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DISTRIBUTIVE STEINER QUASIGROUPS OF ORDER 3^5

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Abstract: All distributive Steiner quasigroups of order 243 are described.

Key words: Distributive, Steiner, quasigroup, order.

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The class of distributive Steiner quasigroups is interesting from the algebraic as well as combinatorial point of view. It plays a prominent rôle in the structure theory of distributive groupoids, distributive quasigroups, trimedial quasigroups, F-quasigroups, etc. Although several relatively deep results concerning distributive Steiner quasigroups are known, the systematic treatment is not available. For example, the complete description of finite distributive Steiner quasigroups (these have necessarily 3^n elements) is known only up to 3^4 elements. It is the purpose of the present note to describe distributive Steiner quasigroups of order 3^5. In particular, an answer to a question formulated in [3 , p. 44] is given.

1. A groupoid G is said to be
- commutative if it satisfies the identity $xy = yx$,

- distributive if it satisfies the identities $x.yz = xy.xz$ and $yz.x = yx.zx$,
- idempotent if it satisfies the identity $x = xx$,
- medial if it satisfies the identity $xy.uv = xu.yv$,
- symmetric if it satisfies the identities $x = y.yx$ and $x = xy.y$.

Obviously, every symmetric groupoid is a commutative quasigroup, every symmetric distributive groupoid is idempotent and every medial idempotent groupoid is distributive. The symmetric idempotent groupoids are called sometimes Steiner quasigroups (due to the obvious equivalence between these groupoids and Steiner triple systems). Thus the distributive Steiner quasigroups are just groupoids satisfying the identitites $xy = yx$, $x = y.yx$ and $x.yz = xy.xz$.

Let G be a groupoid. We denote by p_G the least congruence of G such that the corresponding factor is medial. Further, if M is a non-empty subset of G, then [M] is the subgroupoid generated by M. Finally, o(G) is the least cardinal number with $o(G) = |M|$ for a non-empty generator set M of G.

1.1. Proposition. Let G be a distributive Steiner quasigroup.

(i) If $a,b,c,d \in G$ and $ab.cd = ac.bd$ then the subgroupoid $[a,b,c,d]$ is medial.

(ii) For all $a,b,c, \in G$, the subgroupoid $[a,b,c]$ is medial.

(iii) If $o(G) \leq 3$ then G is medial.

(iv) p_G is just the intersection of all maximal congruence of G.

(v) If G is finite then $o(G) = o(G/p)$ and $|G| = 3^n$ for some $0 \leq n$.

(vi) $|G| \geq 81$, provided G is not medial.

(vii) G is finite, provided it is finitely generated.

Proof. (i) See [1, Theorem 8.6] or [4, § II.5, Théorème 1].

(ii) and (iii) These assertions are immediate consequences of (i).

(iv) See [4, § V.5, Proposition 6].

(v) See [4, § V.5, Proposition 7, Proposition 3].

(vi) See [4, § VI.6, Lemme 2].

(vii) See [4, § V.2, Théorème 2].

In this paper, let Z(3) designate the three-element field with elements 0,1,2. Put $x*y = -x-y$ for all $x,y \in Z(3)$. Obviously, $Z(3)(*)$ is a distributive Steiner quasigroup and we shall denote it by T(2) (it is visible that T(2) is a free Steiner quasigroup of rank 2).

1.2. Proposition. Let G be a medial distributive Steiner quasigroup such that $o(G) = n$ is finite. Then G is isomorphic to the cartesian product $T(2)^{n-1}$.

Proof. The statement is well known and easy.

Let G be a distributive Steiner quasigroup. Define a relation q_G on G as follows: a q b iff the subgroupoid $[a,b,x,y]$ is medial for all $x,y \in G$. According to 1.1, a q b iff $ab.xy = ax.by$ for all $x,y \in G$.

1.3. Proposition. Let G be a distributive Steiner quasigroup.

(i) q_G is a congruence of G.

(ii) If H is a subgroupoid of G such that H is contained in a block of q_G, then H is a block of a congruence of G.
(iii) If G is finite and r is a congruence of G such that $r \cap q_G = d_G$, where d_G is the diagonal congruence of G, then $r = d_G$.

