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ON THE EXISTENCE OF FINITE GENERATORS FOR INVERTIBLE MEASURE-PRESERVING TRANSFORMATIONS

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Abstract: A measure-theoretical version of topological entropy is defined as a new invariant for an invertible measure-preserving transformation of a finite measure space to show that the existence of finite generators is guaranteed for such a transformation if and only if the numerical value of the invariant is finite and the transformation may be decomposed into aperiodic and purely atomic parts, the number of atoms being asymptotically finite.

Key words: Invertible measure-preserving transformation, finite generator, asymptotic rate.

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1. Introduction. Throughout this paper $(\Omega, \mathcal{F}, \mu)$ means a finite measure space with the measure normed, and T is an invertible measure-preserving transformation of the space, which will be referred to as its automorphism. The additive group of integers is denoted by I; hence $(T^i, i \in I)$ means the cyclic group of automorphisms associated with the transformation T.

We shall denote by $\mathcal{Z} = \mathcal{Z}(\mathcal{F})$ the complete lattice (with respect to the relation of inclusion) of sub-σ-algebras of $\mathcal{F}$; for a class of sets $\Delta \subset \mathcal{F}$ the notation

$\sigma\Delta$ means the element in $\mathfrak{Z}$ that is generated by Δ. The lattice operations in $\mathfrak{Z}$ will be designated in a customary way; we shall set

$$\mathcal{B}_T = \bigvee_{i \in I} T^i \mathcal{B} \quad \text{for } \mathcal{B} \in \mathfrak{Z};$$

this is done in accordance with the notation used in Parry's work [1], and we shall make use both of the notations and of the terminology given in the work quoted without further reference.

Especially, we shall employ the mod 0 nomenclature in the spirit of Rohlin's fundamental paper [2], writing mod μ, or even a.s. (μ), whenever more convenient for the sake of clarity. Here a partition is taken as a class of sets, and partitions studied in [2] are referred to as Rohlin measurable.

To define the basic notion of this paper, a modified version of topological entropy, we restrict ourselves to the class $Z = Z(\mathcal{F})$ of finite measurable partitions, which forms a lattice with respect to the relation $\eta \leq \xi$ (ξ is a refinement of η); the lattice Z is taken as a sublattice of $\mathfrak{Z}$, embedded into $\mathfrak{Z}$ by the injective map $\xi \longrightarrow \hat{\xi} = \sigma\xi$. Recall from [1] that $\hat{\xi}_T = (\sigma\xi)_T$; in what follows we shall set

$$\xi^n = \xi^n_T = \bigvee_{i=0}^{n-1} T^{-i}\xi \quad \text{for } \xi \in Z.$$

Given $0 < \varepsilon < 1$, we define, for $\xi \in Z$,

$$L(\varepsilon, \xi) = L_\mu(\varepsilon, \xi) = \min\{\operatorname{card}(\Delta) : \Delta \subset \xi, \sum_{C \in \Delta} \mu(C) > 1 - \varepsilon\},$$

$$H(T,\xi) = H_\mu(T,\xi) = \sup_{0<\varepsilon<1} \limsup_{n} \frac{1}{n} \log L(\varepsilon, \xi^n),$$

$$H(T) = H_\mu(T) = \sup_{\xi \in Z} H(T,\xi);$$

here the symbol card means the cardinality, which is, in the case considered, simply the number of elements. We shall call $H(T)$ the asymptotic rate of automorphism T, referring to $H(T,\xi)$ as the asymptotic rate of the partition ξ with respect to T.

Note. The asymptotic rate was studied by the author first in 1962 for the case of two-sided shifts; the quantity $L(\varepsilon,\xi)$ was introduced in 1959 for investigating the transmission of ergodic information sources over non-ergodic communication channels (cf. [3],[4],[5]). The results concerning the asymptotic rate were extended by Št. Šujan to the case of non-continuous measures (i.e. only additive measures) in a paper to appear as a supplement to the journal Kybernetika this year.

Let us recall that $(\mathcal{F},\mu)$ and the space considered are said to be countably generated if there is a countable class $\Delta \subset \mathcal{F}$ such that $\sigma\Delta = \mathcal{F}$ mod 0; $(\mathcal{F},\mu)$ is (together with the space) said to have a finite generator with respect to T if there is $\xi \in Z$ such that $\hat{\xi}_T = \mathcal{F}$ mod 0; then ξ is called a generator for T.

Since we consider neither Lebesgue measure spaces nor complete measure spaces only, we must make use of a more general concept of aperiodicity, as given, e.g., in [6], Sec. 2. Let us make the convention that, anywhere in the sequel, by N is denoted the set of positive integers. An automorphism T is called aperiodic if $C \neq 0 \bmod 0$ $(C \in \mathcal{F})$, $n \in N \Longrightarrow$ there is $D \subset C$ $(D \in \mathcal{F})$, $D \neq T^{-n} D \bmod 0$.

In this paper a disjoint class of μ-atoms Δ is said to be exhaustive if the complement $(\bigcup \Delta)^c = D_0$ contains no μ-atoms. Assume that $(\mathcal{F}, \mu)$ is countably generated. Then the class Δ is countable; it is uniquely determined mod 0. If Δ is empty, μ is non-atomic; in any case, the non-atomic part ν of measure μ is defined as the probability measure

$$(1.1) \qquad \nu(E) = \frac{\mu(E \cap D_0)}{\mu(D_0)} \quad (E \in \mathcal{F})$$

for $\mu(D_0) > 0$; $\mu(D_0) = 0$ means that μ is purely atomic. Since D_0 is T-invariant, the non-atomic part of (T, μ) may be defined as the pair (T', ν) where T' is the restriction of T to the set D_0; we say briefly that T' is the non-atomic part of T (defined mod 0).

