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THE LUSIN-MENCHOFF PROPERTY OF FINE TOPOLOGIES

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Abstract: The Lusin-Menchoff property and the Zahorski property of general "fine" topology are introduced. These properties are discussed in the cases of the density topology, of the Scheinberg's U-topology, and of the fine topology in potential theory.

Key words: Lusin-Menchoff property, Zahorski property, density topology, fine topology in potential theory, Scheinberg's U-topology, Denjoy theorem, pairwise normality of bitopological space.

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Lusin-Menchoff property. Let (P, ϱ) be a topological space. Considering on P another topology τ finer than ϱ which will be called the "fine topology", we can state the following main theorem (the topological notions referring to the τ-topology will be qualified by the prefix τ to distinguish them from those pertaining to the topology ϱ).

Theorem 1. The following assertions are equivalent:

(i) Given any pair of disjoint subsets F, F_τ of P, F closed, F_τ τ-closed, there are $G, G_\tau \subset P$, G open, G_τ τ-open such that

$$F_\tau \subset G, \; F \subset G_\tau, \; G \cap G_\tau = \emptyset .$$

(ii) For any couple $F_\tau \subset U$ of P, F_τ τ-closed, U open, there is an open set $G \subset P$ such that $F_\tau \subset G \subset \overline{G}^\tau \subset U$.

(iii) For any couple $F \subset U_\tau$, F closed, U_τ τ-open, there is a τ-open set Ω_τ such that $F \subset \Omega_\tau \subset \overline{\Omega}_\tau \subset U_\tau$.

(iv) Given any pair of disjoint subsets F, F_τ of P, F closed, F_τ τ-closed, there is a τ-continuous and upper-semicontinuous function f on P such that

$$0 \leq f \leq 1,\ f = 0 \text{ on } F_\tau,\ f = 1 \text{ on } F.$$

Proof. Obviously, (i),(ii) and (iii) are equivalent, and (iv) implies (i). Assuming (i), the construction of f is similar to that of Urysohn's lemma (see [9], Chap. IV, Lemma 4), so we can sketch it only. Let D be the set of positive dyadic rationals. For $t \in D$, $t > 1$ we put $F(t) = P$, let $F(1) = P \setminus F$, and let $F(0)$ be any open set containing F_τ whose τ-closure is disjoint with F. By induction we associate with any $t \in (0,1) \cap D$ an open set $F(t)$ in such a way that

$$t < s,\ t,s \in (0,1) \cap D \Longrightarrow F(t) \subset \overline{F(t)}^\tau \subset F(s).$$

Putting $f: x \longmapsto \inf\{t;\ x \in F(t)\}$, f has all desirable properties. It is τ-continuous ([9], Chap. IV, Lemma 3) and, moreover, it is upper-semicontinuous since

$$\{x \in P;\ f(x) < \alpha\} = \bigcup \{F(t);\ t \in D,\ t < \alpha\}.$$

Obviously,

$$0 \leq f \leq 1 \text{ on } P,\ f = 0 \text{ on } F_\tau,\ f = 1 \text{ on } F.$$

Definition. We shall say that the topology τ has the Lusin-Menchoff property (with respect to ρ) if any of the equivalent assertions of Theorem 1 is satisfied. Of course, if $\tau = \rho$ then the Lusin-Menchoff property is equivalent to the normality of the space (P,τ). Generally, a topology with the Lusin-Menchoff property need not be normal.

Remark. In the setting of bitopological spaces of J.C. Kelly (Proc. London Math. Soc. (3)13(1963), 71-89) the Lusin-Menchoff property of the fine topology τ with respect to topology φ means nothing else as the pairwise normality of the bitopological space (P, τ, φ).

In what follows, given a function f on P, $Z(f)$ stands for the zero set of f, i.e. $Z(f) = \{x \in P;\ f(x) = 0\}$. As usual, $\mathbb{R}$ will be the set of reals.