Proof. (i) See [4, § IV.4, Proposition 1] or [3, Lemma 2.7].
(ii) See [4, § IV.9, Proposition 3] or [3, Lemma 2.14].
(iii) See [3, Lemma 4.3].

1.4. Proposition. Let G be a finite non-trivial distributive Steiner quasigroup. The following conditions are equivalent:
(i) G is subdirectly irreducible.
(ii) At least one of the blocks of q_G contains exactly 3 elements.
(iii) Every block of q_G contains exactly 3 elements.
(iv) Every block of q_G is a subgroupoid isomorphic to T(2).

Proof. See [3, Satz 4.4] (the proposition is an easy consequence of 1.2 and 1.3).

A distributive Steiner quasigroup G is said to be nilpotent of class at most 2 if the factor G/q is medial, i.e., $p_G \subseteq q_G$. It is visible that the class of distributive Steiner quasigroups nilpotent of class at most 2 is a groupoid variety. This variety is determined in the variety of distributive Steiner quasigroups by the identity $((xy.uv)z)(w(xu.yv)) = ((xy.uv)w)(z(xu.yv))$.

1.5. Proposition. Let G be a distributive Steiner quasigroup with $o(G) \leq 4$. Then G is nilpotent of class at most 2.

Proof. See [4, § V.3, Théorème 1] or [3, Satz 2.4].

The reader is referred to [1],[3] and [4] for further results and details concerning distributive (Steiner) quasigroups.

2.

2.1. Lemma. Let F be a field and $V = F^4$. Let W be a subspace with $\dim W = 3$ of the vector space V. Then there exists a non-zero element $a = \langle a_1, a_2, a_3, a_4 \rangle \in W$ such that the elements $\langle a_1, 0, 0, a_4 \rangle$, $\langle 0, a_2, a_3, 0 \rangle$, $\langle a_3, a_4, 0, 0 \rangle$, $\langle 0, 0, a_1, a_2 \rangle$, $\langle a_2, 0, -a_4, 0 \rangle$ and $\langle 0, a_1, 0, -a_3 \rangle$ belong to W.

Proof. The proof will be divided into three steps.

(i) Suppose that there is $c \in F$ such that $x_2 = cx_1$, whenever $\langle x_1, x_2, x_3, x_4 \rangle \in W$. Let A be the set of all $\langle y_1, y_2, y_3, y_4 \rangle$ from V with $y_2 = cy_1$. Clearly, A is a subspace of V and $W \subseteq A \subseteq V$. But $A \neq V$ and $\dim W = 3$. Hence $A = W$ and we can put $a = \langle 0, 0, 1, c \rangle$.

(ii) Suppose that there is $d \in F$ such that $x_1 = dx_2$, whenever $\langle x_1, x_2, x_3, x_4 \rangle \in W$. Similarly as in (i), we can put $a = \langle 0, 0, d, 1 \rangle$.

(iii) Suppose that neither (i) nor (ii) may be applied. Define a mapping f of W into $B = F^2$ by $f(\langle x_1, x_2, x_3, x_4 \rangle) = \langle x_1, x_2 \rangle$. Clearly, f is a homomorphism and, taking into account the hypothesis, it is easy to see that $\dim f(W) = 2$. Hence $f(W) = B$ and there are two elements $u, v \in W$ such that $u = \langle 1, 0, u_3, u_4 \rangle$ and $v = \langle 0, 1, v_3, v_4 \rangle$. Since u, v are independent, there is $z = \langle z_1, z_2, z_3, z_4 \rangle \in W$ such that $\{u, v, z\}$ is a basis of W. We can assume that $z_1 = 0 = z_2$. Now, we must distinguish the following two cases:

(iii 1) Let $z_3 = 0$. Then $z_4 \neq 0$ and we can assume that $z_4 = 1$ and $u_4 = 0 = v_4$. Put $a = \langle 0,1,v_3,-u_3\rangle$. Then the elements $\langle a_1,0,0,a_4\rangle = \langle 0,0,0,-u_3\rangle = -u_3z$, $\langle 0,a_2,a_3,0\rangle = \langle 0,1,v_3,0\rangle = v$, $\langle a_3,a_4,0,0\rangle = \langle v_3,-u_3,0,0\rangle = v_3u-u_3v$, $\langle 0,0,a_1,a_2\rangle = \langle 0,0,0,1\rangle = z$, $\langle a_2,0,-a_4,0\rangle = \langle 1,0,u_3,0\rangle = u$ and $\langle 0,a_1,0,-a_3\rangle = \langle 0,0,0,-v_3\rangle = -v_3z$ belong to W.

