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COMPACTNESS as $\mathcal{C}$-PSEUDOCOMPACTNESS

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Abstract: We prove that if $\mathcal{C}$ is a class of Tychonoff spaces and the class of $\mathcal{C}$-pseudocompact spaces equals to the class of compact Hausdorff spaces then there exists an $E \in \mathcal{C}$ such that $\mathcal{C} \setminus \{E\}$ has this property, too. This answers a question of A. Šostak.

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1. Introduction. Classes of topological spaces satisfying many of the properties of the class of compact spaces are widely investigated by topologists. This trend includes studies of reflective subcategories of Hausdorff spaces, topological extension properties, $\mathcal{C}$-pseudocompactness and so on. If $\mathcal{C}$ is a class of Hausdorff spaces then an $\mathcal{C}$-regular space X is said to be $\mathcal{C}$-pseudocompact provided that for any $E \in \mathcal{C}$ and for a continuous map f from X to E $cl_E f[X]$ is compact. So $\{\mathbb{R}\}$-pseudocompactness is just pseudocompactness. $\mathcal{C}$-pseudocompactness has many of the properties possessed by pseudocompactness. For example an $\mathcal{C}$-compact space is $\mathcal{C}$-pseudocompact if and only if it is compact. The basic references for this material are in [6] and in [7]. The class of $\mathcal{C}$-pseudocompact spaces

will be denoted by $\mathcal{P}(\mathcal{E})$.

It is necessary to remark that $\mathcal{E}$-pseudocompactness defined in [6] is quite similar to the pseudocompactness property introduced in [7] but is not the same. If $\mathcal{E}$ consists of Tychonoff spaces and contains the space $[0,1] = I$ then $\mathcal{P}(\mathcal{E})$ is a pseudocompactness property in the sense of [7].

Another characterization of $\mathcal{E}$-pseudocompactness is based upon the following assertion. Let f be a continuous map of X to E. Then the following statements are equivalent

(i) $c\ell_E f[X]$ is compact

(ii) If $\mathcal{U}$ is an ultrafilter on X then the filter $f[\mathcal{U}]$ converges in E.

In this paper such classes $\mathcal{E}$ are investigated for which $\mathcal{E}$-pseudocompactness is equal to compactness. This problem was posed by Šostak in [6].

We borrow the notion of k-closure from the theory of k-compact spaces. (See [3].) If a topology on X and an infinite cardinal k are given then the basis consisting of all sets $\bigcap\{G_j : j \in J\}$ (where G_j is open for $j \in J$ and $|J| < k$) defines a new topology. The closure operation in this topology will be denoted by $c\ell_k$.

The space $I^k \setminus \{(1)\}$ can be familiar as an universal k^+-compact space. We need that $\beta(I^k \setminus \{(1)\}) = \beta_k(I^k \setminus \{(1)\}) = I^k$ for $k > \omega$ (where β_k stands for the k-compactification). (See [1] and [4].)

2. Results

Lemma. Let Y be a Tychonoff extension of X and k be an infinite cardinal. If $p \in Y \setminus cl_{k^+} X$ then there is a continuous map $f: Y \to I^k$ such that $f(p) = (1)$ and $f(p) \notin f[X]$.

<u>Proof</u>. Because of $p \notin cl_{k^+} X$ there are neighbourhoods $G_j (j \in J, |J| = k)$ such that $\bigcap \{G_j \cap X : j \in J\} = \emptyset$. Let f_j be a continuous function from X to I such that $f_j(p) = 1$ and vanishes out of G_j $(j \in J)$. Then $f = \langle f_j \rangle_{j \in J}$ satisfies the condition.

<u>Theorem</u>. Let k be an infinite cardinal. X is $\{I^k \setminus \{(1)\}\}$-pseudocompact if and only if $\beta_{k^+} X = \beta X$.

<u>Proof</u>. Assume that X is $\{I^k \setminus \{(1)\}\}$-pseudocompact. Then X is $\{I^k \setminus \{(1)\}\}$-regular so it is a Tychonoff space. If $p \in \beta X \setminus \beta_{k^+} X$ then by the lemma there is a continuous map $f: \beta X \to I^k$ with $f \mid X : X \to I^k \to \{(1)\}$ and $f(p) = (1)$. Consequently $cl f[X]$ is not compact (where cl stands for the closure in $I^k \setminus \{(1)\}$). So $\beta X = \beta_{k^+} X$.

