1} r \sum_{j=1}^{N} (1 + r)^{j-1} \sum_{s=0}^{\infty} \binom{j + s - 1}{s} r^s z^s.$$

Equating coefficients of z^{m+1}, $m \geq N$, we have

$$\sum_{k=1}^{N} (-1)^{N-k} t_{m+1,k} = r^{m-N+1} \sum_{j=0}^{N-1} \binom{m - N + j}{j} (1 + r)^j,$$

and (5) follows.

The remaining inequality in Theorem 1 follows from (7) and the fact that the number of monomials of degree j in N variables is $\binom{j + N - 1}{N - 1}$.

We conclude with a discussion of how good the estimates of Theorem 1 are. We note firstly that they only give information if, roughly speaking, r is small or m is large: otherwise the bounds obtained are not even less than 1. For particular operator norms one can certainly do better; for instance, in the case of the operator norm on N-dimensional Hilbert space bounds for $\| A^N \|$ are obtained in [4] which are always less than 1 when $r < 1$. However, for an arbitrary operator norm, the bounds (5) and (8) are in a sense sharp: (5) and (8) hold with equality for a suitable choice of $\| \cdot \|$ and A and sufficiently small r. This is a consequence of another observation of Dostál's, namely, that (28) holds with equality if we take $\| \cdot \|$ to be the operator norm on $\ell^{\infty}(n)$ and A to be the companion matrix of p. In this case $[t_{m+1,1},\dots,t_{m+1,n}]$ is the first row of T^m, and since $\| A \|$ is the maximum of the absolute row sums of A, we infer the opposite inequality to (28). Thus, if we take the roots of p to be real and non-negative we have equality in (29). Since the right hand side of (29) and (8) are equal, (8) also holds with equality when T = A, and provided we make r small enough, the constraint $\| T \| \leq 1$ will also be satisfied.

I should like to express my thanks to the Mathematical Institute of the Czechoslovak Academy of Sciences for its hospitality and financial support during the year 1978, when this work was carried out.

Note. Z. Dostál has pointed out to me that formula (4) for the quantities $c(\nu, \ell)$ can be simplified. If we write $\binom{\nu}{j} = \binom{\nu - 1}{j - 1} + \binom{\nu - 1}{j}$, $1 \leq j < \nu$, we find that

$$c(\nu,\ell) = \begin{cases} (-1)^{\ell+1} \binom{\nu - 1}{\ell - 1} & \text{if } 1 \leq \ell \leq \nu, \\ 0 & \text{otherwise.} \end{cases}$$

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