In what follows we shall set

$$(1.2) \qquad \Delta_q(T) = \{ D \in \Delta : T^q D = D \bmod 0 \}, \; q \in N.$$

It follows from the finiteness of the measure that

$$\bigcup_{q \in N} \Delta_q(T) = \Delta .$$

We shall say that the number of atoms is essentially bounded with respect to T if, for any $q \in N$, $\Delta_q(T)$ is finite, and

$$(1.3) \qquad \limsup_{q \to \infty} \frac{1}{q} \log (\mathrm{card}(\Delta_q(T)) < +\infty .$$

The entire paper is devoted to developing tools for proving the following theorem.

<u>The Theorem</u>. A necessary and sufficient condition that a probability space have a finite generator with respect to an automorphism T, is that (1) the space be countably generated, (2) the number of atoms be essentially bounded with respect to T, (3) the non-atomic part of T be aperiodic, and (4) the asymptotic rate of the automorphism T be finite: $H(T) < +\infty$.

Moreover, if ξ is a finite generator for the automorphism T, then card $(\xi) \geq \exp H(T)$.

<u>Remark</u>. It will be shown in a paper to appear in the next issue that, under the condition stated in the theorem, there is a finite generator ξ for T with the property that card $(\xi) \leq \exp H(T) + 1$.

2. <u>Invariance of H(T)</u>. Let us notice that the definition of the asymptotic rate $H(T, \xi)$ makes sense for any countable measurable partition ξ ; the lattice of such partitions will be denoted by $Z_0 = Z_0(\mathcal{F})$. If $h_\mu(\xi)$ means the entropy of $\xi \in Z_0$, then the class

$$Z_\mu = Z_\mu(\mathcal{F}) = \{\xi \in Z_0 : h_\mu(\xi) < +\infty\}$$

is known to be a sublattice of Z_0; the entropy of an automorphism T with respect to $\xi \in Z_\mu$ will be denoted by $h_\mu(T, \xi)$.

As immediately seen from the definition, $L(\varepsilon, \xi)$ is monotonic in both variables; especially,

$$\eta \leq \xi \Longrightarrow L(\varepsilon, \eta) \leq L(\varepsilon, \xi), \qquad \xi, \eta \in Z_0.$$

Consequently,

(2.1) $H(T, \eta) \leq H(T, \xi)$ for $\eta \leq \xi$ $(\xi, \eta \in Z_0)$.

In the following proposition we assume that we are given together with the space under consideration another probability space $(\Omega', \mathcal{F}', \mu')$; by $\mathcal{F}/\mu$ we denote the measure algebra associated with $(\mathcal{F}, \mu)$; T' is supposed to be an automorphism on Ω' (on conjugacy cf., e.g., [7]).

Proposition 1. If there is an isomorphism between measure algebras $\mathcal{F}/\mu$ and $\mathcal{F}'/\mu'$ under which T and T' are conjugate, then H(T) = H(T').

It is because (2.1) is valid with $\eta \leq \xi$ mod 0 so that

(2.2) $\eta = \xi \bmod 0 \Longrightarrow H(T, \eta) = H(T, \xi)$; $\xi, \eta \in Z_0$.

In this paper we usually decompose measures instead of the automorphism, keeping both T and $\mathcal{F}$ fixed (compare with the definition (1.1); to be consequent, ν should be restricted to D_0 there). The latter principle is employed in the statement of the following

Proposition 2. If $(\mathcal{F}, \mu)$ is countably generated, and if μ is not purely atomic, then

$$H_\nu(T) = H_\mu(T),$$

where ν is the non-atomic part of μ. If μ is purely atomic, then $H_\mu(T) = 0$.

Proof. Let Δ be an exhaustive class of atoms. Since Δ is countable, $\xi_0 = \Delta \cup \{D_0\}$ where $D_0 = (\bigcup\Delta)^c$ is a partition in Z_0. If μ is purely atomic, then $\zeta \leq \xi_0$ mod 0 for any $\zeta \in Z$ so that $H(T,\zeta) \leq H(T,\xi_0)$ by (2.1) valid mod 0 as well. If $\mu(D_0) > 0$, let $\tilde{\mu}$ be the purely atomic part of μ (defined similarly as ν in (1.1)). Since ξ_0 is T-invariant, $H(T,\xi_0) = 0$; hence $H_{\tilde{\mu}}(T) = 0$. The remainder of the proof is based on the inequalities

$$L_{\mu}(\varepsilon,\xi) \leq L_{\nu}(\varepsilon,\xi) + L_{\tilde{\mu}}(\varepsilon,\xi),$$
$$L_{\nu}(\varepsilon.(\mu(D_0))^{-1},\xi) \leq L_{\mu}(\varepsilon,\xi)$$

valid for any $\xi \in Z_0$ and proved in Lemma 1.3 in [4], p.770 (cf. also the proof of Theorem 5.1, p. 783); hence it follows that, for any $\xi \in Z_0$,

$$H_{\nu}(T,\xi) \leq H_{\mu}(T,\xi) \leq \max(H_{\nu}(T,\xi), H_{\tilde{\mu}}(T,\xi)),$$

which guarantees the validity of the proposition.

