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GRAPHS WITH GIVEN SUBGRAPHS REPRESENT ALL CATEGORIES II.

Václav KOUBEK, Praha

Abstract: We characterize sets $\mathcal{G}$ of graphs for which the category GRA of (all) graphs can be fully embedded into its full subcategory $\mathcal{G}$(GRA) defined as follows: a graph G belongs to $\mathcal{G}$(GRA) if for each edge in G and each graph $H \in \mathcal{G}$ there exists a full subgraph H' of G, isomorphic to H and containing the edge.

Key words: Full subcategory, binding category, graphs with given subgraphs, strong embedding.

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For a singleton set $\mathcal{G} = \{H\}$ with H a finite graph, the categories $\mathcal{G}$(GRA) (denoted by GRA_H) were studied in [10]. The main result stated: GRA can be fully embedded into GRA_H iff H is not discrete and contains no loops. In the present paper we show that GRA can be fully embedded into $\mathcal{G}$(GRA), where $\mathcal{G}$ is a set of (possibly infinite) graphs iff
1) H is not discrete and does not contain loops for $H \in \mathcal{G}$
2) the graphs in $\mathcal{G}$ have the same variance, i.e. they are either all antisymmetric, or all symmetric, or all fail to be antisymmetric or symmetric.

There remains an interesting open problem: if $\mathcal{G}$ is a finite set of finite graphs, does there exist a finite graph in $\mathcal{G}$(GRA) ? (If so then it will easily follow from the results presented in [10] that there is a full embedding GRA into $\mathcal{G}$(GRA) preserving finite graphs).

First we recall some well-known definitions.

<u>Definition [14,15]</u> : Let (K,U), (L,V) be concrete categories. A full embedding $\phi : K \longrightarrow L$ is called a strong embedding if there exists a set functor $F: \mathrm{Set} \longrightarrow \mathrm{Set}$ such that the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\ \phi\ } & L \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathrm{Set} & \xrightarrow{\ F\ } & \mathrm{Set} \end{array}$$

commutes.

<u>Definition [10]</u> : Let X be a set, R, R' be relations (graphs) on X with $R \subset R'$, A, B be subsets of X with a bijection $i: A \longrightarrow B$ such that $i \times i(R \cap (A \times A)) = R \cap (B \times B)$ and $i \times i(R' \cap (A \times A)) = R' \cap (B \times B)$. Then (X,R,R',A,B) is called a šíp.

For a given graph (Y,S) define a šíp-product $(X,R,R',A,B) * * (Y,S) = (Z,Q)$ as follows: $Z = X \times (Y \times Y)/\sim$ where $(x,y_1,y_2) \sim (\bar{x},\bar{y}_1,\bar{y}_2)$ iff either $y_1 = \bar{y}_1$ and $x = \bar{x}$ for $x \in A$, or $y_1 = \bar{y}_2$ and $\bar{x} = i(x)$ for $x \in A$, or $y_2 = \bar{y}_2$ and $x = \bar{x}$ for $x \in B$. Q is a factor-relation of $\bar{Q} = \{((x,y_1,y_2),(\bar{x},\bar{y}_1,\bar{y}_2));$ $(y_1 = \bar{y}_1 \,\&\, y_2 = \bar{y}_2 \,\&\, (x,\bar{x}) \in R)$ or $(y_1 = \bar{y}_1 \,\&\, y_2 = \bar{y}_2 \,\&$ $\& \,(y_1,y_2) \in S \,\&\, (x,\bar{x}) \in R'\}$ by $\sim$.

For $f:(Y,S) \longrightarrow (\bar{Y},\bar{S})$ define $(X,R,R',A,B) * f : (X,R,R',A,B) * * (Y,S) \longrightarrow (X,R,R',A,B) * (\bar{Y},\bar{S})$ as follows: $f([(x,y_1,y_2)]) =$

$= [(x,f(y_1),f(y_2))]$ where $[a]$ is the class of $\sim$ containing a point a. Then $(X,R,R',A,B)*-$ is a functor which is an embedding.