Corollary 2. Suppose that the topology τ has the Lusin-Menchoff property. Then:

(a) τ is completely regular;

(b) If φ is a metric topology, then τ is cometrizable (topological space (P, τ) is cometrizable if there is a metric topology φ on P coarser than τ such that each point of P has a neighborhood base in (P, τ) the elements of which are φ-closed);

(c) For any closed set F and for any τ-open set G_τ, $F \subset G_\tau$, there is a τ-cozero set C, $F \subset C \subset G_\tau$;

(d) For any τ-closed set F_τ and for any set G of type G_δ, $F_\tau \subset G$, there is a τ-continuous and upper-semicontinuous function f on P such that $F_\tau \subset Z(f) \subset G$;

(e) (Zahorski property of τ) Any τ-closed set of type G_δ is zero set of a τ-continuous and upper-semicontinuous function on P;

(f) Any pair of disjoint τ-closed sets of type G_δ can be separated by τ-cozero sets.

Proof. The proofs are immediate consequences of Theorem 1. For (d), if $G = \bigcap_{n=1}^{\infty} G_n$, G_n open, it is sufficient to put $f = \sum_{n=1}^{\infty} 2^{-n} f_n$, where f_n are constructed as in Theorem 1 for F_τ and $P \setminus G_n$. For (f), assume that A,B are disjoint τ-closed sets of type G_δ, and f_A, f_B are τ-continuous functions on P, $0 \leq f_A, f_B \leq 1$, $Z(f_A) = A$, $Z(f_B) = B$. We put $\varphi = f_A/(f_A + f_B)$, $C_A = \{x \in P;\ \varphi(x) < \frac{1}{2}\}$, $C_B = \{x \in P;\ \varphi(x) > \frac{1}{2}\}$. Then C_A, C_B are disjoint τ-cozero sets containing A,B, respectively.

Remarks. (1) Let ρ be a metric topology on P. Assume that any τ-continuous function on P is of Baire class 1 (more on this subject can be found in [13]). Then any τ-zero set is τ-closed and of type G_δ. Zahorski property for τ states the converse assertion, and thus τ-zero sets are completely described. Moreover, the function in question can be chosen to be upper-semicontinuous.

(2) Assume that the topology τ on P has the Lusin-Menchoff property with respect to ρ. Putting

> $\Phi = \{f;\ f$ is non-negative τ-continuous and lower-semicontinuous function on $P\}$,

Φ is a convex cone on P. The coarsest topology τ_Φ on P finer than ρ making all functions from Φ continuous is exactly τ. Indeed, let U be τ-open, $x \in U$. There is f, $0 \leq f \leq 1$, $1 - f \in \Phi$ such that

$$f(x) = 1,\ P \setminus U \subset Z(f).$$

Then

$$x \in \{y \in P;\ 1 - f(y) < \tfrac{1}{2}\} \subset U,$$

On the other hand, τ is obviously finer than τ_{Φ} . Thus the topology with the Lusin-Menchoff property is the <u>fine topology</u> associated with the cone Φ in the sense of M. Brelot [2] and has all corresponding properties. In particular, we obtain the following corollary immediately (see [2], Theorem I,4).

<u>Proposition 3</u>. If τ has the Lusin-Menchoff property with respect to a locally compact space (P,ρ), then (P,τ) is a Baire space.

<u>Tietze's type extension theorem</u>. The classical Tietze's theorem on extension of continuous functions from closed subsets of topological normal space can be transferred in more general situations. Let us mention just the principle of quasi-normality of the fine topology in potential theory (see Fuglede [6]), or the extension theorem from Lebesgue null sets resulting in approximatively continuous functions (see Petruska and Laczkovich [11]). We shall not examine the connection between the Lusin-Menchoff property and the Tietze's type extension theorem, we state the following simple theorem only.

<u>Theorem 4</u>. Assume that τ has the Lusin-Menchoff property with respect to a metric topology ρ . Let F be a τ-closed subset of P, and f be a τ-continuous [bounded] function on F being a restriction on F of a function of Baire class one. Then f has τ-continuous [bounded] extension f^* to the whole space P. Moreover, f^* is of Baire class one.