3. Shifts. A finite or denumerably infinite set A being given, let S_A be the shift in A^I (defined by $(S_A z)_i = z_{i+1}$) and set

$$[\bar{z}] = \{z \in A^I : (z_0, z_1, \ldots, z_{n-1}) = \bar{z}\} \text{ for } \bar{z} \in A^n;$$

an elementary cylinder is defined as a finite-dimensional cylinder of the form $S_A^{-i}[\bar{z}]$, $i \in I$, $\bar{z} \in A^n$, $n \in N$. The class of all elementary cylinders, denoted by V_A, is taken as the open base of a topology which makes from A^I a Polish space. The σ-algebra of Borel sets in this space will be denoted by F_A: $F_A = \sigma V_A$. In our terminology Borel measures are those that are defined on F_A and normed.

For the sake of brevity, let us denote by $[E]$, $E \subset A^n$, the set $\bigcup \{ [\bar{z}] : \bar{z} \in E \}$, and let us set

$$(3.1) \qquad K_q(S_A) = \{ z \in A^I : (S_A)^q z = z \} ;$$

it is Borel because

$$K_q(S_A) = \bigcap_{i \in I} (S_A)^{iq} [A^q] \; ; \; q \in N.$$

<u>Lemma 3.1</u>. A Borel measure μ which is S_A-invariant, is non-atomic if and only if

$$(3.2) \qquad \mu(K(S_A)) = 0 \quad \text{where} \quad K(S_A) = \bigcup_{q=1}^{\infty} K_q(S_A).$$

Condition (3.2) is necessary and sufficient for (S_A, μ) to be aperiodic.

The facts summarized in the lemma are well-known and may be easily established. They show that the definition of aperiodicity given in Sec. 1 coincides for the shift with the usual one (the same is true for Lebesgue measure spaces).

A point $z \in A^I$ is called regular (with respect to the shift S_A) if there is an S_A-invariant Borel measure μ_z which is ergodic with respect to S_A and such that (χ_E is the characteristic function of the set E)

$$(3.3) \qquad \mu_z(E) = \lim_{m} (1/n) \sum_{i=0}^{m-1} \chi_E(S_A^i z)$$

for any $E \in V_A$; the measure μ_z is uniquely determined by the regular point z. The set of all regular points in A^I

will be denoted by R_A; it holds that $R_A \in F_A$,

(3.4) $\mu(R_A) = 1$ for any μ Borel and S_A-invariant.

Cf. [5], Chapter 2, where an elementary theory is built up without making use of topological concepts, for the more difficult case of A countable.

<u>Lemma 3.2</u>. If μ is a Borel measure which is S_A-invariant and non-atomic, then

$$\mu\{z \in R_A : \mu_z \text{ is non-atomic}\} = 1.$$

Proof. Making use of (3.4) and lemma 3.1, we obtain that $\mu(R_A - K(S_A)) = 1$; set

(3.5) $R_A^z = \{y \in R_A : \mu_y = \mu_z\}, \quad z \in R_A.$

It follows from the theory of regular points (cf.[5], loc. cit.) that $\mu_z(R_A^z) = 1$ (R_A^z Borel). Then the assumptions $z \in R_A$, $z \notin K(S_A)$, μ_z is not non-atomic lead to a contradiction because the last implies that $\mu_z(K(S_A)) > 0$ so that $\mu_z(R_A^z \cap K(S_A)) > 0$. From the definition (3.1) it follows that $z_1 \in K(S_A)$, $z_2 \in R_A$, $\mu_{z_1} = \mu_{z_2} \Longrightarrow z_2 \in K(S_A)$. Summarizing all these facts we obtain the desired result.

The partition $\gamma_A = \{[a] : a \in A\}$ represents the "alphabet" A in the space A^I; it is a generator of the space in the strict sense. In what follows we shall set, for μ Borel and S_A-invariant,

$$H(\mu) = H_A(\mu) = H_\mu(S_A, \gamma_A),$$

$$h(\mu) = h_A(\mu) = h_\mu(S_A, \gamma_A).$$

If the condition

(3.6) $h_{\mu}(\gamma_A) < +\infty$, i.e. $\gamma_A \in Z_{\mu}(F_A)$

is satisfied, then

(3.7) $$h(\mu) = \int_{R_A} h(\mu_z)\, d\mu(z)$$

as proved by Jacobs and by Parthasarathy independently (for A finite; the case of A countable is a trivial extension; cf.[4], Chapter 8). The basic tool for our investigation will be Theorem 9.3 from [4], proved later by making use of a more direct method as Theorem II in [5]; we shall restate it as

Lemma 3.3. If $h_{\mu}(\gamma_A) < +\infty$ then
$H(\mu) = \text{ess.sup}\{h(\mu_z) : z \in R_A \bmod \mu\}$.

A correction. Since the proofs of Parthasarathy's theorems given in [8] are not valid, and since the case $\gamma_A \notin Z_{\mu}(F_A)$ was treated by the author both in [4] and [5] with their aid, the condition (3.6) must be added to the assumptions of Theorems 8.2, 8.3, 9.1 - 9.4, 10.1, 11.1, and 11.3 in [4], and to the assumptions of Theorems I and II (together with Lemma II) in [5].