Definition [10] : A šíp (X,R,R', A,B) is called strongly rigid if for every graph (Y,S) and for every compatible mapping

$$f:(X,R)\longrightarrow (X,R,R',A,B)*(Y,S) \quad (\text{or } f:(X,R')\longrightarrow (X,R,R',A,B)*(Y,S))$$

there exists $(y_1,y_2)\in Y\times Y$ (or $(y_1,y_2)\in S$) such that $f(x)$ is the class containing (x,y_1,y_2) for every $x\in X$.

Proposition 1: If (X,R,R',A,B) is strongly rigid then $(X,R,R',A,B)*-$ is a strong embedding from GRA to GRA. Proof see [10].

Before the main construction, we give a construction of special infinite rigid graphs (i.e. such graphs which have no endomorphism different from the identity). We recall that for every set there exists a rigid connected graph on it, see [17]. If we use the results in [5,14] we get that for every infinite set X, there exists a rigid symmetric connected graph, say P_X, on X. Further for a set X, denote by C_X, the complete graph on X without loops, i.e.
$C_X = \{(x_1,x_2);\ x_1,x_2\in X,\ x_1\neq x_2\}$.

The following statement was first proved by L. Babai and J. Nešetřil in [1], and they told me this result via conversation: for every cardinal α there exists a rigid graph in $\mathcal{G}$(GRA), where $\mathcal{G} = \{(\alpha,C_\alpha)\}$. I give here an independent construction.

<u>Construction 2:</u> Let α be an infinite cardinal. We shall construct a sequence of triples $(Z_i, S_i, \bar{S}_i)$ where Z_1 is a set, S_i and $\bar{S}_i$ are relations on Z_i. First define a sequence $\{\alpha_i\}_{i=0}$ such that $\alpha_0 = \alpha$ and α_{i+1} is a successor cardinal of α_i. Define $Z_0 = \alpha_0$ (we identify a cardinal α with the set of all ordinals smaller than α), $S_0 = C_{\alpha_0}$, $\bar{S}_0 = P_{\alpha_0}$. $Z_{i+1} = Z_i \cup (\alpha_{i+1} \times \bar{S}_i)$ (we assume that $Z_i \cap (\alpha_{i+1} \times \bar{S}_i) = \emptyset$), $S_{i+1} = S_i \cup (\bigcup \{ C_{[\alpha_{i+1} \times \{(x,y)\}]} \cup \{x,y\}$; $(x,y) \in \bar{S}_i \}$, $\bar{S}_{i+1} = \bigcup \{ P_{\alpha_{i+1} \times \{(x,y)\}}$; $(x,y) \in \bar{S}_i \}$. Put $Z = \bigcup \{ Z_i ;\ i = 0,1,\dots \}$, $\bar{S} = \bigcup \{ S_i ;\ i = 0,1,\dots \}$. Then clearly:

1) $Z_i \subset Z_{i+1}$, $S_i \subset S_{i+1}$ for all i;

2) if $(x,y) \in S_{i+1} - S_i$ (or $(x,y) \in S_0$) then there exists a full subgraph of (Z_{i+1}, S_{i+1}) (and of $(Z, \bar{S})$, too) isomorphic to $(\alpha_{i+1}, C_{\alpha_{i+1}})$ containing the edge (x,y) and there exists no full subgraph of $(Z, \bar{S})$ isomorphic to (α_j, C_{α_j}) for $j > i + 2$ containing the edge.

3) For every couple $\{x,y\}$ of points of Z there exists a finite sequence $T_0, T_1, \dots, T_n$ of subsets of Z with $x \in T_0$, $y \in T_n$ such that $\bar{S} \cap (T_i \times T_i) = C_{T_i}$ for every $i = 0, 1, \dots, n$ and card $(T_i \cap T_{i+1}) \geq 2$ for every $i = 0, 1, \dots, n-1$.