(iii 2) Let $z_3 \neq 0$. We can assume that $z_3 = 1$ and $u_3 = 0 = v_3$. Put $a = \langle 1,z_4,-v_4,u_4\rangle$. Then the elements $\langle a_1,0,0,a_4\rangle = \langle 1,0,0,u_4\rangle = u$, $\langle 0,a_2,a_3,0\rangle = \langle 0,z_4,-v_4,0\rangle = z_4v-v_4z$, $\langle a_3,a_4,0,0\rangle = \langle -v_4,u_4,0,0\rangle = -v_4u+u_4v$, $\langle 0,0,a_1,a_2\rangle = \langle 0,0,1,z_4\rangle = z$, $\langle a_2,0,-a_4,0\rangle = \langle z_4,0,-u_4,0\rangle = z_4u-u_4z$ and $\langle 0,a_1,0,-a_3\rangle = \langle 0,1,0,v_4\rangle = v$ belong to W.

3. Throughout this paragraph, let $G(+)$ be an abelian group such that $3x = 0$ for every $x \in G$ and F be a trilinear mapping of $G(+)$ (i.e., F is a ternary operation on G such that $G(+,F)$ is a ternary ring). Consider the following conditions:

(1) $F(x,x,y) = 0$ for all $x,y \in G$.

(2) $F(x,F(x,y,x-y),x-y) = 0$ for all $x,y \in G$.

(3) $F(x,y-F(x,y,x-y),F(x,y,x-y)) = 0$ for all $x,y \in G$.

(4) $F(x,y,x-y) + F(y,x,x-y) = 0$ for all $x,y \in G$.

(5) $F(x,y,z) + F(y,x,z) = 0$ for all $x,y,z \in G$.

(6) $F(F(x,y,x-y),z,u) = 0 = F(z,u,F(x,y,x-y))$ for all $x,y,z,u \in G$.

(7) $F(F(x,y,z),u,v) = 0 = F(u,v,F(x,y,z))$ for all $x,y,z,u,v \in G$.

Further, we shall define a new binary operation $*$ on G by $x \; y = -x - y + F(x,y,x-y)$ for all $x,y \in G$.

3.1. Proposition. (i) $G(*)$ is a Steiner quasigroup, provided the conditions (1),(2),(3),(4) are satisfied.
(ii) $G(*)$ is a distributive Steiner quasigroup, provided the conditions (5),(6) are satisfied.

Proof. Easy.

Put $K(x,y,z) = F(x,y,z) + F(y,z,x) + F(z,x,y)$ for all $x,y,z \in G$. It is visible that K is a trilinear mapping of $G(+)$.

3.2. Lemma. Let the conditions (5),(6) be satisfied. Then:
(i) For all $x,y,u,v \in G$, $((x*y)*(u*v)) - ((x*u)* *(y*v)) = K(x,y,u-v) + K(u,v,x-y)$.
(ii) For $a,b \in G$, $a \; q_{G(*)} \; b$ iff $K(a-b,x,y) = 0$ for all $x, y \in G$.
(iii) $q_{G(*)}$ is a congruence of both $G(*)$ and $G(+)$; the corresponding subgroup is equal to $\{ x \in G \mid K(x,y,z) = 0$ for all $y,z \in G \}$.

Proof. Easy.

3.3. Proposition. Let the conditions (5) and (7) be satisfied. Then $G(*)$ is a distributive Steiner quasigroup nilpotent of class at most 2.

Proof. Use 3.1 and 3.2.

3.4. Lemma. Let $H(+)$ be a subgroup of $G(+)$ such that $F(a,x,y)$, $F(x,a,y)$, $F(x,y,a) \in H$ for all $a \in H$ and $x,y \in G$. Define a relation r on G by $x \; r \; y$ iff $x-y \in H$. Then r is a congruence of both $G(*)$ and $G(+)$.

Proof. Evident.