<u>Proof</u>. The proof is a slight modification of the classical proof of the Tietze's theorem. Assume $-1 \leq f \leq 1$ on F. Let G be a function of the first class of Baire on P which extends

f. We put

$$G_1 = \{x \in P;\ G(x) \leq -\frac{1}{3}\ ,\ G^1 = \{x \in P;\ G(x) \geq \frac{1}{3}\},$$

$$F_1 = \{x \in F;\ f(x) \leq -\frac{1}{3}\}\ ,\ F^1 = \{x \in F;\ f(x) \geq \frac{1}{3}\}.$$

By Corollary 2.d there is a τ-continuous function φ_1 of Baire class one on P, $-\frac{1}{3} \leq \varphi_1 \leq \frac{1}{3}$, $\varphi_1 = -\frac{1}{3}$ on F_1, $\varphi_1 = \frac{1}{3}$ on F^1. As usual, setting $f_1 = f - \varphi_1$ on F we repeat the process.

Limits of finely continuous functions

Theorem 5. Let (P, ρ) be a metric space, and let τ be a topology on P finer than ρ satisfying

(i) the Lusin-Menchoff property,

(ii) any set of type F_σ (in ρ) is of type G_δ in τ.

Then any function (possibly infinite) of the second class of Baire on P is the limit of a sequence of τ-continuous functions.

Proof. It is known that a function f is the limit of a sequence of τ-continuous functions (i.e. f belongs to the first class of Baire in the topology τ) if and only if for any real c, the sets $\{x \in P;\ f(x) \geq c\}$, $\{x \in P;\ f(x) \leq c\}$ are the countable intersections of τ-cozero sets. Thus, it is sufficient to prove that any set of type F_σ is the countable intersection of τ-cozero sets. Let F be such a set of type F_σ. Using (ii), there are τ-open sets G_n such that $F = \bigcap_{n=1}^{\infty} G_n$. Fix now a natural n. There are closed sets F^j, $F = \bigcup_{j=1}^{\infty} F^j$. For any j we can find τ-cozero set C_n^j such that $F^j \subset C_n^j \subset G_n$. Then $F = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} C_n^j$, and $\bigcup_{j=1}^{\infty} C_n^j$ is τ-cozero set. The proof of Theorem 5 is now straightforward.

Remark. If any τ -continuous function on P is of the first class of Baire and the assumptions of Theorem 5 are satisfied, then the set of all functions of the second class of Baire coincides with the set of all functions which are in the τ-topology of the first class of Baire.

Density topology. Let us consider the usual ordinary density topology on an euclidean space R^n (the ordinary density topology is formed by measurable sets having any of its points as a point of ordinary density; approximately continuous functions are exactly functions continuous in this topology. See, e.g., [8]). The history of discovery of the Lusin-Menchoff property is interesting. It seems that the first attempt is due to V.S. Bogomolova 1924 (Sur une classe des fonctions asymptotiquement continues, Matem. Sbornik 32(1924),152-171, Russian with French summary). She writes word for word: "Les théorèmes sur les points d'épaisseur étaient démontrés d'abord par M. N.N. Lusin et M. D.E. Menchoff. N'ayant aucune idée de leur méthode j'ai obtenu quelques jours plus tard une autre démonstration". The generalization of the results of V. Bogomolova can be found in the paper of I. Maximoff 1940,[15]. In [8] the proof of the Lusin-Menchoff property is simplified and generalized to R^n. We must underline that in these papers the obtained results are a little deeper than the just defined Lusin-Menchoff property.

Thus, the density topology has the Lusin-Menchoff property. Moreover, any Borel set in density topology is of type G_σ in this topology. Indeed, any density-Borel set is Lebesgue measurable, so it is of the form $G \setminus N$, where G is of type G_σ

and N has Lebesgue measure zero. It follows that any such a set is G_δ in density topology. Of course, any continuous function in the density topology is of Baire class one (cf. [13]), thus our Theorem 5 gives the following theorem which was proved by D. Preiss 1971, [16], and independently by G. Petruska and M. Laczkovich 1973, [10].