Choose a sequence $\{ \varphi_n : Z_0 \longrightarrow \bar{S}_n ;\ n \geq 5 \}$ of one-to-one mappings (such sequence is called suitable for α). Define $G(\alpha, \{\varphi_n ;\ n \geq 5\}) = (Z, S(\alpha, \{\varphi_n ;\ n \geq 5\})$ where $S(\alpha, \{\varphi_n ;\ n \geq 5\}) = \bar{S} \cup \{ (x,y), (y,x) ;\ x \in Z_0,\ y \in \alpha_{n+1} \times \varphi_n(x),\ n \geq 5 \}$. We shall write only S instead $S(\alpha, \{\varphi_n ;\ n \geq 5\})$ if a misunderstanding cannot occur. Then it holds:

4) every edge $(x,y) \in S$ lies in a full subgraph of (Z,S) which is isomorphic to (α, C_{α});

5) for every point $x \in Z_{i+1} - Z_i$, $i \geq 0$, there exists no full subgraph of (Z,S) containing x which is isomorphic to $(\alpha_{i+3}, C_{\alpha_{i+3}})$;

[Proof: $(Z,\bar{S})$ has this property and for every $x \in Z_{i+1} - Z_i$, $\mathrm{card}\{y;\ (y,x) \in S - \bar{S}\} \leq \alpha$.]

6) for every point $x \in Z_0$ and every cardinal β such that $\beta = \alpha_i$ for some i, there exists a full subgraph of (Z,S) isomorphic to (β, C_β) containing x;

7) if $f:(Z,S) \longrightarrow (Z,S)$ is a compatible mapping, then $f(Z_0) \subset Z_0$, $f(Z_{i+1} - Z_i) \subset Z_{i+1} - Z_i$, for $i \geq 0$;

[Proof: Since (Z,S) has not loops, by 2),5) and 6) we get that $f(Z_i) \subset Z_i$ for every $i \geq 0$. Further, if $H = (U,T)$ is a full subgraph of (Z,S) isomorphic to $(\alpha_{i+1}, C_{\alpha_{i+1}})$, then $\mathrm{card}\,(U \cap Z_i) \leq 2$. Let $x \in Z_n - Z_{n-1}$ for $n > 0$, then there exist two distinct points $u,v \in Z_{n-1}$ and a full subgraph $H = (U,T)$ of (Z,S) isomorphic to (α_n, C_{α_n}) with $u,v,x \in U$. Then f/U is one-to-one and $\mathrm{card}\,(f(U) \cap Z_{n-1}) \leq 2$, but $f(u) \neq f(v)$ and $f(u), f(v) \in f(U) \cap Z_{n-1}$, therefore $f(x) \notin Z_{n-1}$, and hence $f(Z_n - Z_{n-1}) \subset Z_n - Z_{n-1}$ for all $n > 0$.]

8) (Z,S) is a rigid graph.

[Proof: Let $f:(Z,S) \longrightarrow (Z,S)$ be a compatible mapping. We prove by induction over n that $f/Z_n = 1_{Z_n}$.

a) $f/Z_0 = 1_{Z_0}$. Let $(u,v) \in P_{\alpha_0}$, then $((\alpha_1 \times \{(u,v)\}) \cup \{u,v\},\ C_{(\alpha_1 \times \{(u,v)\}) \cup \{u,v\}})$ is a subgraph of (Z,S). Further, if $H = (U,T)$ is a subgraph of (Z,S) isomorphic to (α_1, C_{α_1}) and $\mathrm{card}\,(U \cap Z_0) \geq 2$, then $T \subset \bar{S}$ and hence there exists $(\bar{u},\bar{v}) \in P_{\alpha_0}$ with $\{\bar{u},\bar{v}\} = U \cap Z_0$. So $(f(u),f(v)) \in P_{\alpha_0}$ and therefore $f/Z_0 : (\alpha_0, P_{\alpha_0}) \longrightarrow (\alpha_0, P_{\alpha_0})$ is a compatible mappings, thus $f/Z_0 = 1_{Z_0}$.