4. Let $4 \le n$ be a natural number and $m = n - 1 + \binom{n-1}{3}$. Denote by M the set of all ordered triples $\langle j,k,l\rangle$ such that $1 \le j < k < l \le n-1$. Then there exists just one bijective mapping f of the set $\{n,n+1,\dots,m\}$ onto M such that if $n \le i < i' \le m$, $f(i) = \langle j,k,l\rangle$ and $f(i') = \langle j',k',l'\rangle$ then either $j<j'$ or $j = j'$, $k<k'$ or $j = j'$, $k = k'$, $l<l'$. Put $G = Z(3)^m$ and define a trilinear mapping F of the group $G(+)$ as follows: Let $a = \langle a_1,\dots,a_m\rangle$, $b = \langle b_1,\dots,b_m\rangle$, $c = \langle c_1,\dots,c_m\rangle \in G$. If $n \le i \le m$ and $f(i) = \langle j,k,l\rangle$ then the i-th component of $F(a,b,c)$ is equal to $(a_jb_k - b_ja_k)c_l$; if $1 \le i < n$ then the i-th component of $F(a,b,c)$ is equal to 0. It is visible that F satisfies the conditions (5) and (7). Now, consider the groupoid $G(*)$ defined by $x*y = -x-y + F(x,y,x-y)$. By 3.3, $G(*)$ is a distributive Steiner quasigroup nilpotent of class at most 2. In the following, we shall use the notation $T(n)$ for $G(*)$.

4.1. Proposition. (i) $T(n)$ is a free distributive Steiner quasigroup nilpotent of class at most 2 of rank n. (ii) If $a = \langle a_1,\dots,a_m\rangle$, $b = \langle b_1,\dots,b_m\rangle \in T(n)$ then $a\ q\ b$ iff $a_1 = b_1,\dots,a_{n-1} = b_{n-1}$.

Proof. Apply [3, 5.4 - 5.8] and 3.2 (ii) (see also [2, Theorem 9A]).

4.2. Proposition. Let $n = 5$ and $r \subseteq q_{G(*)}$ be a congruence of $G(*) = T(5)$. Then every block of $q_{G(*)/r}$ contains at least four elements.

Proof. We have $m = 8$ and $f(5) = \langle 1,2,3\rangle$, $f(6) =$

$= \langle 1,2,4 \rangle$, $f(7) = \langle 1,3,4 \rangle$, $f(8) = \langle 2,3,4 \rangle$. Put $A = \{\langle a_1,\dots,a_8 \rangle \in G \mid a_1 = a_2 = a_3 = a_4 = 0\}$ and $B = \{a \in G \mid a\ r\ 0\}$. By 4.1 (ii), A is the block of $q_{G(*)}$ containing the element 0. Since $r \subseteq q_{G(*)}$, $B \subseteq A$. Moreover, $A(+)$ is a subspace of the vector space $G(+)$ over $Z(3)$ and $\dim A(+) = 4$. On the other hand, $F(a,x,y) = F(x,a,y) = F(x,y,a) = 0$ for all $a \in A$, $x,y \in G$ and it is easy to see that $B(+)$ is a subspace of $G(+)$, too. It follows from 3.4 that $x\ r\ y$ iff $x-y \in B$. Denote by g the natural homomorphism of $G(*)$ onto $G(*)/r = H(*)$. Obviously, $g(A)$ is contained in a block of $q_{H(*)}$, and hence the assertion is clear in case $\dim B(+) \le 2$. Further, if $\dim B(+) = 4$, then $B = A$, $r = q_{G(*)}$ and $H(*)$ is medial, since $G(*)$ is nilpotent of class at most 2. However, $|H| = 81$ and $q_{H(*)} = H(*) \times H(*)$. Thus we can assume in the rest of the proof that $\dim B(+) = 3$. In that case, $g(A)$ contains exactly 3 elements. By 2.1, there exists a non-zero element $\langle 0,0,0,0,a_1,a_2,a_3,a_4 \rangle \in B$ such that the elements $a = \langle 0,0,0,0,a_3,a_4,0,0 \rangle$, $b = \langle 0,0,0,0,a_2,0,-a_4,0 \rangle$, $c = \langle 0,0,0,0,a_1,0,0,a_4 \rangle$, $d = \langle 0,0,0,0,0,a_2,a_3,0 \rangle$, $e = \langle 0,0,0,0,0,a_1,0,-a_3 \rangle$, $h = \langle 0,0,0,0,0,0,a_1,a_2 \rangle$ belong to B. Put $x = \langle a_1,a_2,a_3,a_4,0,0,0,0 \rangle$. Then $x \notin A$ and it suffices to show that $g(x)\ q_{H(*)}\ g(0)$. We must prove that $(0*x)*(y*z)\ r\ (0*y)*(x*z)$, i.e., $((0*x)*(y*z)) - ((0*y)*(x*z)) \in B$ for all $y,z \in G$. However, $((0*x)*(y*z)) - ((0*y)*(x*z)) = F(z,y,x) + F(x,z,y) + F(y,x,z)$ by 3.2 (i). Let $y = \langle y_1,\dots,y_8 \rangle$, $z = \langle z_1,\dots,z_8 \rangle$ and $w = \langle w_1,\dots,w_8 \rangle = F(x,y,x) + F(x,z,y) + F(y,x,z)$. We have $w_1 = w_2 = w_3 = w_4 = 0$ and