Conversely, let $\beta_{k^+} X = \beta X$. Then X is a Tychonoff space so it is $\{I^k \setminus \{(1)\}\}$-regular. Let $f: X \to I^k \setminus \{(1)\}$ be a continuous map. Using the k^+-compactness of $I^k \setminus \{(1)\}$ f admits an $\tilde{f}: \beta_{k^+} X \to I^k \setminus \{(1)\}$ extensions. So $cl f[X] \subset cl \tilde{f}[\beta_{k^+} X] = cl \tilde{f}[\beta X] = \tilde{f}[\beta X]$ and we have $cl f[X]$ to be compact.

<u>Corollary</u>. If $\mathcal{C}$ is a proper class of spaces of the form of $I^k \setminus \{(1)\}$ then $\mathcal{C}$-pseudocompactness is equal to compactness.

<u>Corollary</u>. There are two disjoint classes $\mathcal{C}_1$ and

$\mathcal{C}_2$ such that $\mathcal{C}_i$-pseudocompactness is equal to compactness for $i = 1,2$.

<u>Theorem</u>. There is no Tychonoff space E such that $\mathcal{P}(\{E\})$ is identical with the class of compact Hausdorff spaces.

<u>Proof</u>. Indirectly, if $\mathcal{P}(\{E\})$ is the class of compact Hausdorff spaces then $\{E\}$-regularity is the same as complete regularity. So for every cardinal k $I^k \setminus \{(1)\}$ is $\{E\}$-regular. We can choose a $k > \omega$ such that E is k-compact. If $f: I^k \setminus \{(1)\} \to E$ is a continuous map then it admits an extension $\tilde{f}: \beta_k(I^k \setminus \{(1)\}) \to E$ i.e. $\tilde{f}: I^k \to E$. Hence $c\ell_E f[I^k \setminus \{(1)\}]$ is compact and so $I^k \setminus \{(1)\}$ belongs to $\mathcal{P}(\{E\})$. The contradiction proves the theorem.

<u>Corollary</u>. There is not set of Tychonoff spaces $\mathcal{C}$ such that $\mathcal{P}(\mathcal{C})$ is the class of compact Hausdorff spaces.

<u>Theorem</u>. Let $\mathcal{C}$ be a class of Tychonoff spaces. If the class of $\mathcal{C}$-pseudocompact spaces is identical with the class of compact Hausdorff spaces then there is a space E belonging to $\mathcal{C}$ with $\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathcal{C} \setminus \{E\})$.

<u>Proof</u>. By the previous corollary $\mathcal{C}$ is a proper class. Hence there is an E from $\mathcal{C}$ such that $\mathcal{C} \setminus \{E\}$-regularity is equal to $\mathcal{C}$-regularity. So the inclusion $\mathcal{P}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C} \setminus \{E\})$ is evident. Let X be a space not belonging to $\mathcal{P}(\mathcal{C})$. If X is not a Tychonoff space then X does not belong to $\mathcal{P}(\mathcal{C} \setminus \{E\})$. If X is a Tychonoff space then there are an F from $\mathcal{C}$ and a continuous map f of X to E such that $Y = c\ell_F[X]$ is not compact. We take a cardinal

$k > \omega$ such that F and Y are k-compact. Using the lemma we get a map $g: \beta Y \longrightarrow I^k$ with $g(p) = (1)$ for some $p \in \beta Y \setminus Y$ and $g[Y] \subset I^k \setminus \{(1)\}$. $I^k \setminus \{(1)\}$ is not compact hence there are a G from $\mathcal{C}$ and a continuous map h from $I^k \setminus \{(1)\}$ to G such that $c\ell_G\, h[I^k \setminus \{(1)\}]$ is not compact. h has an extension $\tilde{h}: \beta_k(I^k \setminus \{(1)\}) \longrightarrow \beta_k G$ i.e. $\tilde{h}: I^k \longrightarrow \beta_k G$. We observe that $\tilde{h}((1)) \in \beta_k G \setminus G$ so G is not k-compact, consequently G is different from E. $h \circ g \circ f: X \longrightarrow G$ and $h \circ g \circ f[X]$ is not relatively compact in G. So we have that X is not $\mathcal{C} \setminus \{E\}$-pseudocompact.

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