b) Assume that $f/Z_i = 1_{Z_i}$ for all $i < n$. Then $((\alpha_n \times \{(x,y)\}) \cup \{x,y\}, C_{(\alpha_n \times \{(x,y)\}) \cup \{x,y\}})$ for $(x,y) \in \bar{S}_{n-1}$ is a subgraph of (Z,S) and $f(x) = x$, $f(y) = y$. It means that $f(\alpha_n \times \{(x,y)\}) \subset \alpha_n \times \{(x,y)\}$ for all $(x,y) \in \bar{S}_{n-1}$, therefore it suffices to prove $f/\alpha_n \times \{(x,y)\} = 1$. If (U,T) is a subgraph of (Z,S) isomorphic to $(\alpha_{n+1}, C_{\alpha_{n+1}})$ with card $(U \cap (\alpha_n \times \{(x,y)\})) \geq 2$, then there exists $(u,v) \in P_{\alpha_n \times \{(x,y)\}}$ with $\{u,v\} = U \cap (\alpha_n \times \{(x,y)\})$ and thus $f/\alpha_n \times \{(x,y)\} : (\alpha_n \times \{(x,y)\}, P_{\alpha_n \times \{(x,y)\}}) \longrightarrow (\alpha_n \times \{(x,y)\}, P_{\alpha_n \times \{(x,y)\}})$ is a compatible mapping, it means that $f/Z_n = 1_{Z_n}$. The proof is concluded.]

Summarize these properties in the following theorem:

Theorem 3: For every infinite cardinal α and every suitable sequence $\{\varphi_n;\ n \geq 5\}$ for α, the graph $G(\alpha, \{\varphi_n;\ n \geq 5\})$ is a rigid object of $\mathcal{G}(GRA)$, where $\mathcal{G} = \{(\alpha, C_\alpha)\}$. Moreover, for every distinct points x,y of the underlying set of $G(\alpha, \{\varphi_n;\ n \geq 5\})$ there exists a finite sequence $T_0, T_1, \dots, T_n$ of subsets with $x \in T_0$, $y \in T_n$ such that for every $i = 0,1,\dots,n$, (T_i, C_{T_i}) there is a subgraph of $G(\alpha, \{\varphi_n;\ n \geq 5\})$ and for every $i = 0,1,\dots,n-1$, card $(T_i \cap T_{i+1}) \geq 2$.

Proposition 4: Let α be an infinite cardinal, $\{\varphi_n;\ n \geq 5\}$, $\{\psi_n;\ n \geq 5\}$ be different suitable sequences for α. Then there exists no compatible mapping from $G(\alpha, \{\varphi_n;\ n \geq 5\})$ to $G(\alpha, \{\psi_n;\ n \geq 5\})$.

Proof: Let $f:G(\alpha, \{\varphi_n; n \geq 5\}) \longrightarrow G(\alpha, \{\psi_n; n \geq 5\})$ be a compatible mapping. Then by 2),5) and 6) we prove $f(Z_i) \subset Z_i$ and analogously as in 7) $f(Z_{i+1} - Z_i) \subset \subset Z_{i+1} - Z_i$ for all $i \geq 0$. Since for the proof of 8) we use only 7) and the properties of $(Z,\overline{S})$, we get that $f = 1_Z$, hence it follows that $\varphi_n = \psi_n$ for every $n \geq 5$ - a contradiction.

Now we shall describe the main construction. For a given connected graph G we shall construct a strongly rigid šíp (V,T,T',A,B) such that $(V,T),(V,T') \in \mathcal{G}(GRA)$ where $\mathcal{G} = \{G\}$. This šíp is constructed so that to a suitable sum of graphs $G(\alpha, \{\varphi_n; n \geq 5\})$ we add edges to get the required šíp. More precisely:

<u>Construction 5</u>: Let $G = (X,R)$ be a connected graph without loops with card $X > 2$. Choose an infinite cardinal $\alpha >$ card X. Let a be a point with $a \notin X \times \{0,1\}$ and choose an edge $(x,y) \in R$ and a one-to-one mapping Ψ from $((X - \{x,y\}) \times \{0,1\}) \cup \{a\}$ to the set of all suitable sequences for α (which has power bigger than $\alpha >$ card X). Now, denote $G(\alpha, \Psi(b)) = (Z,S(\alpha, \Psi(b)))$ (the underlying set is the same for all $G(\alpha, \Psi(b))$) for all $b \in ((X - \{x,y\}) \times \{0,1\}) \cup \{a\}$. Further choose a total ordering $\leq$ on Z and define $Q(\alpha, \Psi(b)) = \{(u,v); (u,v) \in S(\alpha, \Psi(b)), u \leq v\}$. For every $b \in ((X - \{x,y\}) \times \{0,1\}) \cup \{a\}$ choose a bijection $\psi(b)$ from $Q(\alpha, \Psi(b))$ to Z. Define subsets T_0, T_1, T_2, T_3 of $V \times V$ where $V = Z \times [((X - \{x,y\}) \times \{0,1\}) \cup \{a\}]$ as follows:

$T_0 = \{((u,b),(v,b)); (u,v) \in Q(\alpha, \Psi(b)),$

$b \in ((X - \{x,y\}) \times \{0,1\}) \cup \{a\}\}$;

$T_1 = \{((u,b),(v,b));\ (u,v) \in S(\alpha, \Psi(b)),$
$b \in ((X - \{x,y\}) \times \{0,1\}) \cup \{a\}\}$;

$T_2 = \{((z,u,i),(z,v,i));\ (u,v) \in R,\ i = 0,1,\ z \in Z,\ u,v \notin \notin \{x,y\}\} \cup \{((u,w,i),(\psi(w,i)(u,v),t,1-i)),((v,w,i), (\psi(w,i)(u,v),s,1-i));\ (u,v) \in Q(\alpha, \Psi(w,i)),\ w \in X - - \{x,y\},\ i = 0,1,\ (x,t),(y,s) \in R,\ t \neq y,\ s \neq x\} \cup \cup \{((\psi(w,i)(u,v),t,1-i),(u,w,i)),\ ((\psi(w,i)(u,v),s, 1-i),(v,w,i));\ (u,v) \in Q(\alpha, \Psi(w,i)),\ w \in X - \{x,y\},$
$i = 0,1,\ (t,x),(s,y) \in R,\ t \neq y,\ s \neq x\} \cup \{((u,a),(\psi(a) (u,v),t,0)),\ ((v,a),(\psi(a)(u,v),s,0));\ (u,v) \in Q(\alpha, \Psi(a)),\ (x,t),(y,s) \in R,\ y \neq t,\ s \neq x\} \cup \{((\psi(a)(u,v), t,0),(u,a)),((\psi(a)(u,v),s,0),(v,a));\ (u,v) \in Q(\alpha, \Psi(a)),\ (t,x),(s,y) \in R,\ y \neq t,\ x \neq s\}$;

$T_3 = \{((u,a),(\psi(a)(u,v),t,1)),((v,a),(\psi(a)(u,v),s,1));$
$(u,v) \in Q(\alpha, \Psi(a)),\ (x,t),(y,s) \in R,\ y \neq t,\ s \neq x\} \cup$
$\cup \{((\psi(a)(u,v),t,1),\ (u,a)),\ ((\psi(a)(u,v),s,1), (v,a));\ (u,v) \in Q(\alpha, \Psi(a)),\ (t,x),\ (s,y) \in R,\ y \neq t,$
$s \neq x\}$.

Define $T = T_0 \cup T_2$, $T' = T_0 \cup T_2 \cup T_3 = T \cup T_3$ if $(y,x) \notin R$, $T = T_1 \cup T_2$, $T' = T_1 \cup T_2 \cup T_3 = T \cup T_3$ if $(y,x) \in R$. Then it holds:

9) there exists no full subgraph of (V,T_2) or (V,T_3) which is isomorphic to (α, C_α);

10) for every $b \in ((X - \{x,y\}) \times \{0,1\}) \cup \{a\}$, there exist no $z,z' \in Z$ with $((z,b),(z',b)) \in T_2 \cup T_3$;

11) if $\mathcal{G} = \{G\}$ then (V,T), (V,T') are objects of $\mathcal{G}(GRA)$;

12) $T \subsetneqq T'$.

Choose $z_1, z_2 \in Z$, $z_1 \neq z_2$ and put $A = \{(z_1, a)\}$, $B = \{(z_2, a)\}$. Clearly, (V,T,T',A,B) is a šíp.

Proposition 6: The šíp (V,T,T',A,B) is strongly rigid.