<u>Theorem</u>. Any function (possibly infinite) on R^n is of the second class of Baire if and only if it is the limit of a sequence of approximately continuous functions.

There are variety of mathematical papers devoted the study of the approximate continuity and of the density topology beginning with the significant investigation by A. Denjoy in 1915. The summary of the most important facts about the density topology is collected in the recent paper of F.D. Tall 1976, [18]. We add simple remarks only.

It is not difficult to prove that the Borel subsets of the real line in the density topology are precisely the Lebesgue measurable sets. Further, the rare (= nowhere dense) subsets in density are always closed, and coincide with the Lebesgue null sets. These observations lead easily to another proof of the classical result of A. Denjoy.

<u>Theorem</u> (Denjoy). A real function on $\mathbb{R}$ is Lebesgue measurable if and only if it is approximately continuous almost everywhere.

<u>Proof</u>. In density topology, the meagre sets (= sets of first category) are closed. Therefore, the Borel sets coincide with almost open sets (= sets with the Baire property). It follows that f is a Borel function in density topology if and only if it is almost open (= measurable with respect to the

system of all almost open sets), and this is the case if and only if there is a null set $N \subset \mathbb{R}$ such that the restriction of f to $\mathbb{R} \setminus N$ is approximately continuous. Thus, a function f is Lebesgue measurable iff it is approximately continuous almost everywhere (i.e. approximately continuous on a density-open set $\mathbb{R} \setminus N$).

The previous considerations are closely related to many constructions of functions with required properties. As a simple application only, we draw the attention to the so-called "functions of Pompeiu" whose lengthy and detailed study can be found in S. Marcus 1963, [14]. Some questions raised there were answered in the papers of A. Bruckner 1863, [3] and of J.S. Lipiński 1963, [12]. Using the Extension theorem 4 we are able to construct the simple counterexamples as well. (It seems that the similar problems motivated the investigation of Petruska and Laczkovich.)

(a) Let A be a countable dense subset of $\mathbb{R}$ containing 0 disjoint with $B = \{n^{-1};\ n \text{ natural}\}$. There is a bounded approximately continuous function f such that $A \subset Z(f)$, $f = 1$ on B. Thus, at the point $0 \in Z(f)$ the function f is not continuous.

(b) Assume that A_1, A_2, A_3 are disjoint, dense, countable subsets of $\mathbb{R}$. There is a bounded approximately continuous function f such that

$$A_1 \subset Z(f),\ A_2 \subset \{x \in \mathbb{R};\ f(x) > 0\},\ A_3 \subset \{x \in \mathbb{R};\ f(x) < 0\}.$$

(It is sufficient to prescribe the values of f at the points of A_1, A_2, A_3 as follows:

$f(x) = 0$ if $x \in A_1$, $f(a_n^i) = (-1)^i n^{-1}$, if $A_i = \{a_1^i, a_2^i, \dots\}$, $i = 1,2$.)

The primitive of f on R is thus the example of function which is differentiable everywhere and nowhere monotone.

Remark. There are some generalizations of the Lusin-Menchoff theorem in more general setting (see, e.g., [4]). On the other hand, the so-called strong density topology introduced in [8] has not the Lusin-Menchoff property.

Scheinberg's U-Topologies. In Scheinberg's paper [17], 1971 the density topology on the real line was strengthened to extremally disconnected topologies using ultrafilters on the collection of sets of positive Lebesgue measure. More precisely, let $\mathcal{U}$ be an ultrafilter in the family of all measurable subsets of $\mathbf{R}$ containing the filter of all measurable sets having the density 1 at the point 0. A set A is a $\mathcal{U}$-neighborhood of 0 if A contains some member of $\mathcal{U}$. By translation of $\mathcal{U}$ we define $\mathcal{U}$-neighborhoods of any point in $\mathbf{R}$. In [19], F.D. Tall asked for the cometrizability of the $\mathcal{U}$-topology. The answer is contained in the following theorem.