4. Properties of H(T). Let $Z_a = \{\zeta \in Z : \text{card}(\zeta) \le a\}$. If $\zeta \in Z_0$, then a sequence $\mathcal{U} = (U_n, n \in N)$ of mutually disjoint sets belonging to the class $\zeta \cup \{\emptyset\}$ will be called an ordering of the partition ζ if

$\{\mathcal{U}_n : n \in N\} \supset \zeta - \{\emptyset\}$, $\zeta \in Z_a \Rightarrow \mathcal{U}_n = \emptyset$ for $n > a$ $(a \in N)$.

Throughout this paper we shall assume that we have assigned to every $\xi \in Z_o$ an ordering $\mathcal{U}(\xi)$ of the partition ξ.

Let $A_a = \{n \in N : n \leq a\}$. For $A = A_a$, $\xi \in Z_a$ or $A = N$, $\xi \in Z_o$, let $\psi_\xi^A : \Omega \longrightarrow A^I$ be defined by

$$(4.1) \qquad (\psi_\xi^A \omega)_i = n \text{ iff } T^i\omega \in \mathcal{U}_n(\xi); \quad i \in I.$$

Then $(\psi_\xi^A)^{-1}$ establishes a 1-1 correspondence between $F_A \cap \cap\, \psi_\xi^A(\Omega)$ and $(\sigma\xi)_T$, $(\psi_\xi^A)^{-1}\gamma_A^n = \xi_T^n$ so that

$$(4.2) \qquad \mu^\xi = \mu(\psi_\xi^A)^{-1} \quad \text{on } F_A$$

is Borel and S_A-invariant, and

$$(4.3) \qquad H_\mu(T,\xi) = H_{\mu^\xi}(S_A, \gamma_A) = H(\mu^\xi).$$

It follows from the construction of μ^ξ that it is valid

<u>Lemma 4.1</u>. If ξ is a generator for T, $\xi \in Z_a$ or $\xi \in Z_o$, then $\mathcal{F}/\mu$ and F_A/μ^ξ are isomorphic measure algebras for an isomorphism under which T and S_A are conjugate, where $A = A_a$ and $A = N$, respectively.

In the remainder of this section it is supposed that $A = N$, $\psi_\xi = \psi_\xi^N$. If $\eta \leq \xi$ $(\eta, \xi \in Z_o)$, let $\tau_o : N \longrightarrow N$ be defined by

$$\tau_o(n) = m \text{ iff } \mathcal{U}_n(\xi) \subset \mathcal{U}_m(\eta).$$

Then the transformation $\tau = \tau[\xi, \eta] : N^I \longrightarrow N^I$ given by

$$\tau z = (\tau_o z_i, i \in I), \quad z \in N^I$$

is Borel measurable and

(4.4) $$\mu^{\eta} = \mu^{\xi}\tau^{-1} \text{ for } \tau = \tau[\xi, \eta].$$

Lemma 4.2. If $\xi \in Z$ then

$$H_{\mu}(T, \eta) = H_{\mu^{\xi}}(S, \tau^{-1}\gamma) = \text{ess.sup}\{h(\mu_z\tau^{-1}) : z \in R \bmod \mu^{\xi}\}$$

where $S = S_N$, $\gamma = \gamma_N$, $R = R_N$, $\tau = \tau[\xi, \eta]$.

Proof. Taking into account that

(4.5) $$\frac{1}{n} h_m(\gamma^n) \downarrow h(m)$$

for $m = \mu_z$, $z \in R$, and that

$$h_{\mu_z\tau^{-1}}(\gamma^n) = h_{\mu_z}(\tau^{-1}\gamma^n),$$

$$\mu_z\tau^{-1} = \mu_{\tau z} \text{ for } z \in R \cap \tau^{-1}(R), \quad \mu^{\xi}(R \cap \tau^{-1}(R)) = 1,$$

we obtain the equalities

$$h(\mu_z\tau^{-1}) = h(\mu_{\tau z}) = h_{\mu_z}(S, \tau^{-1}\gamma), \quad z \in R \cap \tau^{-1}(R).$$

By making use of the latter equalities and of Lemma 3.3 we get the desired result.

Lemma 4.3. If $\xi_n \uparrow \xi$, $\xi \in Z_{\mu}$ then $H(T, \xi_n) \uparrow \uparrow H(T, \xi)$.

Proof. The monotonicity of the convergence is guaranteed by (2.1). From Lemma 4.2 we obtain that

$$\sup_n H(T, \xi_n) = \text{ess.sup}\{\sup_n h(\mu_z\tau_n^{-1}) : z \in R \bmod \mu^{\xi}\}$$

for $\tau_n = \tau[\xi, \xi_n]$. By (4.4) (cf. the proof of Lemma 4.2)

$$\sup_n h(\mu_z\tau_n^{-1}) = \sup_n h_{\mu_z}(S, \tau_n^{-1}\gamma) = h_{\mu_z}(S, \gamma),$$

the latter equality being a consequence of the relation $\gamma \in Z_{\mu\xi}(F_N)$. Since $h_{\mu_z}(S,\gamma) = h(\mu_z)$ by definition, the assertion of the lemma follows from (4.3) and Lemma 3.3.

Proposition 3. $H(T) = \sup_{\xi \in Z_\mu} H(T,\xi)$.