Proof: Let (X',R') be a graph. Let $f:(V,T) \longrightarrow (V,T,T',A,B) * (X',R')$ be a compatible mapping. By 3), 8) and 9) we get that for every $b \in ((X - \{x,y\}) \times \{0,1\}) \cup \{a\}$ there exists $\bar{x}_b \in X' \times X'$ such that the class $f(z,b)$ of $\sim$ contains $(z,b,\bar{x}_b)$. Choose $w \in X - \{x,y\}$, $i = 0,1$, $(z,z') \in Q(\alpha, \Psi(w,i))$, then it holds:

$y \neq t,\ (x,t) \in R \Longrightarrow ((z,w,i),(\psi(w,i)(z,z'),t,1-i)) \in T$

$y \neq t, (t,x) \in R \Longrightarrow ((\psi(w,i)(z,z'),t,1-i),(z,w,i)) \in T$

$x \neq t,\ (y,t) \in R \Longrightarrow ((z',w,i),(\psi(w,i)(z,z'),t,1-i)) \in T$

$x \neq t,\ (t,y) \in R \Longrightarrow ((\psi(w,i)(z,z'),t,1-i),(z',w,i)) \in T$

Since card $X > 2$ and G is connected we have that there exists $t \in X$ with $x \neq t \neq y$ such that either $(x,t) \in R$, or $(t,x) \in R$, or $(y,t) \in R$, or $(t,y) \in R$. Hence $\bar{x}_{(w,0)} = \bar{x}_{(s,1)}$ for all $w,s \in X - \{x,y\}$. Further, the foregoing implication holds, too, if we substitute a in place of (w,i) and 0 in place of $1-i$ and choose (z,z') with $z,z' \notin \{z_1,z_2\}$, hence we get that $\bar{x}_a = \bar{x}_{(w,0)}$ for all $w \in X - \{x,y\}$. Hence, f has the required form. If $f:(V,T') \longrightarrow (V,T,T',A,B) * (X',R')$ is a compatible mapping then the proof is the same.

Lemma 7: If G is a graph with at least one edge without loops then there exists a connected graph G' without loops, with at least three-point underlying set such that for every edge of G' there exists a full subgraph of G', isomorphic to G, containing this edge.

Proof: Let $G = (X,R)$. If card $X = 1$ then all is obvious. Therefore we can assume that card $X > 2$. Let $X = \{X_i\}_{i\in I}$ be a decomposition of X to components of G. Since $R \neq \emptyset$ and G has not loops there exists $i_0 \in I$ with card $X_{i_0} > 1$. Choose $x \in X_{i_0}$ and for every $i \in I$ choose $y_i \in X_i$ with $x \neq y_{i_0}$. Define $G_1 = (X_1,R_1)$ as follows:

$X_1 = (X - \{x\}) \cup \{(x,i);\ i \in I\}$, $R_1 = \{(v,z);\ (v,z) \in R,\ v \neq x \neq z\} \cup \{(v,(x,i)),((x,i),z);\ (v,x),(x,z) \in R\}$. Let $2 = (\{0,1\}, \{(0,0),(1,1)\})$.

Define an equivalence $\sim$ on $X_1 \times \{0,1\}$ as follows: $((x,i),j) \sim (y_i,1-j)$ for every $i \in I$, $j = 0,1$. Obviously, $G' = G_1 \times 2/\sim$ (where $\times$ is the categorical product) has the required properties.

<u>Definition</u>: A graph (X,R) without loops is

a) symmetric if $(x,x') \in R$ implies $(x',x) \in R$;

b) antisymmetric if $(x,x') \in R$ implies $(x',x) \notin R$;

c) mixed if it is neither symmetric nor antisymmetric.

We say that graphs (X,R) and (X',R') have the same variance if both are either symmetric, or antisymmetric, or mixed.

<u>Construction 8</u>: Let (X,R), (Y,S) be connected graphs without loops with the same variance such that card $X > 2$, card $Y > 2$.