Theorem 6. The $\mathcal{U}$-topology has the Lusin-Menchoff property (with respect to the euclidean topology on $\mathbf{R}$).

Proof (the idea is due to L. Zajíček): Given a closed set $F \subset \mathbf{R}$ and a $\mathcal{U}$-closed set $F_\tau \subset \mathbf{R}$, it is sufficient to construct an open set G containing F_τ in such a way that any point of F is point of dispersion of $G \setminus F_\tau$. Indeed, then the closure of $G \setminus F_\tau$ in the density topology is disjoint with F, and therefore the $\mathcal{U}$-closure of G does not meet F.

Let $\{(a_n, b_n)\}$ be the sequence of all contiguous intervals of F. Put

$\alpha_n = a_n$ if $a_n \in \mathbb{R}$, and $\alpha_n = b_n - 1$ if $a_n = -\infty$,

$\beta_n = b_n$ if $b_n \in \mathbb{R}$, and $\beta_n = a_n + 1$ if $b_n = +\infty$,

$c_n^o = \frac{1}{2}\,(\alpha_n + \beta_n).$

For positive integers k define points c_n^k by the relation

$$c_n^{k-1} - c_n^k = \frac{1}{n+k+1}\,(c_n^{k-1} - \alpha_n).$$

Obviously, $c_n^k \searrow \alpha_n$. For negative integers k define c_n^k in such a way that c_n^o would be the center of the segment c_n^{-k}, c_n^k .
It is easy to see that there is an open set G containing F_τ such that

$$\frac{\mu((c_n^k, c_n^{k-1}) \cap (G \setminus F_\tau))}{c_n^{k-1} - c_n^k} < \frac{1}{n+k} \quad \text{for positive } k,$$

and

$$\frac{\mu((c_n^{k+1}, c_n^k) \cap (G \setminus F_\tau))}{c_n^k - c_n^{k+1}} < \frac{1}{n+|k|} \quad \text{for negative } k.$$

If $x \in (\alpha_n, \beta_n)$, choose j such that $x \in [c_n^{j+1}, c_n^j)$.
Then

$$(*) \quad \frac{\mu((\alpha_n, x) \cap (G \setminus F_\tau))}{x - \alpha_n} \leqq \frac{\mu((\alpha_n, c_n^j)\ (G \setminus F_\tau))}{c_n^{j+1} - \alpha_n} \leqq$$

$$\leqq \frac{\mu((\alpha_n, c_n^{j+1}) \cap (G \setminus F_\tau))}{c_n^{j+1} - \alpha_n} + \frac{c_n^j - c_n^{j+1}}{c_n^{j+1} - \alpha_n} \leqq \frac{2}{n} \quad .$$

Similarly, if m is a positive integer and $j > m$, then

$$(**) \qquad \frac{\mu((\alpha_n, x) \cap (G \setminus F_\tau))}{x - \alpha_n} \leqq \frac{2}{m} \quad .$$

Now, any point $x \in F$ is a point of dispersion from the right of $G \setminus F_\tau$. This follows easily from $(**)$ in the case $x = a_n$

for some n, and from ($*$) in the opposite case. Using symmetry of our construction, any point $x \in F$ is in fact a point of dispersion of $G \setminus F_{\tau}$. The proof is complete.

In [17] it is proved that the $\mathfrak{U}$-Borel sets coincide with the Lebesgue measurable sets, and that a set is $\mathfrak{U}$-rare if and only if its measure is zero. The same observations as above give the following result.

Theorem. A real function on R is Lebesgue measurable if and only if it is $\mathfrak{U}$-continuous almost everywhere.

Remark. The important theorem from [17] asserts that any bounded measurable function is almost everywhere equal to a unique determined $\mathfrak{U}$-continuous function. Thus, it is easy to see that the $\mathfrak{U}$-topology serves an example of "fine" topology in which not every $\mathfrak{U}$-continuous function is of the first class of Baire.