The assertion is an immediate consequence of Lemma 4.3.

Proposition 4. $H_\mu(T) \geq h_\mu(T)$; if T is ergodic then $H_\mu(T) = h_\mu(T)$.

The assertion follows from (4.3), (3.7), and Lemma 3.3 immediately, because then it is guaranteed that

$$H_\mu(T,\xi) \geq h_\mu(T,\xi) \text{ for } \xi \in Z_\mu .$$

Proposition 5. If ξ is a generator for T then

$$H(T,\xi) = H(T) \text{ if } \xi \in Z_\mu .$$

Proof. We are to show that $\eta \leq \xi$, $\eta \in Z_\mu$ (it suffices $\eta \in Z$) $\Longrightarrow H(T,\eta) \leq H(T,\xi)$, and after to apply Proposition 3 to get the equality asserted. Let $\eta \in Z_\mu$, then $\xi \vee \eta \in Z_\mu$. Writing m for $\mu^{\xi\vee\eta}$ and τ for $\tau[\xi \vee \eta, \xi]$, we obtain from Lemma 4.2 that

$$H(T,\xi) = H_m(S,\tau^{-1}\gamma) = \text{ess.sup}\{h_{\mu_z}(S,\tau^{-1}\gamma) : z \in \\ \in R \bmod m\} .$$

From the implications

ξ is a generator for $T \Longrightarrow \tau^{-1}\gamma$ is a generator for $(S,m) \Longrightarrow m\{z \in R : \tau^{-1}\gamma$ is a generator for $(S,\mu_z)\} = = 1$

we conclude that

$$h_{\mu_z}(S, \tau^{-1}\gamma) = h_{\mu_z}(S) = h_{\mu_z}(S, \gamma),\ z \in R \bmod m,$$

which leads to the equality $H(T, \xi) = H(T, \xi \vee \eta)$, Q.E.D.

5. Preparatory Lemmas. By P we shall denote the space of probability vectors $p = (p_n, n \in N)$; i.e. $p_n \geq 0$, $\sum_m p_n = 1$. For $p \in P$, $h(p)$ is the entropy of p. We shall set

$$P_0 = \{p \in P : h(p) < +\infty\}.$$

For $p \in P$, let $N_p = \{n \in N : p_n > 0\}$. Setting, for $a \in N$,

$$P_a = \{p \in P : \text{card}\,(N_p) = a\},\ P_0' = P_0 - \bigcup_{a \in N} P_a,$$

we shall define $t = t_p : A_a \longrightarrow N_p$ (as to the notation A_a cf. Sec. 4) for $p \in P_a$ and $t = t_p : N \longrightarrow N_p$ for $p \in P_0'$ by the requirement that

$$p_{t(n)} \geq p_{t(n+1)},\ p_{t(n)} = p_{t(n+1)} \Longrightarrow t(n) < t(n+1).$$

In the remainder of this paper we shall assume that logarithms are taken to base 2: $\log = \log_2$. Let $\{u\}$ be the integer associated with a real number u by the condition that $u - 1 < \{u\} \leq u$. Define

$$b_p(n) = \{-\log p_{t(n)}\},\quad t = t_p,$$

$n \in A_a$ for $p \in P_a$, and $n \in N$ for P_0', respectively. Setting $r_p(1) = 1$,

$$r_p(n) = \min\{n > r_p(n-1) : b_p(n) > b_p(r_p(n-1))\},$$

$n > 1$, in case $p \in P_0'$, and similarly in case $p \in P_a$ (with the restriction that both $r_p(n-1)$ and $n \leq a$), we easily find that

$$r_p(n+1) - r_p(n) < \exp_2 [2b_p(r_p(n))] ,$$

because t_p orders N_p monotonically (compare with the construction given by Krieger in [9]).

Given a triple (k,α,β) of positive integers such that $\alpha < \beta$, $\beta - \alpha < 2^k$, let $g_{k,\alpha,\beta}$ be a 1-1 mapping of $\{n \in N: : \alpha \leq n < \beta\}$ into $\{1,2\}^k$. Let us assign to $p \in P_o$ the mapping $g_p : A_a \to A_o'$ for $p \in P_a$, and $g_p : N \to A_o'$ for $p \in P_o'$ given by

$$g_p(n) = g_{k,\alpha,\beta}(n) \text{ for } r_p(s) \leq n < r_p(s+1),$$

$\alpha = r_p(s)$, $\beta = r_p(s+1)$, $k = 2b_p(r_p(s))$, where

$$A_o' = \bigcup_{k=1}^{\infty} \{1,2\}^k.$$

Setting

(5.1) $g_p^o(n) = (x_1,..,x_k,3)$ for $g_p(n) \in \{1,2\}^k$,

$(x_1,\dots,x_k) = g_p(n)$, we have a map g_p^o into

(5.2) $A_o = \bigcup_{k=1}^{\infty} \{(x_1,\dots,x_k,3) : (x_1,\dots,x_k) \in \{1,2\}^k\}$.

Given $w \in N$, let

(5.3) $B_w = \bigcap_{k \in I} \bigcup_{s \in I} \{ x \in A_o^I : x = (x_i, i \in I), \ x_i = (x_{i1}, \dots \dots x_{id_i}), \ \sum_{i=k}^{k+s} (d_i - w) \leq 0 \}$.

Put $S_o = S_{A_o}$, $A = \{1,2,3\}^w$. It was shown in [9] that the following assertion holds.