Denote $R_1 = \{(x_1,x_2);\ (x_1,x_2),(x_2,x_1) \in R\}$, $R_2 = R - R_1$

and analogously

$$S_1 = \{(y_1,y_2);\ (y_1,y_2),(y_2,y_1) \in S\},\ S_2 = S - S_1.$$

Choose $(y_1,y_2) \in S_1$ (if it exists) and $(y_3,y_4) \in S_2$ (if it exists) and define $\overline{Y}_1 = [(Y - \{y_1,y_2\}) \cup (\{y_1,y_2\} \times R_1)] \times \{1\}$,

$$\overline{Y}_2 = [(Y - \{y_3,y_4\}) \cup (\{y_3,y_4\} \times R_2)] \times \{2\},$$

$\overline{S}_1 = \{((u,1),(v,1));\ (u,v) \in S,\ u,v \notin \{y_1,y_2\}\} \cup \{((u,1),((y_1,r)1)),((v,1),((y_2,r),1)),\ (((y_1,r),1),(w,1)),\ (((y_2,r),1),(z,1));\ (u,y_1),(y_1,w),(v,y_2),(y_2,z) \in S,\ u \neq y_2 \neq w,\ v \neq y_1 \neq z,\ r \in R_1\} \cup \{(((y_1,r),1),((y_2,r),1)),\ (((y_2,r),((y_1,r),1));\ r \in R_1\}$,

$\overline{S}_2 = \{((u,2),(v,2));\ (u,v) \in S,\ u,v \notin \{y_3,y_4\}\} \cup \{((u,2),((y_3,r),2)),\ ((v,2),((y_4,r),2)),\ (((y_3,r),2),(w,2)),\ (((y_4,r),2),(z,2));\ (u,y_3),\ (y_3,w),(v,y_4),(y_4,z) \in S,\ u \neq y_4 \neq w,\ v \neq y_3 \neq z,\ r \in R_2\} \cup \{(((y_3,r),2),((y_4,r),2));\ r \in R_2\}$.

Assume that $\overline{Y}_1$, $\overline{Y}_2$ and X are disjoint sets. Choose total ordering $\leq$ on X and define an equivalence $\approx$ on $\overline{Y}_1 \cup \overline{Y}_2 \cup X$:

$$x \approx (y_1,(x,\overline{x}),1),\ \overline{x} \approx (y_2,(x,\overline{x}),1) \text{ if } (x,\overline{x}) \in R_1,\ x \leq \overline{x},$$
$$x \approx (y_3,(x,\overline{x}),2),\ \overline{x} \approx (y_4,(x,\overline{x}),2) \text{ if } (x,\overline{x}) \in R_2$$

Put $(Z,T) = (X \cup \overline{Y}_1 \cup \overline{Y}_2, R \cup \overline{S}_1 \cup \overline{S}_2)/\approx\ = (X,R) \otimes (Y,S)$. Further define $\varphi_{(X,R)\otimes(Y,S)} : (X,R) \longrightarrow (X,R) \otimes (Y,S)$ such that $\varphi_{(X,R)\otimes(Y,S)}(x)$ is the class of $\approx$ containing x. Since $x \approx y$ implies $(x,y) \notin R \cup \overline{S}_1 \cup \overline{S}_2$ we get that

$\varphi_{(X,R)\otimes(Y,S)}$ is a full embedding. Moreover $(X,R) \otimes (Y,S)$ is connected and has the same variance as (X,R) and card $Z>2$. Further for every edge of $(X,R)\otimes(Y,S)$ there exists a full subgraph of $(X,R)\otimes(Y,S)$, isomorphic to (Y,S), containing the edge.

Proposition 9: Let $\mathcal{G}$ be a set of graphs without loops with the same variance such that each graph of $\mathcal{G}$ has at least one edge. Then there exists a connected graph $G=(X,R)\in\mathcal{G}(GRA)$ without loops with card $X>2$.