$$w_5 = (y_1a_2 - a_1y_2)z_3 + (a_1z_2 - z_1a_2)y_3 + (z_1y_2 - y_1z_2)a_3,$$
$$w_6 = (y_1a_2 - a_1y_2)z_4 + (a_1z_2 - z_1a_2)y_4 + (z_1y_2 - y_1z_2)a_4,$$
$$w_7 = (y_1a_3 - a_1y_3)z_4 + (a_1z_3 - z_1a_3)y_4 + (z_1y_3 - y_1z_3)a_4,$$
$$w_8 = (y_2a_3 - a_2y_3)z_4 + (a_2z_3 - z_2a_3)y_4 + (z_2y_3 - y_2z_3)a_4.$$

Consequently

$$w_5 = (y_3z_2 - y_2z_3)a_1 + (y_1z_3 - y_3z_1)a_2 + (y_2z_1 - y_1z_2)a_3,$$
$$w_6 = (y_4z_2 - y_2z_4)a_1 + (y_1z_4 - y_4z_1)a_2 + (y_2z_1 - y_1z_2)a_4,$$
$$w_7 = (y_4z_3 - y_3z_4)a_1 + (y_1z_4 - y_4z_1)a_3 + (y_3r_1 - y_1z_3)a_4,$$
$$w_8 = (y_4z_3 - y_3z_4)a_2 + (y_2z_4 - y_4z_2)a_3 + (y_3z_2 - y_2z_3)a_4.$$

Finally, $w = (y_2z_1 - y_1z_2)a + (y_1z_3 - y_3z_1)b + (y_3z_2 -$ $- y_2z_3)c + (y_1z_4 - y_4z_1)d + (y_4z_2 - y_2z_4)e + (y_4z_3 -$ $- y_3z_4)h \in B$.

5.

5.1. <u>Proposition</u>. There is no subdirectly irreducible distributive Steiner quasigroup G such that $o(G) = 5$ and G is nilpotent of class at most 2.

<u>Proof</u>. Suppose, on the contrary, that such a quasigroup G exists. By 4.1 (i), there is a congruence r of T(5) such that G is isomorphic to T(5)/r and we can assume that $G = T(5)/r$. First, we are going to show that $r \subseteq q_{T(5)}$. For, let g be the natural homomorphism of T(5) onto G. There is a congruence s of T(5) with $r \subseteq s$ and $s/r = p_G$. But $o(G) =$ $= 5 = o(G/p)$ (apply 1.1 (v),(vii)). According to 1.2, $|G/p| = 3^4 = 81$. Consequently $|T(5)/s| = 81$. Since G/p is medial, $p_{T(5)} \subseteq s$. However $|T(5)/p| = 81$ by the same argu-

ment and we see that $s = p_{T(5)}$. Finally, since $T(5)$ is nilpotent of class at most 2, $p_{T(5)} \subseteq q_{T(5)}$. Thus $r \subseteq q_{T(5)}$. Now, with respect to 4.2, every block of q_G contains at least 4 elements, a contradiction with 1.4.