<u>Lemma 5.1</u>. There is a 1-1 Borel measurable mapping

$f_w : B_w \longrightarrow A^I$ such that $f_w(S_o x) = S_A f_w(x), x \in B_w$.

For any A at most countable, let $\wp_A$ be the Rohlin measurable partition of the space A^I defined by

$$\wp_A = \{ R_A^z : z \in R_A \} \cup \{ (R_A)^c \} \qquad \text{(cf. (3.5))},$$

and let, for μ Borel and S_A-invariant, (F_m, m) be the completion of (F_A, μ). By $(Q^A, \mathcal{B}^A, m_o^A)$ we shall denote the factor space of (A^I, F_m, m) with respect to $\wp_A$, and by Γ^A the corresponding homomorphism (cf. [1], Chapter IV). Let $(J, \mathcal{L}, \lambda)$ be the unit interval with Lebesgue measure. It follows from Lemma 3.2 and from Rohlin's theorem given in [2], § 4, par. 3 that

<u>Lemma 5.2</u>. If μ is non-atomic then there is a mod 0 isomorphism between (A^I, F_m, m) and

$$(Q^A, \mathcal{B}^A, m_o^A) \times (J, \mathcal{L}, \lambda).$$

In the following lemma we have set (F_m, m) for the completion of a Borel measure in N^I and $(Q, \mathcal{B}, m_o)$ for the factor space of $\wp = \wp_N$ (the factor space of the homomorphism $\Gamma = \Gamma^N$); A being an arbitrary finite set, A^I endowed by a completed measure $(F_{\tilde{m}}, \tilde{m})$ which is S_A-invariant, we write $(\tilde{Q}, \tilde{\mathcal{B}}, \tilde{m}_o)$ for the factor space with respect to $\wp_A$ corresponding to homomorphism $\tilde{\Gamma}$. Setting $S = S_N$ everywhere in the sequel, we shall assume that m is S-invariant. The canonical system of measures with respect to $\wp$ and that with respect to $\wp_A$ are denoted

$$(m_X, X \in Q) \quad \text{and} \quad (m_Y, Y \in \tilde{Q}).$$

From (3.3) it follows (cf. [5], Chapter 2) that

(5.4) $m_X = \mu_z$ on $F_N \cap X$, $z \in X$, $X \in Q = \varphi$,

$m_Y = \mu_{z'}$ on $F_A \cap Y$, $z' \in Y$, $Y \in \tilde{Q} = \tilde{\varphi} = \varphi_A$.

Lemma 5.3. If $\Psi : Q_0 \longrightarrow \tilde{Q}$ ($Q_0 \in \mathcal{B}$, $m_0(Q_0) = 1$) is injective and such that, for any $E' \in F_A$ and u real,

$$\Psi^{-1}\{\tilde{\Gamma} z' : \mu_{z'}(E') < u\} \in \mathcal{B},$$

then Ψ is $(\mathcal{B}, \tilde{\mathcal{B}})$-measurable, and if $\tilde{m}_0 = m_0 \Psi^{-1}$, Ψ is a mod 0 isomorphism between the factor space of φ and that of $\tilde{\varphi}$.

Proof. Making use of the properties of the canonical measures m_Y and of (5.4), we obtain that, for $M' \in \tilde{\mathcal{B}}$, $z' \in Y$, $Y \in \tilde{Q}$,

$$m_Y(\tilde{\Gamma}^{-1}M' \cap Y) = \chi_{M'}(Y) \bmod \tilde{m}_0,$$

where $\chi_{M'}$ means the characteristic function of M', and

$$\mu_{z'}(\tilde{\Gamma}^{-1}M') = m_Y(\tilde{\Gamma}^{-1}M' \cap Y)$$

supposed that $\tilde{\Gamma}^{-1}M' \in F_A$. Hence the class of sets of the form

$$\{\tilde{\Gamma} z' : \mu_{z'}(E') < u\}, \quad E' \in V_A, \text{ u rational}$$

generates $\tilde{\mathcal{B}}$ mod $\tilde{m}_0$. From here the assertion of the lemma follows.

6. Theorems. The preceding lemmas will be used together with the notations given in Sec. 5 to proving

Theorem 1. If μ is a non-atomic Borel measure in N^I

which is invariant with respect to the shift S and such that

$$h_\mu(\gamma) < +\infty \ , \ H_\mu(S) < +\infty \ ,$$

then there is a finite set A and a Borel measure μ' in A^I which is S_A-invariant, together with a mod 0 isomorphism between (A^I, F_A, μ') and (N^I, F_N, μ) under which the shifts S_A and S are isomorphic.

Proof. Applying Proposition 5, we get $H_\mu(S) =$ $= H_\mu(S, \gamma) = H(\mu) < +\infty$. Let

$$R_\mu = \{ z \in R : h(\mu_z) \leq H(\mu), \quad \mu_z \text{ non-atomic} \} .$$

From lemmas 3.2 and 3.3 it follows that $\mu(R_\mu) = 1$. Since $R_\mu \in \tilde{\Gamma}^{-1}(\mathcal{B}_0)$, there is a strict correspondence between $Q_0 = \Gamma(R_\mu)$ and $R_\mu = \Gamma^{-1} Q_0$ so that $Q_0 \in \mathcal{B}$, $m_0(Q_0) = 1$. Choose $\delta > 0$ and $w \in N$ such that

$$w > 2(H(\mu) + \delta) + 2.$$

Assign to every $X \in Q_0$ the least integer $q = q(X) \in N$ for which

$$\frac{1}{q} h_{m_X}(\gamma^q) < H(\mu) + \delta ;$$

this is possible by (5.4) and (4.5). Since m_X is X-measurable, the set

$$M_q = \{ X \in Q_0 : q(X) = q \} \in \mathcal{B} ,$$

i.e. it is X-measurable.