Proof: By Lemma 7 we can assume that every $H\in\mathcal{G}$ is connected and its underlying set has at least three points. Choose a well-ordering on $\mathcal{G}=\{H_i;\ i\in\alpha\}$ where $\alpha=$ = card $\mathcal{G}$. We shall construct a chain of graphs $\{\psi_{i,j}:$ $:G_i\longrightarrow G_j;\ i\leq j\leq\omega_0\cdot\alpha\}$ such that $\psi_{i,j}$ are full embeddings and for every $k\leq i\leq j\leq\omega_0\cdot\alpha$, $\psi_{i,j}\cdot\psi_{k,i}=\psi_{k,j}$ and $\psi_{i,i}=1$.

a) Put $G_0=G_1=H_0$, $\psi_{0,1}=1$;

b) put $G_{i+1}=G_i\otimes H_k$, $\psi_{i,i+1}=\varphi_{G_i\otimes H_k}$ where $k<\alpha$ and $i=n\cdot\alpha+k$ for some $n<\omega_0$;

c) if i is limit, put $\{G_i,\ \psi_{j,i};j<i\}=\operatorname{colim}\{\psi_{j,k}:G_j\to G_k;$ $j\leq k<i\}$. Since $\varphi_{j,k}$ is a full embedding for every $j\leq k<i$, we get that $\varphi_{j,i}$ is a full embedding, too, for every $j<i$. Put $G=G_{\omega_0\cdot\alpha}$. Then G is connected without loops and its underlying set has at least three points. We are to prove $G\in\mathcal{G}(GRA)$. Let $H_i\in\mathcal{G}$ and let (x,y) be an edge of G. Then there exists $j<\omega_0\cdot\alpha$ such that (x,y) is an edge of G_j. Clearly there exists $n<\omega_0$ with $j<n\cdot\alpha$, then (x,y) is an edge of $n\cdot\alpha+1$. By Construction 8, there exists a full subgraph of $G_{n\cdot\alpha+i+1}$ isomorphic to H_i contai-

ning (x,y). Since $\varphi_{m\cdot\alpha+i+1,\omega_0\cdot\alpha}$ is a full embedding we get that $G \in \mathcal{G}(GRA)$.

<u>Main Theorem 10</u>: Let $\mathcal{G}$ be a set of graphs. There exists a strong embedding of GRA to $\mathcal{G}(GRA)$ (i.e. $\mathcal{G}(GRA)$ is binding) if and only if every graph in $\mathcal{G}$ has at least one edge, has not loops and all graphs in $\mathcal{G}$ have the same variance.

Proof: Sufficiency follows from Propositions 6 and 9 and Construction 5, because if (V,T,T', A,B) is a šíp with $(V,T) \in \mathcal{G}(GRA)$ and $(V,T') \in \mathcal{G}(GRA)$, then for every graph (X,R), $(V,T,T',A,B) * (X,R) \in \mathcal{G}(GRA)$.
Necessity. If some graph in $\mathcal{G}$ has not an edge or graphs in $\mathcal{G}$ have not the same variance, then $(X,R) \in \mathcal{G}(GRA)$ iff $R = \emptyset$. If some graph in $\mathcal{G}$ has a loop, then $(X,R) \in \mathcal{G}(GRA)$ implies either $R = \emptyset$ or (X,R) has a loop. In both cases for every $(X,R),(Y,S) \in \mathcal{G}(GRA)$ there exists a compatible mapping between (X,R) and (Y,S), hence $\mathcal{G}(GRA)$ is not binding: the two-object discrete category cannot be embedded into $\mathcal{G}(GRA)$.

<u>Corollary 11</u>: Denote $GRA_G = \mathcal{G}(GRA)$, if $\mathcal{G} = \{G\}$. Then GRA_G is binding iff G has not loops and has at least one edge.

<u>Corollary 12:</u> For every set $\mathcal{G}$ of graphs with the same variance such that every graph in $\mathcal{G}$ has at least one edge and has not loops and for every monoid M and for every cardinal α there exist graphs G_i, $i \in \alpha$ such that

1) $G_i \in \mathcal{G}(GRA)$ for every $i \in \alpha$;

2) the endomorphism monoid of G_i is isomorphic to M;

3) there exists no compatible mapping between G_i and G_j whenever $i \neq j$.

Moreover, there exist strong embeddings $\Phi_i : GRA \longrightarrow \mathcal{G}(GRA)$, $i \in \alpha$ such that for every couple of graphs (X,R) and (Y,S) there exists no compatible mapping between $\Phi_i(X,R)$ and $\Phi_j(Y,S)$ whenever $i \neq j$.

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Matematicko-fyzikální fakulta
Universita Karlova
Malostranské nám. 25
Praha 1
Československo

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