Fine topology in potential theory. In this section, X will denote an abstract $\mathfrak{P}$-harmonic space with countable base in the sense of the axiomatics C. Constantinescu and A. Cornea. For all notions we refer to [5]. The fine topology on X is defined as the coarsest topology on X which is finer than the initial topology and which makes any hyperharmonic function on X continuous. The fine topology is always completely regular, and X endowed with the fine topology is a Baire space. On the other hand, the fine topology has many pathological properties - it is seldom metrizable, generally, it is neither normal nor Lindelöf, it has not the Blumberg property. Recently, B. Fuglede 1974, [7] proved that any finely continuous function on X is of the Baire class one (the simplified proof of this fact can be found in [13]). Hence, fine-zero sets are fine closed and of type G_δ. Conversely, we shall show that the fine topology (under certain restricted assumptions) has the Zahorski property. Even, it has the Lusin-Menchoff property.

Theorem 7. If the axiom of polarity holds on X, then the fine topology on X has the Lusin-Menchoff property.

Proof. Assume that F,Q are disjoint sets, F closed, Q fine closed. First, we find a zero set Z of a finely continuous function such that $Q \subset Z \subset X \setminus F$. Denoting by b(A) the set of all points of X where A is not thin, we have

$$Q = b(Q) \cup [Q \setminus b(Q)] .$$

If p is a finite, continuous strict potential on X, then

$$b(Q) = \{x \in X;\ \hat{R}_p^Q(x) = p(x)\} .$$

Obviously, $\hat{R}_p^Q$ and p are finely continuous functions on X, and $p - \hat{R}_p^Q$ is upper-semicontinuous. Therefore, b(Q) is a zero set of a finely continuous and upper-semicontinuous function. The set $Q \setminus b(Q)$ is polar, so it is contained in a polar set P of type G_δ ([5], Corollary 7.2.3). We can suppose that $P \subset X \setminus F$. By a theorem of M. Brelot 1958 (cf. [5] , Exercise 6.2.1), there exists a potential q on X which is $+\infty$ on P and finite on $X \setminus P$. Then

$$P = \{x \in X;\ \frac{\pi}{2} - \operatorname{arctg} q(x) = 0\},$$

and therefore P is a zero set of a finely continuous and upper-semicontinuous function f on X, $0 \leq f \leq 1$. Now, we put $Z = b(Q) \cup \cup P$. If we denote by h the continuous function on X such that

$0 \leq h \leq 1$, $h = 0$ on F, $h > 0$ on $X \setminus F$,

then the function $f/f + h$ has all properties from Theorem 1.

Corollary (Zahorski property of the fine topology). If the axiom of polarity holds on X, then the zero sets of finely continuous functions are exactly fine closed sets of type G_δ.

Remark. Even in the case of harmonic functions derived from the Laplace equation, the F_σ sets need not be of type G_δ in fine topology. Nevertheless, we shall show that in the fine topology (under certain assumptions) any fine closed subset of X is of type G_δ in this topology. We restrict to harmonic spaces only in which the axiom of thinness holds. This axiom was introduced in the theory of harmonic spaces by J. Bliedtner and W. Hansen 1975 [1] and says that any semi-polar set is totally thin, i.e. it is thin at every point of X.

Proposition 8 (perfectness of the fine topology). If the axiom of thinness holds on X, then any fine closed set is of type G_δ in the fine topology.

Proof. Let M be a totally thin subset of X. Using Corollary 7.2.3 of [5], there is a totally thin set M^* of type G_δ containing M. Let G_n be open sets such that $\bigcap_{n=1}^{\infty} G_n = M^*$. Since $M^* \setminus M$ is totally thin, $M^* \setminus M$ is fine closed ([5], Corollary 6.3.5). It follows that $M = \bigcap_{n=1}^{\infty} [G_n \setminus (M^* \setminus M)]$ is of type G_δ in fine topology. Now, let F be a fine closed subset of X. So, $F = b(F) \cup (F \setminus b(F))$. The set $F \setminus b(F)$ is semi-polar, and, in view of the axiom of thinness, it is totally thin. The set $b(F)$ is always of type G_δ.

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