Given $q \in N$, $X \in M_q$, we have

$$p_X = (m_X(\mathcal{U}_n(\gamma^q)), \ n \in N) \in P_0 ;$$

$\mathcal{U}(\gamma^q)$ is the ordering associated to γ^q by the convention given in Sec. 4. Let $\bar{z}_n$ be the element in N^q for which $\mathcal{U}_n(\gamma^q) = [\bar{z}_n]$. Define $g_X^o : N^q \longrightarrow A_o$ (cf. (5.1),(5.2)) by

$$g_X^o(\bar{z}_n) = g_{p_X}^o(n), \quad n \in N_{p_X},$$

and $g_X : N^I \longrightarrow (A_o)^I$ as the sequential coding

$$(g_X(z))_i = g_X^o(z_i, z_{i+1}, \ldots, z_{i+q-1}), \; z \in N^I, \; i \in I.$$

It follows from our constructions given in the preceding section that $g_X(z)$ is (X,z)-measurable, and that

$$S_o(g_X(z)) = g_X(Sz);$$

g_X is defined mod m_X. Set

$$f_X^o(z) = f_w(g_X(z)) \text{ on } g_X^{-1}(B_w);$$

cf. (5.3). Since m_X is ergodic,

$$\lim \frac{1}{nq} \sum_{i=0}^{n-1} \sum_{j=0}^{q-1} b_{p_X}(\varphi(z_{iq+j}, \ldots, z_{iq+q+j-1})) \leq$$

$$\leq \frac{1}{q} h_{m_X}(\gamma^q), \; z \in N^I \text{ a.s. } (m_X),$$

where $\varphi : \bar{z}_n \longrightarrow n$ (compare with [9], proof p. 457) so that $m_X(g_X^{-1}(B_w)) = 1$; consequently, f_X^o is defined on N^I mod m_X and, by definition, $f_X^o(z)$ is (X,z)-measurable. It follows from Lemma 5.1 that f_X^o is a 1-1 Borel measurable mapping of $g_X^{-1}(B_w)$ into A^I, $A = \{1,2,3\}^w$ which commutes with the shifts S and S_A (it is, of course, bimeasurable because A^I and N^I are Polish); notice that $f_X^o(g_X^{-1}(B_w))$ is the Borel set $f_w(B_w)$, which we shall denote by B_o.

Let $\Psi : Q_0 \longrightarrow \widetilde{Q}$ be given by

$$\Psi X = \{ z' \in R_A : \mu_{z'} = \mu_z (f_X^0)^{-1} \} \ , \ X \in Q_0, \ z \in X,$$

and set (cf. (5.4))

$$\mu'(E') = \int_{Q} m_X((f_X^0)^{-1}(E') \cap X) \ dm_0(X), E' \in F_A;$$

the sense of the definition is guaranteed by the (X,z)-measurability of $f_X^0(z)$; μ' is an S_A-invariant Borel measure in A^I and its completion will be denoted by $(F_{\widetilde{m}}, \widetilde{m})$. Since the assumptions of Lemma 5.3 are satisfied for Ψ defined above because of the (X,z)-measurability of f_X^0, then, setting $\widetilde{m}_0 = m_0 \Psi^{-1}$, we find that Ψ is a mod 0 isomorphism between the spaces $\wp$ and $\widetilde{\wp} = \wp_A$ (more precisely, Ψ is first shown to be Borel measurable, and then $\widetilde{m}_0$ is the completion of the measure constructed with the aid of Ψ and the restriction of m_0 to Borel sets lying in $\mathcal{B}$, i.e. sets M for which $\Gamma^{-1} M \in F_N$).

Let $B_X = (f_X^0)^{-1}(B_0 \cap \Psi X)$, and let f_X be the restriction of f_X^0 to B_X. It is easy to show that

$$\widetilde{m}(E') = \int_{\widetilde{Q}} m_Y(E' \cap Y) \ d\widetilde{m}_0(Y), E' \in F_m \ .$$

Applying Lemma 5.2 both to μ and μ', we conclude that there are mod 0 isomorphisms $\widetilde{\varphi}, \varphi$ making A^I isomorphic to $\widetilde{Q} \times J$ and N^I to $Q \times J$. We shall set mod 0

$$\overline{\psi}(Y,y) = (\Psi^{-1} Y, \widetilde{f}_Y(y)), \ Y \in \widetilde{Q}, \ y \in J,$$

$$\widetilde{f}_Y(y) = \varphi_X^{-1} \ f_X^{-1}(\widetilde{\varphi}_Y \ y), \ X = \Psi^{-1} Y$$

where φ_X, $\widetilde{\varphi}_Y$ are sections of φ, $\widetilde{\varphi}$; $\widetilde{f}_Y(y)$ is (Y,y)-

measurable because of the measurability of $f_X(z)$ and of the sections so that $\overline{\psi}$ is a mod 0 isomorphism between

$$(\tilde{Q},\tilde{\mathcal{B}},\tilde{m}_0)\times(J,\mathcal{L},\lambda) \text{ and } (Q,\mathcal{B},m_0)\times(J,\mathcal{L},\lambda);$$

hence $\psi = \varphi\overline{\psi}\varphi^{-1}$ is a mod 0 isomorphism between $(A^I,F_{\tilde{m}},\tilde{m})$ and (N^I,F_m,m) which commutes with the shifts S_A and S as follows from its construction; its restriction to a Borel subset of B_0 of μ'-measure one on which it is defined yields the desired mod 0 isomorphism between (A^I,F_A,μ') and (N^I,F_N,μ).