5.2. <u>Remark</u>. If $1 \leq n$, $n \neq 3, 5$, then, by [3, Satz 5.12], there exists a subdirectly irreducible Steiner quasigroup G such that G is distributive, nilpotent of class at most 2 and $o(G) = n$. It is clear that $|G| = 3^n$, provided $4 \leq n$, and $|G| = 3^{n-1}$ for $n = 1,2$. For $n = 3,5$ such a quasigroup does not exist as it follows from 1.2 and 5.1.

5.3. <u>Theorem</u>. (i) If G is a finite distributive Steiner quasigroup then $|G| = 3^n$ for some $0 \leq n$.
(ii) $T(2)$ is up to isomorphism the only distributive Steiner quasigroup of order $3^1 = 3$.
(iii) $T(2)^2$ is up to isomorphism the only distributive Steiner quasigroup of order $3^2 = 9$.
(iv) $T(2)^3$ is up to isomorphism the only distributive Steiner quasigroup of order $3^3 = 27$.
(v) $T(2)^4$ and $T(4)$ are up to isomorphism the only distributive Steiner quasigroups of order $3^4 = 81$.
(vi) $T(2)^5$ and $T(2) \times T(4)$ are up to isomorphism the only distributive Steiner quasigroups of order $3^5 = 243$.

<u>Proof</u>. (i) See 1.1 (v).
(ii), (iii) and (iv). These assertions follow from 1.1 (vi) and 1.2.
(v) Let G be a distributive Steiner quasigroup of order 81. With regard to 1.2, we can assume that G is not medial. Then $|G/p| \leq 27$, and so $o(G) \leq 4$. By 1.5, G is nilpotent of

class at most 2, and therefore G is a homomorphic image of $T(4)$ (use 4.1 (i)). However, both G and $T(4)$ have the same number of elements, and consequently G is isomorphic to $T(4)$,

(vi) Let G be a distributive Steiner quasigroup of order 243. We can assume that G is not medial. If $o(G) \leq 4$ then G is a homomorphic image of $T(4)$, and so $|G| \leq 81$, a contradiction. Hence $o(G) = o(G/p) \geq 5$. According to 1.2, $|G/p| \geq 81$. Since $|G| = 243$, $|G/p| = 81$ and every block of p_G contains just 3 elements. By 1.1 (v) and 1.2, $o(G) = o(G/p) = 5$. On the other hand, every block of p_G is isomorphic to $T(2)$ and we see that p_G is a minimal congruence of G. It follows from 1.3 (iii) that p_G is contained in q_G. Consequently, G is nilpotent of class at most 2. If $p_G = q_G$ then G is subdirectly irreducible by 1.4, a contradiction with 5.1. Hence $p_G \neq q_G$ and there are $a,b \in G$ such that $a \ q_G \ b$ and $\langle a,b \rangle \notin p_G$. Put $A = \{a,b,ab\}$. Then A is a subgroupoid of G and A is contained in a block of q_G. In view of 1.3 (ii), A is a block of a congruence r of G. Clearly, r is a minimal congruence of G and r is not contained in p_G. By 1.1 (iv), there is a maximal congruence s of G such that r is not contained in s. Due to the minimality of r, $r \cap s = d_G$ and G is isomorphic to a subgroupoid of the cartesian product $G/r \times G/s$. Since every block of r contains exactly 3 elements, $|G/r| = 81$. Further, $p_G \subseteq s$, G/s is medial and G/s is isomorphic to $T(2)$, since s is maximal (apply 1.2). In particular, $|G/s| = 3$ and $|G/r \times G/s| = 243 = |G|$. Thus G is isomorphic to $G/r \times T(2)$. Finally, since G is not medial, G/r is not medial and G/r is isomorphic to $T(4)$ by (v).

5.4. Remark. As it is proved in [3], there exists a subdirectly irreducible distributive Steiner quasigroup of order $3^6 = 729$. Hence there are at least 3 non-isomorphic distributive Steiner quasigroups of order 729.

5.5. Remark. Combining 5.2 with 5.3, we see that there exists a subdirectly irreducible Steiner quasigroup which is distributive and has order 3^n iff $0 \leq n$ and $n \neq 2,3,5$.

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