<u>Theorem 2</u>. If $(\Omega,\mathcal{F},\mu)$ is countably generated and such that the number of atoms is essentially bounded with respect to an automorphism T and the asymptotic rate of T is finite, i.e. $H(T)< +\infty$, then if ν is the non-atomic part of measure μ such that (T,ν) is aperiodic, there is a finite generator for the transformation T of $(\mathcal{F},\mu)$.

Proof. Since (T,ν) is aperiodic, it has a generator $\xi_0\in Z_0$ (cf., e.g., [6], Sec. 2); then, according to Lemma 4.1, $\mathcal{F}/\nu$ and F_N/ν^{ξ_0} where

$$\nu^{\xi_0} = \nu(\psi^N_{\xi_0})^{-1}$$

are isomorphic, with the isomorphism making T and S ($S = S_N$) conjugate. By making use of Proposition 1, we conclude that $H_\nu(T) = H_{\nu^{\xi_0}}(S)$; the latter number in general differs from $H_{\nu_0}(S,\gamma)$, $\gamma = \gamma_N$, $\nu_0 = \nu^{\xi_0}$. An application of Proposition 2 together with Proposition 4 yields the relations

$$+\infty > H_{\mu}(T) = H_{\nu}(T) = H_{\nu_0}(S) \geq h_{\nu_0}(S),$$

which makes possible to apply Rohlin's theorem (cf. [10], § 10) to the aperiodic (S, ν_0), and to assert that (S, ν_0) has a generator $\xi \in Z_{\nu_0}(F_N)$, i.e. with finite entropy (to be precise, ν_0 is first completed and the found generator replaced by a Borel one). Applying Lemma 4.1, we conclude that F_N/ν_0 and F_N/ν^{ξ} are isomorphic, where

$$\nu^{\xi} = \nu_0(\psi_{\xi}^{N})^{-1}, \quad \psi_{\xi}^{N}: N^I \longrightarrow N^I$$

constructed on the basis given by the system (N^I, F_A, ν_0, S), for an isomorphism under which (S, ν_0) and (S, ν^{ξ}) are conjugate. Similarly as above we conclude that

$$H_{\nu^{\xi}}(S) = H_{\nu_0}(S) < +\infty .$$

Since $\xi \in Z_{\nu_0}(F_N)$ is equivalent to $\gamma \in Z_{\nu^{\xi}}(F_N)$, we may apply Theorem 1 (cf. Lemma 3.1: ν^{ξ} is non-atomic) to the system (N^I, F_A, ν^{ξ}, S). Let $\nu_1 = \nu^{\xi'}$, and let ψ be an isomorphism between (S, ν_1) and (S_A, ν') for some A finite, ν' Borel on A^I.

Now we take into account that the number of atoms is essentially bounded with respect to T. It follows from (1.3) that there is a natural number $d \in N$ such that

$$\text{card}\,(\Delta_q) \leq d^q$$

Setting $a = \max\,(d, \text{card}\,(A))$, we immediately find that there is a measure algebra isomorphism between S_A/ν' and $S_{A'}/\nu''$ for some $S_{A'}$-invariant measure on $F_{A'}$, $A' = A_a =$

$=\{n\in N: n\leq a\}$. Since $F_{A'}/\nu''$ is isomorphic to $\mathcal{F}/\nu$ by construction, with conjugacy between $S_{A'}$ and T, and since, by Lemma 3.1,

$$\nu''(K(S_{A'}))=0 \text{ and, moreover, card } (K_q(S_{A'}))=a^q$$

for $q\in N$, there is an injective map of Δ_q into $K_q(S_{A'})$, say ψ_q, with the property that

$$\psi_q(TD)=S_{A'}\ \psi_q(D),\ D\in\Delta_q;$$

it is because $d\leq a$. Hence we conclude that there is a measure μ' on $F_{A'}$, Borel and $S_{A'}$-invariant, with ν'' as its non-atomic part, and an isomorphism between $F_{A'}/\mu'$ and $\mathcal{F}/\mu$ under which $S_{A'}$ and T are conjugate. The desired generator corresponds to $\gamma_{A'}$ mod 0 if use is made of the measure algebra isomorphism.

Proof of the Theorem. The sufficiency was established in Theorem. Conversely, if ξ is a finite generator for T, there is a finite set A, an S_A-invariant Borel measure μ' in A^I, and a measure algebra isomorphism between F_A/μ' and $\mathcal{F}/\mu$ so that S_A and T are conjugate with respect to it. Making use of this isomorphism we conclude that card $(\Delta_q)\leq(\text{card}(A))^q$, and that (T,ν), ν the non-atomic part, is aperiodic. Q.E.D.

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